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A New Generalized Lomax Model: Statistical Properties And Applications

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ABSTRACT

In this paper, a new version of the Poisson Lomax distributions is proposed and studied. The new density is expressed as a linear mixture of the Lomax densities. The failure rate function of the new model can be increasing-constant, increasing, U shape, decreasing and upside down-increasing. The statistical properties are derived and four applications are provided to illustrate the importance of the new density. The method of maximum likelihood is used to estimate the unknown parameters of the new density. Adequate fitting is provided by the new model.

Keywords: Truncated Poisson Distribution; Lomax Distribution; Moments, Moment Generating Function; Maximum Likelihood; Simulation; Modelling Real Data.

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1. Introduction and physical motivation

The cumulative distribution function (CDF) of the two parameters generalized Lomax distribution (GL) is given as

$$G_{\theta,\gamma}(y) = [1-(y+1)^{-\gamma}]^{\theta} \;\forall\, y>0 \tag{1}$$

where $\theta > 0$ and $\gamma > 0$ are the shape parameters. The corresponding probability density function (PDF) is given by

$$g_{\theta,\gamma}(y) = \theta\gamma(y+1)^{-1-\gamma}[1-(y+1)^{-\gamma}]^{\theta-1} \;\forall\, y>0 \tag{2}$$

For $\theta = 1$, we have the one parameter Lomax (L) (or Pareto type II (PaII)) model. Clearly, the GL model is a special case from the generalized Burr XII (GBXII) model. Following Yousof et al. (2017), we derive a new model called the Burr X generalized Lomax (BXGL) model defined by the CDF given by

$$\boldsymbol{W}_{\delta,\theta,\gamma}(y) = \left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)\right]^{\delta} \;\forall\, y>0 \tag{3}$$

where $\delta > 0$ is refer to shape parameter. Assume having a system of Nindependent functioning subsystems at a given time where N has zero truncated Poisson (ZTP) distribution with parameter a. The probability mass function (PMF) of N is formulated as follows

$$p_{ZTP}^{(a)}(N=n)|_{(a\in\boldsymbol{R}^{+}-\{0\} and\ n=1,2,\ldots)} = [exp(-a)a^{n}]/\left(n!\,\boldsymbol{v}_{(a)}\right). \tag{4}$$

Note that for ZTP random variable (r.v.), the expected value $\boldsymbol{E}(N|a)$ and the variance $Var(N|a)$ are, respectively, given by

$$\boldsymbol{E}(N|a) = \frac{a}{\boldsymbol{v}_{(a)}},$$

and

$$Var(N|a) = \frac{a+a^{2}}{\boldsymbol{v}_{(a)}} - \frac{a^{2}}{\left[\boldsymbol{v}_{(a)}\right]^{2}}$$

where $\boldsymbol{v}_{(a)} = -exp(-a)+1$.Suppose that the failure time of each subsystem has the BXGL.

Let Z_i denotes the failure time of the i^{th} subsystem and let

$$Y = min\{Z_1, Z_2, \cdots, Z_N\}.$$

Then the conditional CDF of Y given N is

$$F(y \mid N) = 1 - Pr(Y > y \mid N) = 1 - \left[1-\boldsymbol{W}_{\delta,\theta,\gamma}(y)\right]^{N}.$$

Therefore, the unconditional CDF of the Poisson Burr X generalized Lomax (PBXGL) CDF can be expressed as

$$F_{\underline{\boldsymbol{\Psi}}}(y)|_{(a\in\boldsymbol{R}-\{0\})} = \boldsymbol{v}_{(a)}\left(1-exp\left\{-a\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)\right]^{\delta}\right\}\right), \tag{5}$$

where $\underline{\boldsymbol{\Psi}} = (a,\delta,\theta,\gamma)$ refer to the parameters vector. The corresponding PDF is

$$f_{\underline{\boldsymbol{\Psi}}}(y)|_{(a\in\boldsymbol{R}-\{0\})} = 2a\delta\theta\gamma \frac{(y+1)^{-1-\gamma}[1-(y+1)^{-\gamma}]^{2\theta-1}}{\boldsymbol{v}_{(a)}\{1-[1-(y+1)^{-\gamma}]^{\theta}\}^{3}}$$
$$\times \frac{exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)}{[1-exp(-\{[1-(y+1)^{-\gamma}]^{-\theta}-1\}^{-2})]^{1-\delta}}$$
$$\times\, exp\left\{-a\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)\right]^{\delta}\right\}. \qquad (6)$$

The hazard rate function (HRF) can be calculated by $f_{\underline{\boldsymbol{\Psi}}}(y)/\left[1-F_{\underline{\boldsymbol{\Psi}}}(y)\right]$.The PBXGL density can be right skewed and unimodal (see Figure 1) whereas the PBXGL HRF can be increasing, **J** shape, **U** shape, decreasing or upside down (see Figure 2). We are also motivated to introduce the PBXGL distribution that contains many new lifetime models as illustrated in Table 1

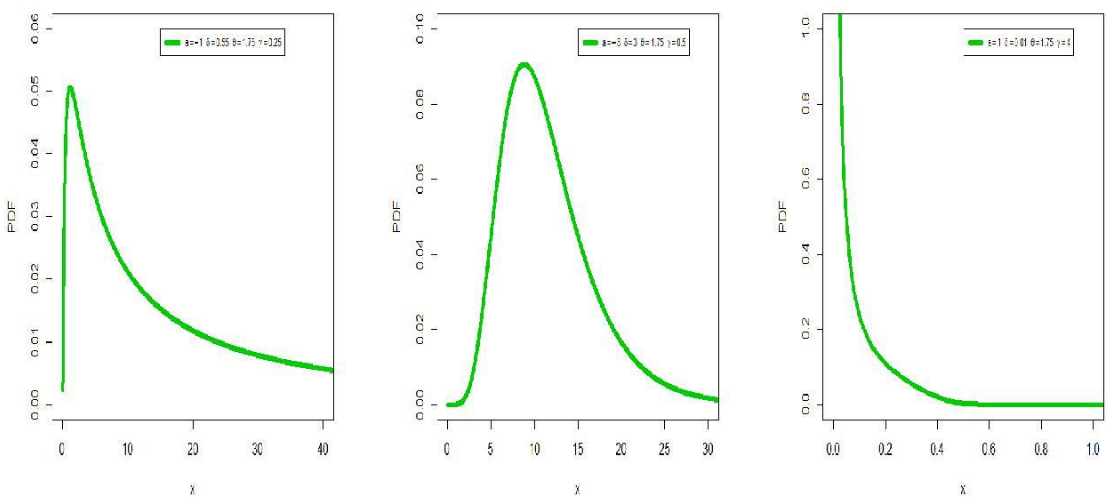


Figure 1: Plots of the PBXGL PDF at some parameter values.

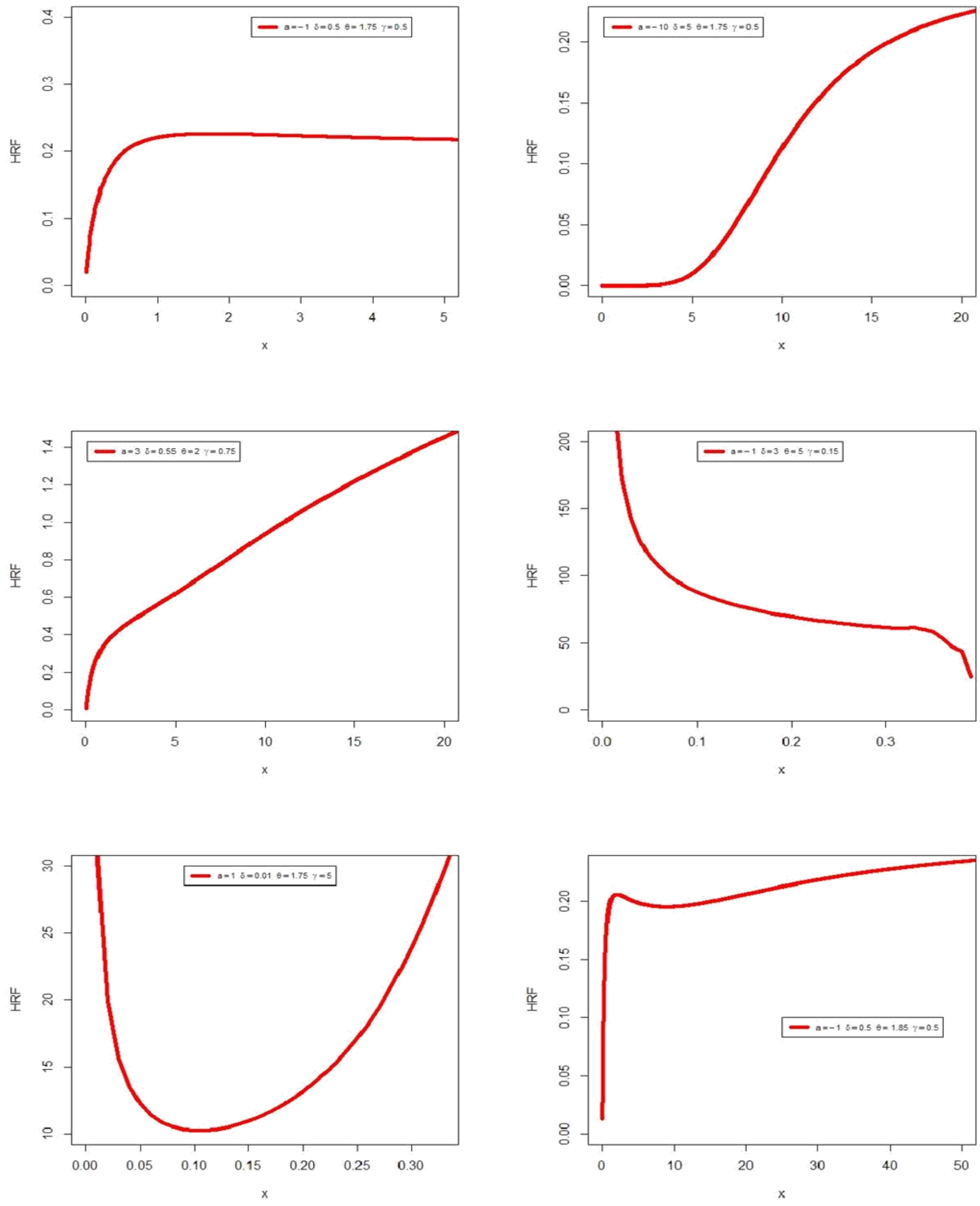


Figure 2: Plots of the PBXGL HRF at some parameter values.

Table 1: CDF for sub-models of the PBXGL model.

	a	δ	θ	γ	Reduced model	CDF
1	ξ				BXGL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)\right]^{\delta}\right\}}{v_{(\xi)}}$
2	ξ			1	QBXGL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-1}]^{-\theta}-1\right\}^{-2}\right)\right]^{\delta}\right\}}{v_{(\xi)}}$
3	ξ		1		BXL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-1}-1\right\}^{-2}\right)\right]^{\delta}\right\}}{v_{(\xi)}}$
4	ξ		1	1	QBXL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-1}]^{-1}-1\right\}^{-2}\right)\right]^{\delta}\right\}}{v_{(\xi)}}$
5	ξ	1	1		RL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-1}-1\right\}^{-2}\right)\right]\right\}}{v_{(\xi)}}$
6	ξ	1	1	1	QRL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-1}-1\right\}^{-2}\right)\right]\right\}}{v_{(\xi)}}$
7	ξ	1			RGL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)\right]\right\}}{v_{(\xi)}}$

	a	δ	θ	γ	Reduced model	CDF
8	ξ	1		1	QRGL	$\frac{1-exp\left\{-\xi\left[1-exp\left(-\left\{\left[1-(y+1)^{-1}\right]^{-\theta}-1\right\}^{-2}\right)\right]\right\}}{\boldsymbol{v}_{(\xi)}}$
9		1			PRGL	$\frac{1-exp\left\{-a\left[1-exp\left(-\{[1-(y+1)^{-\gamma}]^{-\theta}-1\}^{-2}\right)\right]\right\}}{\boldsymbol{v}_{(a)}}$
10		1		1	QPRGL	$\frac{1-exp\left\{-a\left[1-exp\left(-\left\{\left[1-(y+1)^{-1}\right]^{-\theta}-1\right\}^{-2}\right)\right]\right\}}{\boldsymbol{v}_{(a)}}$
11			1		PBXL	$\frac{1-exp\left\{-a\left[1-exp\left(-\{[1-(y+1)^{-\gamma}]^{-1}-1\}^{-2}\right)\right]^{\delta}\right\}}{\boldsymbol{v}_{(a)}}$
12			1	1	QPBXL	$\frac{1-exp\left\{-a\left[1-exp\left(-\left\{\left[1-(y+1)^{-1}\right]^{-1}-1\right\}^{-2}\right)\right]^{\delta}\right\}}{\boldsymbol{v}_{(a)}}$
13		1	1		PRL	$\frac{1-exp\left\{-a\left[1-exp\left(-\{[1-(y+1)^{-\gamma}]^{-1}-1\}^{-2}\right)\right]\right\}}{\boldsymbol{v}_{(a)}}$
14		1	1	1	QPRL	$\frac{1-exp\left\{-a\left[1-exp\left(-\left\{\left[1-(y+1)^{-1}\right]^{-1}-1\right\}^{-2}\right)\right]\right\}}{\boldsymbol{v}_{(a)}}$
15				1	Quasi PBXGL	$\frac{1-exp\left\{-a\left[1-exp\left(-\left\{\left[1-(y+1)^{-1}\right]^{-\theta}-1\right\}^{-2}\right)\right]^{\delta}\right\}}{\boldsymbol{v}_{(a)}}$

Q=Quasi, R=Rayleigh and $\xi = a|_{(a\to 0)}$

This work is organized as follows: In Section 2, we derive some new properties of the PBXGL model. The maximum likelihood method is addressed in Section 3. Simulation studies are presented in Section 4. In Section 5, the potentiality of the proposed model is illustrated using three real data sets. Section 6 gives some concluding remarks.

2 Mathematical properties

2.1 Useful expansions

Using the power series

$$exp(\zeta_1) = \sum_{\zeta_2=0}^{\infty} \frac{\zeta_1^{\zeta_2}}{\zeta_2!}$$

the $f_{\underline{\boldsymbol{\Psi}}}(y)$ in (6) can be written as

$$f_{\underline{\boldsymbol{\Psi}}}(y) = \sum_{\tau=0}^{\infty} \frac{2a^{1+\tau}\delta\theta\gamma}{\boldsymbol{v}_{(a)}}(-1)^{\tau}$$
$$\times\, exp\left(-\left\{[1-(y+1)^{-\gamma}\%]^{-\theta}-1\right\}^{-2}\right)$$
$$\times \frac{(y+1)^{-1-\gamma}[1-(y+1)^{-\gamma}]^{2\theta-1}}{\{1-[1-(y+1)^{-\gamma}]^{\theta}\}^{3}}$$

$$\times\left\{1-exp\left(-\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right)\right\}^{\delta(\tau+1)-1}. \tag{7}$$

Consider the following power series

$$(1-\zeta_1)^{\zeta_2}=\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(1+\zeta_2)}{m!\Gamma(1+\zeta_2-m)}\zeta_1^m\,|_{|\zeta_1|<1\;and\;\zeta_2>0}\text{ is a real non-integer} \tag{8}$$

Applying (8) to (7) we have

$$f_{\underline{\Psi}}(y)=\frac{2\delta\gamma(y+1)^{-1-\gamma}[1-(y+1)^{-\gamma}]^{2\theta-1}}{\boldsymbol{v}_{(a)}\{1-[1-(y+1)^{-\gamma}]^{\theta}\}^3}\times\sum_{\tau,i=0}^{\infty}\frac{a^{1+\tau}(-1)^{\tau+i}\Gamma\left(\delta(\tau+1)\right)}{i!\,\Gamma(\delta(\tau+1)-i)}exp\left(-(i+1)\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right). \tag{9}$$

Applying the power series to the term

$$exp\left(-(i+1)\left\{[1-(y+1)^{-\gamma}]^{-\theta}-1\right\}^{-2}\right),$$

then, equation (9) becomes

$$f_{\underline{\Psi}}(y)=\sum_{\tau,i,d=0}^{\infty}\frac{2\delta a^{1+\tau}(-1)^{\tau+i+d}(i+1)^d\Gamma\left(\delta(\tau+1)\right)}{i!\,d!\,\boldsymbol{v}_{(a)}\Gamma(\delta(\tau+1)-i)}\times\frac{\theta\gamma(y+1)^{-1-\gamma}[1-(y+1)^{-\gamma}]^{2\theta(d+1)-1}}{\{1-[1-(y+1)^{-\gamma}]^{\theta}\}^{3+2d}}. \tag{10}$$

Consider the series expansion

$$(1-\zeta_1)^{-\zeta_2}=\sum_{\omega=0}^{\infty}\frac{\Gamma(\zeta_2+\omega)}{\omega!\,\Gamma(\zeta_2)}\zeta_1^{\omega}\,|_{(|\zeta_1|<1,\,\zeta_2>0)}, \tag{11}$$

and, applying (11) to (10) for the term

$$\left\{1-[1-(y+1)^{-\gamma}]^{\theta}\right\}^{-(3+2d)},$$

equation (10) can be written as

$$f_{\underline{\Psi}}(y)=\sum_{\tau,i,d,\kappa=0}^{\infty}\frac{2\delta a^{1+\tau}(-1)^{\tau+i+d}(i+1)^d}{i!\,d!\,\kappa!\,\boldsymbol{v}_{(a)}}\frac{\Gamma\left(\delta(\tau+1)\right)\Gamma(3+2d+\kappa)}{\Gamma(\delta(\tau+1)-i)\Gamma(2d+3)}\times\theta\gamma(y+1)^{-1-\gamma}[1-(y+1)^{-\gamma}]^{[2(d+1)+\kappa]\theta-1},$$

which can be expressed as

$$f_{\underline{\Psi}}(y)=\sum_{r=0}^{\infty}c_r g_{\gamma(1+r)}(y), \tag{12}$$

where

$$c_r=\frac{(-1)^r\Gamma([2(1+d)+\kappa]\theta)}{r!\,\Gamma([2(1+d)+\kappa]\theta-r)}\times\sum_{d,\kappa=0}^{\infty}\frac{2\delta a^{1+\tau}(-1)^d\Gamma(3+2d+\kappa)}{d!\,\kappa!\,\boldsymbol{v}_{(a)}\Gamma(2d+3)}\times\sum_{\tau,i=0}^{\infty}\frac{(-1)^{\tau+i}\Gamma\left(\delta(\tau+1)\right)(i+1)^d}{i!\,\Gamma(\delta(\tau+1)-i)},$$

and

$$g_{\gamma(1+r)}(y)=\gamma(1+r)(y+1)^{-1-\gamma(1+r)},$$

is the PDF of the Lomax model with parameter $\gamma(1+r)$. Equation (12) reveals that the density of the PBXGL model can be expressed as a linear mixture of the Lomax densities. Thus, several mathematical properties of the PBXGL model can be obtained from the Lomax densities. Similarly, the CDF of the PBXGL model can also be expressed as a mixture of the Lomax CDFs and given by

$$F_{\underline{\Psi}}(y) = \sum_{r=0}^{\infty} c_r \, G_{\gamma(1+r)}(y),$$

where

$$G_{\gamma(1+r)}(y) = 1 - (y+1)^{-\gamma(1+r)}$$

is the CDF of the Lomax model with parameters $\gamma(1+r)$.

2.2 Quantile and random number generation

The quantile function (QF) of Y; where $Y \sim PBXGL(a, \delta, \theta, \gamma)$,is obtained by inverting (5) as

$$Q(U) = \sqrt{-1 + \sqrt[\gamma]{\left\{1 - \sqrt[\theta]{1 + \sqrt{-\ln\left[1 - \sqrt[\delta]{-\frac{1}{a}\ln\left(1 - U\,\boldsymbol{v}_{(a)}\right)}\right]}}\right\}^{-1}}}, \qquad (13)$$

simulating the PBXGL r.v. is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the r.v. $y = Q(U)$ follows (5).

2.3 Ordinary moments

The n^{th} ordinary moment of Y, say $\mu_n^{'}$, follows from (12) as

$$\mu_n^{'} = \boldsymbol{E}(Y^n) = \sum_{r=0}^{\infty} c_r \; (1+r)\gamma B\big((1+r)\gamma - 1, n+1\big)|_{[n<\gamma(1+r)]}, \qquad (14)$$

where

$$B(\zeta_1, \zeta_2) = \int_0^{\infty} \omega^{\zeta_1 - 1}\,(1+\omega)^{-(\zeta_1+\zeta_2)} d\omega,$$

setting $n = 1$ in (14) gives the mean of Y as

$$\mu_1^{'} = \boldsymbol{E}(Y) = \sum_{r=0}^{\infty} c_r \; (1+r)\gamma B\big((1+r)\gamma - 1,2\big)|_{[1<\gamma(1+r)]}$$

2.4 Incomplete moment

The n^{th} incomplete moment of Y is defined by

$$m_n(t) = \int_{-\infty}^{t} y^n\, f(y) dy.$$

Based on (12), the n^{th} incomplete moment of Y becomes

$$m_n(t) = \sum_{r=0}^{\infty} c_r \; (1+r)\gamma B(t; (1+r)\gamma - 1, n+1)|_{[n<\gamma(1+r)]},$$

where

$$B(\cdot; \zeta_1, \zeta_2) = \int_0^{\cdot} \omega^{\zeta_1 - 1}\,(1+\omega)^{-(\zeta_1+\zeta_2)} d\omega,$$

is incomplete beta functions of the second type, respectively.

2.5 Moment generating function (MGF)

The MGF of Y , say $M_Y(t) = \boldsymbol{E}[exp(tY)]$, can be obtained via (12) as

$$M_Y(t; a, \delta, \theta, \gamma) = \sum_{r,n=0}^{\infty} \frac{t^n}{n!} c_r\ (1+r)\gamma B\big((1+r)\gamma - 1, n+1\big)|_{[n<\gamma(1+r)]}.$$

3 Parameter estimation

Consider the estimation of the unknown parameters $(a, \delta, \theta, \gamma)$ of the PBXGL model from the complete samples by maximum likelihood (ML) method. Suppose that $Y_1, \cdots, Y_n$ be a random sample from the PBXGL model with parameter vector $\underline{\boldsymbol{\Psi}} = (a, \delta, \theta, \gamma)^{\intercal}$. The log-likelihood function ($\ell_n(\underline{\boldsymbol{\Psi}})$) for $\underline{\boldsymbol{\Psi}}$ is given by

$$\begin{aligned}\ell_n(\underline{\boldsymbol{\Psi}}) &= n\, log\, 2 + n\, log\, \delta + n\, log\, a + n\, log\, \gamma - n\, log\big[\boldsymbol{v}_{(a)}\big] - (\gamma+1)\sum_{i=1}^{n} log(1+y_i) \\ &+(2\theta - 1)\sum_{i=1}^{n} log[1-(1+y_i)^{-\gamma}] - 3\sum_{i=1}^{n} log(1-q_i) \\ &-\sum_{i=1}^{n} m_i + (\delta - 1) - \sum_{i=1}^{n} log[1-exp(-m_i)] - a\sum_{i=1}^{n}[1-exp(-m_i)]^{\delta},\end{aligned}$$

where

$$q_i = [1-(1+y_i)^{-\gamma}]^{\theta} \text{and } m_i = \left(\frac{q_i}{1-q_i}\right)^2.$$

The function $\ell_n(\underline{\boldsymbol{\Psi}})$ can be maximized numerically via SAS (PROC NLMIXED) or R (optim) or (Ox) program (via sub-routine MaxBFGS). The components of the score vector

$$\boldsymbol{U}(\underline{\boldsymbol{\Psi}}) = \frac{\partial \ell}{\partial \underline{\boldsymbol{\Psi}}} = \left(\frac{\partial \ell_n(\underline{\boldsymbol{\Psi}})}{\partial a}, \frac{\partial \ell_n(\underline{\boldsymbol{\Psi}})}{\partial \delta}, \frac{\partial \ell_n(\underline{\boldsymbol{\Psi}})}{\partial \theta}, \frac{\partial \ell_n(\underline{\boldsymbol{\Psi}})}{\partial \gamma}\right)^{\intercal}$$

can easily be derived.

4 Simulation study

Using (13), we simulate the PBXGL model by taking n =20, 50, 150, 500 and 1000. For each sample size, we evaluate the ML estimations (MLEs) of the parameters using the optim function of the Mathcad software. Then, we repeat this process 1000 times and compute the averages of the estimates (AEs) and mean squared errors (MSEs). Table 2 gives the simulation results. The values in Table 2 indicate that the MSEs of $\hat{a}, \hat{\delta}, \hat{\theta}$ and $\hat{\gamma}$ decay toward zero when n increases for all settings of a, δ, θ and γ as expected under first-under asymptotic theory. The AEs of (**I**: $a = 0.5$, $\delta = 1.5, \theta = 0.9$ and $\gamma = 1.$ and **II**: $a = 2.5, \delta = 0.8, \theta = 1.6$ and $\gamma = 0.6$ when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs.

Table 2: The AEs, biases and MSEs based on 1000 simulations.

n	Θ	AEs	MSE	Θ	AEs	MSE
	I			**II**		
20	a	0.550554	0.687748	a	2.633217	1.137718
	δ	1.535720	0.161429	δ	0.813639	0.019832
	θ	0.896758	0.004083	θ	1.606293	0.026339
	γ	1.212598	0.008217	γ	0.605537	0.004376
50	a	0.479577	0.275757	a	2.502158	0.409679
	δ	1.533873	0.064385	δ	0.812062	0.007323
	θ	0.901872	0.001714	θ	1.610576	0.010553
	γ	1.200769	0.003230	γ	0.599028	0.001679
150	a	0.495784	0.086247	a	2.489394	0.134721
	δ	1.509371	0.019408	δ	0.805320	0.002347
	θ	0.900356	0.000549	θ	1.605307	0.003486
	γ	1.200597	0.001031	γ	0.598894	0.000561
500	a	0.494325	0.025012	a	2.511130	0.041126
	δ	1.505033	0.005616	δ	0.799685	0.000695
	θ	0.900495	0.000161	θ	1.599278	0.001054
	γ	1.199632	0.000299	γ	0.600599	0.000173
1000	a	0.499884	0.012859	a	2.493011	0.019928
	δ	1.501229	0.002891	δ	0.800670	0.000348
	θ	0.900020	0.000083	θ	1.600639	0.000526
	γ	1.200135	0.000154	γ	0.599901	0.000085

5 Modeling real data

Four real data sets are provided to illustrate the importance, potentiality and flexibility of the PBXGL model. According to these data, we compare the PBXGL distribution with BXII, Marshall-Olkin BXII (MOBXII), Topp Leone BXII (TLBXII), Zografos-Balakrishnan BXII (ZB-BXII), Five Parameters beta BXII (FBBXII), BBXII, B exponentiated BXII (BEBXII), Five Parameters Kumaraswamy BXII (FKwBXII) and KwBXII distributions given in Brito et al. (2017), Hamedani et al. (2017), Hamedani et al. (2018), Yousof et al. (2018), Altun et al. (2018 a), Korkmaz et al. (2018), Altun et al. (2018 b), Elbiely and Yousof (2019), Hamedani et al. (2019), Goual and Yousof (2019), Goual et al. (2019), Gad et al. (2019), Ibrahim (2019), Yousof et al. (2019 a) and Yousof et al. (2019 b).

Data set **I**: {0.98, 5.56, 2.83, 3.68, 2.00, 3.51, 0.85, 1.61, 3.28, 2.95, 5.08, 0.39, 1.57, 3.19, 4.90, 2.74, 2.73, 2.50, 3.60, 3.11, 2.93, 2.85, 2.77, 2.76, 1.73, 2.48, 3.22, 3.70, 3.27, 2.87, 1.47, 3.11, 4.42,2.81, 3.15, 1.92, 1.84, 1.22, 2.17, 1.61, 2.12, 3.09, 2.97, 4.20, 2.35, 1.41, 1.59, 1.12, 1.69, 2.79, 1.89,1.87, 3.39, 3.33, 2.55, 3.68, 3.19, 1.71, 1.25, 4.70, 2.88, 3.68, 1.08, 3.22, 3.75, 2.96, 2.55, 2.59, 2.97,1.57, 2.17, 4.38, 2.03, 2.82, 2.53, 3.31, 2.38, 1.36, 0.81, 1.17, 1.84, 12.40, 3.15, 2.67,3.31, 2.81, 2.56,2.17, 4.91, 1.59, 1.18, 2.48, 2.03, 1.69, 2.43, 3.39, 3.56, 0.80, 2.05, 3.65} called breaking stress data. This data set consists of 100 observations of breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006).

Data set **II**: {0.1, 0.33, 1.08, 1.08, 1.08, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 2.54, 2.78, 2.93, 3.27, 3.42, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.09, 1.12, 1.13, 1.15,1.36, 1.39, 1.44, 1.83, 1.95, 1.96, 1.97, 2.02, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 2.13, 1.46,1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53,2.54, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55} called survival times in days of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal (1960).

Data set **III**: {5.9, 20.4, 13.3, 8.5, 21.6, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 18.5, 5.1,6.7, 17, 9.2, 26.2, 21.9,16.7,21.3, 35.4, 14.3, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7,18.1, 16.5, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 11.9, 7, 8.6,12.5, 10.3, 11.2, 6.1, 8.9, 7.1, 10.8} called taxes revenue data (in 1000 million Egyptian pounds).

Data set **IV**: {65, 56, 26, 22, 1, 1, 5, 65, 16, 22, 3, 4, 2, 3, 56, 65, 17, 7, 156, 8, 4, 3, 30, 4, 100, 134, 16, 108, 121, 4, 39, 143, 43} called leukaemia data. This real data set gives the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia (see Feigl and Zelen (1965)).

The total time test (TTT) plots (see Aarset (1987)) for the four real data sets is presented in Figure 3. These plots indicate that the empirical HRFs of data sets **I**, **II** and **III** are increasing. and U shape for data set **IV**. We consider the following goodness-of fit statistics: the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), consistent Akaike Information Criterion (CAIC), where

$$AIC = 2m_{(p)} - 2\ell(\underline{\widehat{\boldsymbol{\Psi}}}),$$

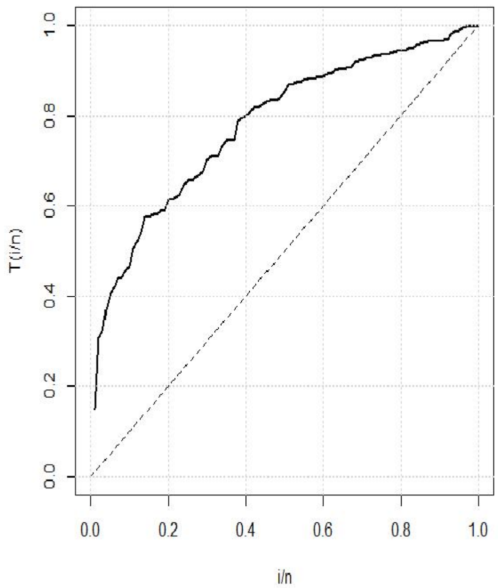
$$BIC = m_{(p)} \log(n) - 2\ell(\underline{\widehat{\boldsymbol{\Psi}}}),$$

$$HQIC = 2m_{(p)} \log[\log(n)] - 2\ell(\underline{\widehat{\boldsymbol{\Psi}}})$$

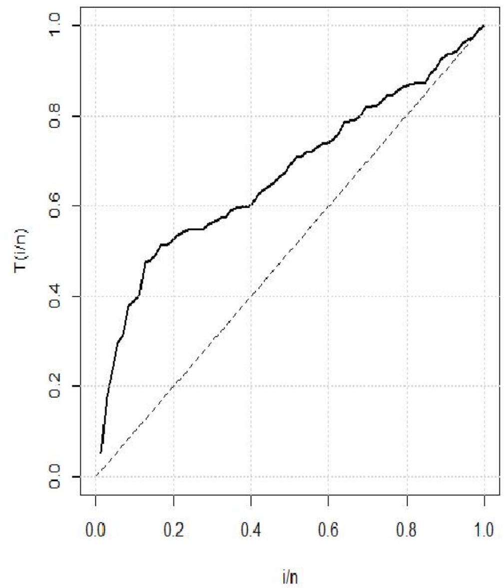
and

$$CAIC = \frac{2nm_{(p)}}{-1 + n - m_{(p)}} - 2\ell(\underline{\widehat{\boldsymbol{\Psi}}}),$$

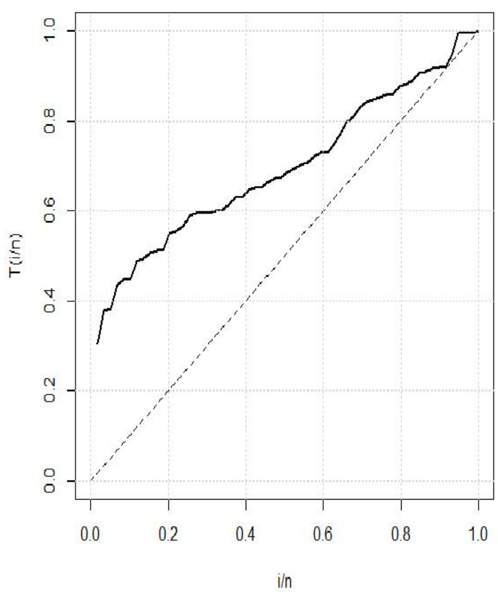
where $m_{(p)}$ is the number of parameters, n is the sample size, $\ell(\underline{\widehat{\boldsymbol{\Psi}}})$ is the maximized log-likelihood. Generally, the smaller statistics indicates the better fit. Tables 3, 5, 7 and 9 gives the MLEs, standard errors (SEs) and confidence interval (CIs) for the data set **I**, **II** ， **III** and **IV** respectively. Based on the values in Tables 4, 6, 8 and 10 and Figures 4-8 the PBXGL model provides the best fits compared to other extensions of the BXII model for the three applications with smallest values for BIC, AIC, CAIC and HQIC.



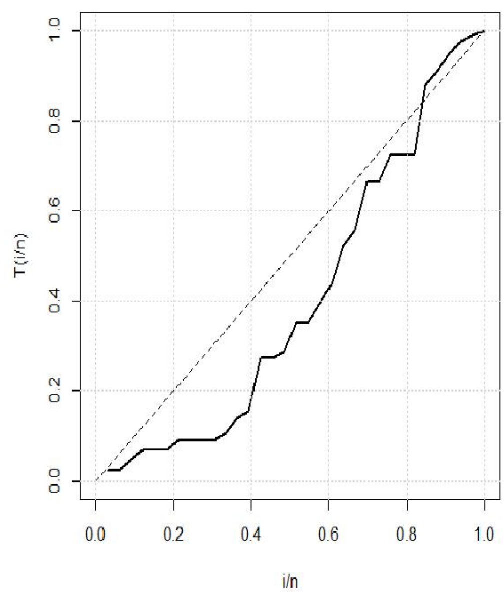
Data **I**



Data **II**



Data **III**



Data **IV**

Figure 3: TTT plots.

Table 3: MLEs, SEs and (CIs) for the data set **I**.

Model	Estimates
BXII(α,β)	5.941, 0.187 (1.279), (0.044) (3.43,8.45), (0.10,0.27)
MOBXII(α,β,γ)	1.192, 4.834, 838.73 (0.952), (4.896), (229.34) (0, 3.06), (0, 14.43), (389.22,1288.24)
TLBXII(α,β,γ)	1.350, 1.061, 13.728 (0.378), (0.384), (8.400) (0.61, 2.09), (0.31,1.81), (0, 30.19)
KwBXII(a,δ,α,β)	48.103 ,79.516 ,0.351 ,2.730 (19.348), (58.186), (0.098), (1.077) (10.18,86.03), (0,193.56), (0.16,0.54), (0.62,4.84)
BBXII(a,δ,α,β)	359.683, 260.097, 0.175, 1.123 (57.941), (132.213), (0.013), (0.243) (246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6)
BEBXII$(a,\delta,\alpha,\beta,\gamma)$	0.381, 11.949, 0.937, 33.402, 1.705 (0.078), (4.635), (0.267), (6.287),(0.478) (0.23,0.53), (2.86,21), (0.41,1.5), (21,45), (0.8,2.6)
FBBXII$(a,\delta,\alpha,\beta,\gamma)$	0.421, 0.834, 6.11, 1.67, 3.450 (0.011), (0.943), (2.314), (0.226), (1.957) (0.4, 0.44), (0. 2.7), (1.6, 10.7), (1.23, 2.1), (0, 7)
FKwBXII$(a,\delta,\alpha,\beta,\gamma)$	0.542, 4.223, 5.313, 0.411, 4.152 (0.137), (1.882), (2.318), (0.497), (1.995) (0.3, 0.8), (0.53, 7.9), (0.9, 9), (0, 1.7), (0.2, 8)
ZBBXII(a,α,β)	123.101, 0.368, 139.247 (243.011), (0.343), (318.546) (0, 599.40), (0, 1.04), (0, 763.59)
PBXGL(a,δ,θ,γ)	**-1.37, 1.105, 0.97, 1.93** **(1.6), (0.648), (0.385), (1.551)** **(-1.8, 4.57), (0, 2.3), (0.21, 1.73), (0, 5.03)**

Table 4: AIC, BIC, CAIC and HQIC values for the data set **I**.

Model	AIC, BIC, CAIC, HQIC
BXII	382.94, 388.15, 383.06, 385.05
MOBXII	305.78, 313.61, 306.03, 308.96
TLBXII	323.52, 331.35, 323.77, 326.70
KwBXII	303.76, 314.20, 304.18, 308.00
BBXII	305.64, 316.06, 306.06, 309.85
BEBXII	305.82, 318.84, 306.46, 311.09
FBBXII	304.26, 317.31, 304.89, 309.56
FKwBXII	305.50, 318.55, 306.14, 310.80
ZBBXII	302.96, 310.78, 303.21, 306.13
PBXGL	**290.41, 300.83, 290.83, 294.63**

Table 5: MLEs, SEs and (CIs) for the data set **II**.

Model	Estimates
BXII(α,β)	3.102, 0.465 (0.538), (0.077) (2.05,4.16), (0.31,0.62)
MOBXII(α,β,γ)	2.259,1.533, 6.760 (0.864), (0.907), (4.587) (0.57,3.95), (0,3.31), (0, 15.75)
TLBXII(α,β,γ)	2.393,0.458,1.796 (0.907), (0.244),(0.915) (0.62,4.17),(0, 0.94),(0.002,3.59)
$KwBXII(a,\delta,\alpha,\beta)$	14.105,7.424, 0.525, 2.274 (10.805), (11.850), (0.279),(0.990) (0, 35.28), (0.30.65), (0, 1.07),(0.33, 4.21)
BBXII(a,δ,α,β)	2.555, 6.058,1.800,0.294, (1.859), (10.391), (0.955),(0.466) (0, 6.28), (0, 26.42), (0, 3.67),(0, 1.21)
BEBXII$(a,\delta,\alpha,\beta,\gamma)$	1.876,2.991, 1.780, 1.341, 0.572 (0.094), (1.731), (0.702), (0.816), (0.325) (1.7,2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)
FBBXII$(a,\delta,\alpha,\beta,\gamma)$	0.621, 0.549,3.838, 1.381, 1.665 (0.541), (1.011), (2.785), (2.312), (0.436) (0, 1.7), (0, 2.5), (0, 9.3), (0, 5.9), (0.8, 4.5)
FKwBXII$(a,\delta,\alpha,\beta,\gamma)$	0.558,0.308, 3.999, 2.131, 1.475 (0.442), (0.314), (2.082), (1.833), (0.361) (0, 1.4), (0, 0.9), (0, 3.1), (0, 5.7), (0.76, 2.2)
PBXGL(a,δ,θ,γ)	**3.0514, 2.188, 0.369, 0.65** **(3.51), (0.98), (0.21), (0.29)** **(0, 10.05), (0.24, 4.16), (0, 0.78), (0.07, 1.23)**

Table 6: AIC, BIC, CAIC and HQIC values for the data set **II**.

Model	AIC, BIC, CAIC, HQIC
BXII	209.60, 214.15, 209.77, 211.40
MOBXII	209.74, 216.56, 210.09, 212.44
TLBXII	211.80, 218.63, 212.15, 214.52
KwBXII	208.76, 217.86, 209.36, 212.38
BBXII	210.44, 219.54, 211.03, 214.06
BEBXII	212.10, 223.50, 213.00, 216.60
FBBXII	206.80, 218.20, 207.71, 211.30
FKwBXII	206.50, 217.90, 207.41, 211.00
PBXGL	**205.52, 214.62, 206.11, 209.14**

Table 7: MLEs, SEs and (CIs) for the the data set **III**.

Model	Estimates
BXII(α,β)	5.615, 0.072 (15.048), (0.194) (0, 35.11), (0, 0.45)
MOBXII(α,β,γ)	8.017, 0.419, 70.359 (22.083), (0.312), (63.831) (0, 51.29), (0, 1.03), (0, 195.47)
TLBXII(α,β,γ)	91.320, 0.012, 141.073 (15.071), (0.002), (70.028) (61.78,120.86) (0.008, 0.02) (3.82,278.33)
$KwBXII(a,\delta,\alpha,\beta)$	18.130, 6.857, 10.694, 0.081 (3.689), (1.035), (1.166), (0.012) (10.89,25.36), (4.83,8.89), (8.41,12.98), (0.06,0.10)
BBXII(a,δ,α,β)	26.725, 9.756, 27.364, 0.020 (9.465), (2.781), (12.351), (0.007) (8.17,45.27), (4.31,15.21), (3.16,51.57), (0.006,0.03)
BEBXII$(a,\delta,\alpha,\beta,\gamma)$	2.924, 2.911, 3.270, 12.486, 0.371 (0.564), (0.549), (1.251), (6.938), (0.788) (1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)
FBBXII$(a,\delta,\alpha,\beta,\gamma)$	30.441, 0.584, 1.089, 5.166, 7.862 (91.745), (1.064), (1.021), (8.268), (15.036) (0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)
FKwBXII$(a,\delta,\alpha,\beta,\gamma)$	12.878, 1.225, 1.665, 1.411, 3.732 (3.442), (0.131), (0.034), (0.088), (1.172) (6.13,19.62), (0.97,1.48), (1.56,1.73), (1.24,1.58), (1.43,6.03)
PBXGL(a,δ,θ,γ)	**-11.135, 35.442, 0.052, 0.16** **(7.822), (41.1), (0.023), (0.04)** **(-26.7, 4.5), (0, 117.6), (0.1, 0.09), (0.08, 0.24)**

Table 8: AIC, BIC, CAIC and HQIC values for the data set **IV**.

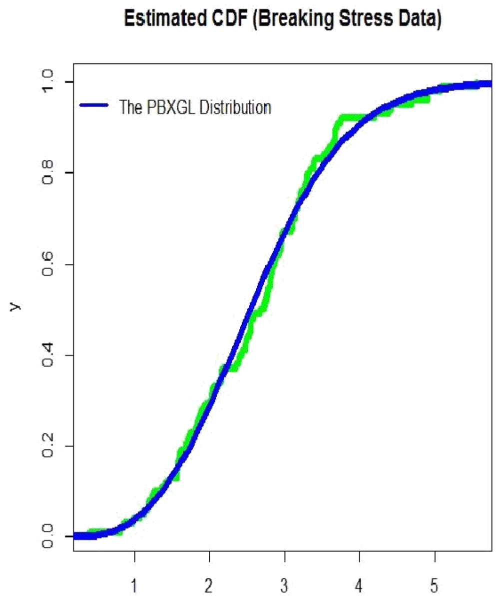
Model	AIC, BIC, CAIC, HQIC
BXII	518.46, 522.62, 518.67, 520.08
MOBXII	387.22, 389.38, 387.66, 389.68
TLBXII	385.94, 392.18, 386.38, 388.40
KwBXII	385.58, 393.90, 386.32, 388.86
BBXII	385.56, 394.10, 386.30, 389.10
BEBXII	387.04, 397.42, 388.17, 391.09
FBBXII	386.74, 397.14, 387.87, 390.84
FKwBXII	386.96, 397.36, 388.09, 391.06
PBXGL	**384.91, 393.22, 385.65, 388.16**

Table 9: MLEs, SEs and (CIs) for the data set **IV**.

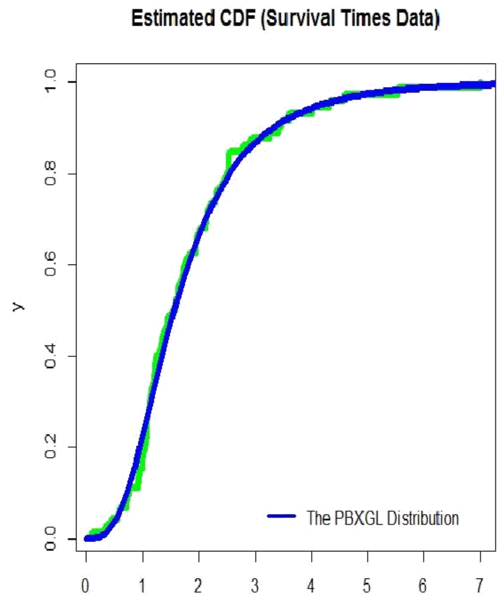
Model	Estimates
BXII(α,β)	58.711,0.006 (42.382), (0.004) (0, 141.78), (0, 0.01)
MOBXII(α,β,γ)	11.838, 0.078, 12.251 (4.368), (0.013), (7.770) (0, 141.78), (0, 0.01), (0, 27.48)
TLBXII(α,β,γ)	0.281, 1.882 ,50.215 (0.288), (2.402), (176.50) (0, 0.85), (0, 6.59), (0, 396.16)
$KwBXII(a,\delta,\alpha,\beta)$	9.201, 36.428, 0.242,0.941 (10.060), (35.650), (0.167), (1.045) (0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99)
BBXII(a,δ,α,β)	96.104, 52.121, 0.104, 1.227 (41.201), (33.490), (0.023), (0.326) (15.4,176.8),(0, 117.8), (0.6, 0.15), (0.59,1.9)
BEBXII$(a,\delta,\alpha,\beta,\gamma)$	0.087, 5.007, 1.561, 31.270, 0.318 (0.077), (3.851), (0.012), (12.940), (0.034) (0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)
FBBXII$(a,\delta,\alpha,\beta,\gamma)$	15.194, 32.048, 0.233, 0.581, 21.855 (11.58), (9.867), (0.091), (0.067), (35.548) (0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)
FKwBXII$(a,\delta,\alpha,\beta,\gamma)$	14.732, 15.285, 0.293, 0.839, 0.034 (12.390), (18.868), (0.215), (0.854), (0.075) (0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)
ZBBXII(a,α,β)	41.973,0.157, 44.263 (38.787),(0.082), (47.648) (0, 117.99),(0, 0.32,) (0, 137.65)
PBXGL(a,δ,θ,γ)	**-5.1, 0.035, 0.62, 12.6** **(0.00), (0.00), (0.03), (0.00)** **–, –, (0.65, 0.68), –**

Table 10: AIC, BIC, CAIC and HQIC values for the data set **IV**.

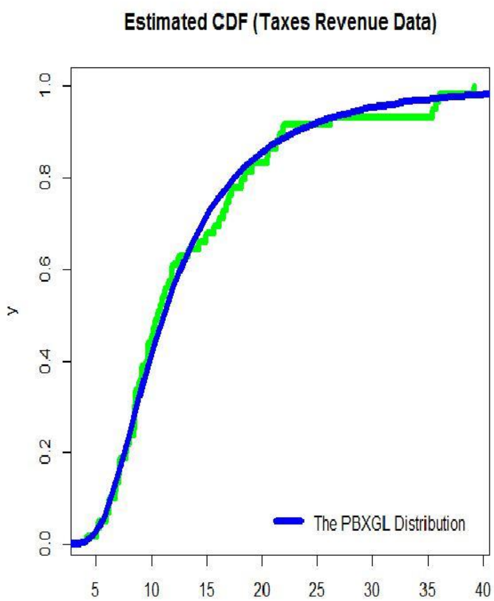
Model	AIC, BIC, CAIC, HQIC
BXII	328.20, 331.19, 328.60, 329.19
MOBXII	315.54, 320.01, 316.37, 317.04
TLBXII	316.26, 320.73, 317.09, 317.76
KwBXII	317.36, 323.30, 318.79, 319.34
BBXII	316.46, 322.45, 317.89, 318.47
BEBXII	317.58, 325.06, 319.80, 320.09
FBBXII	317.86, 325.34, 320.08, 320.36
FKwBXII	317.76, 325.21, 319.98, 320.26
ZBBXII	313.86, 318.35, 314.39, 315.36
PBXGL	**311.23, 317.22, 312.66, 313.25**



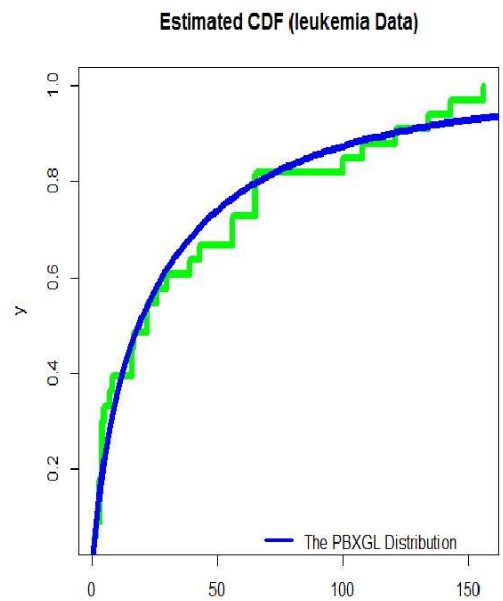
Data **I**



Data **II**

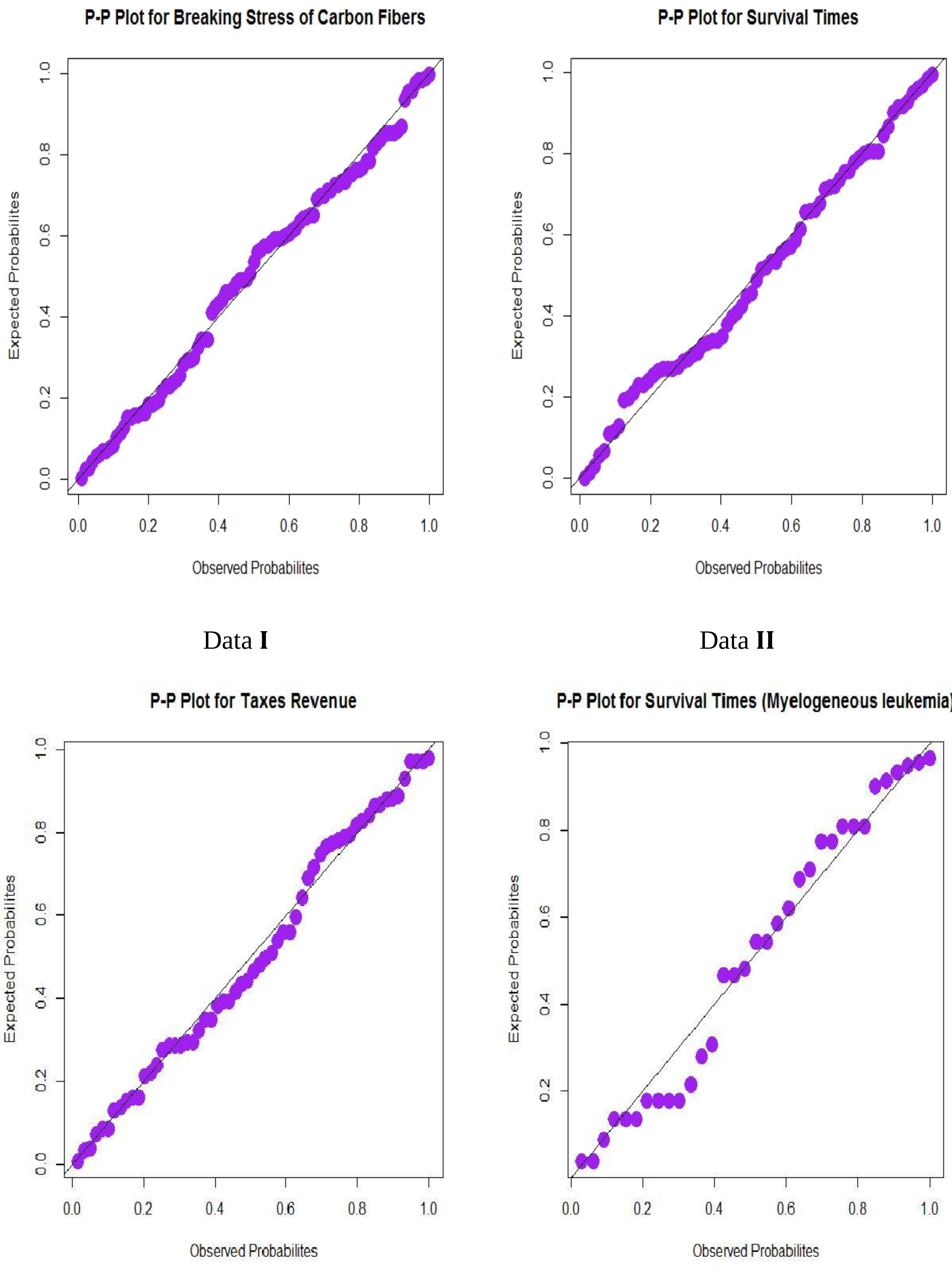


Data **III**



Data **IV**

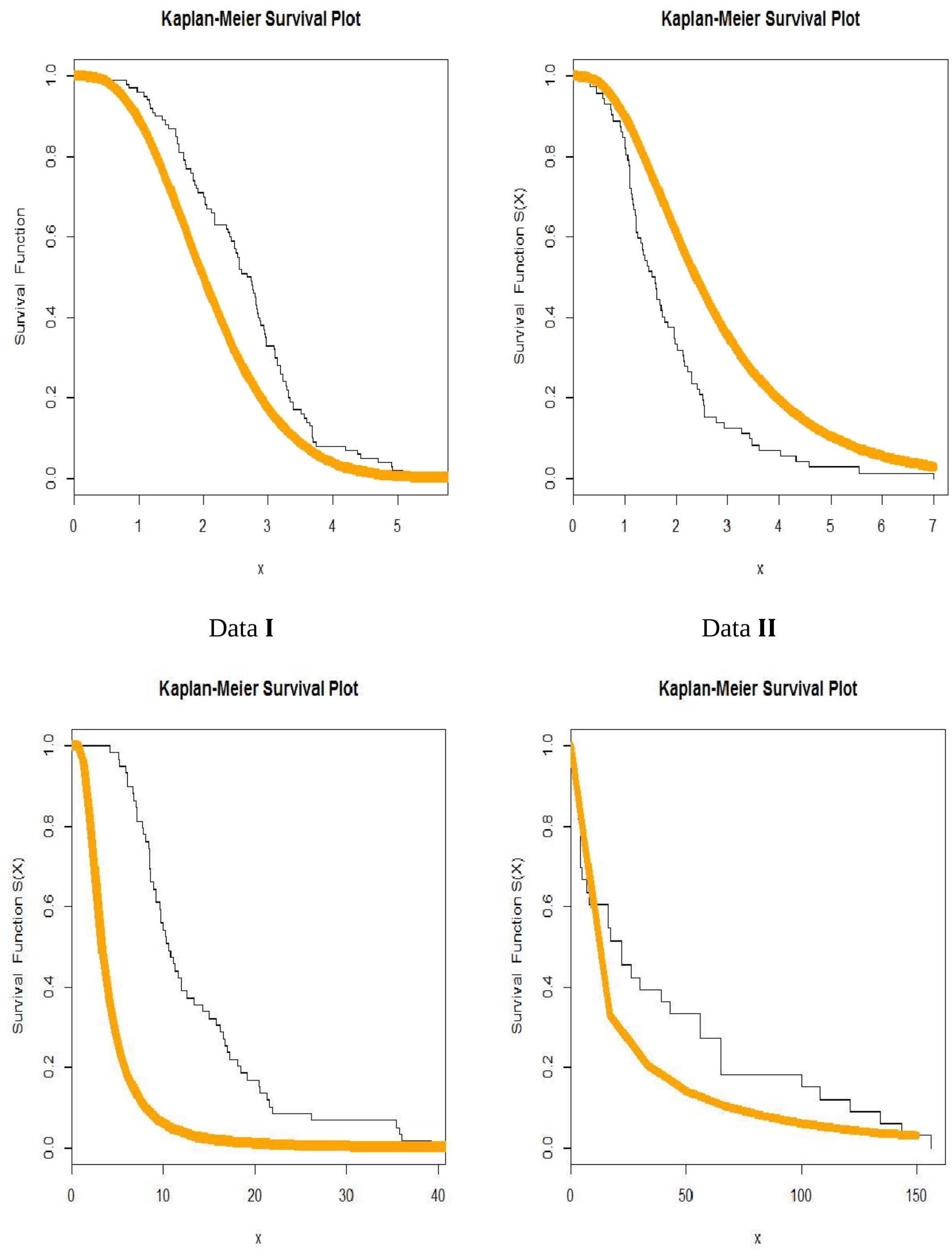
Figure 4: Estimated CDFs.



Data **I** Data **II**

Data **III** Data **IV**

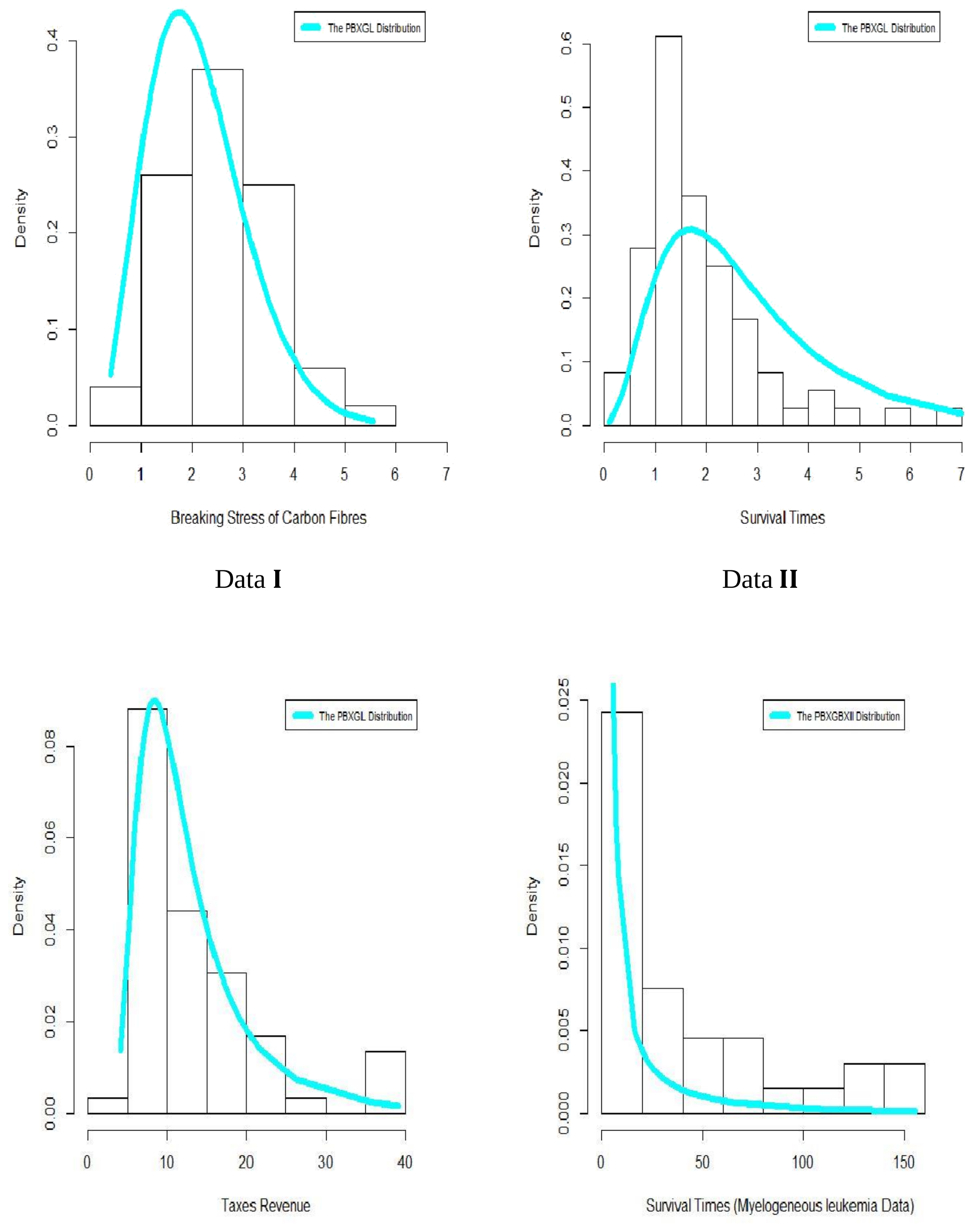
Figure 5: P-P plots.



Data **I** Data **II**

Data **III** Data **IV**

Figure 6: Kaplan-Meier survival plots.



Data **I** Data **II**

Data **III** Data **IV**

Figure 7: Estimated PDFs.

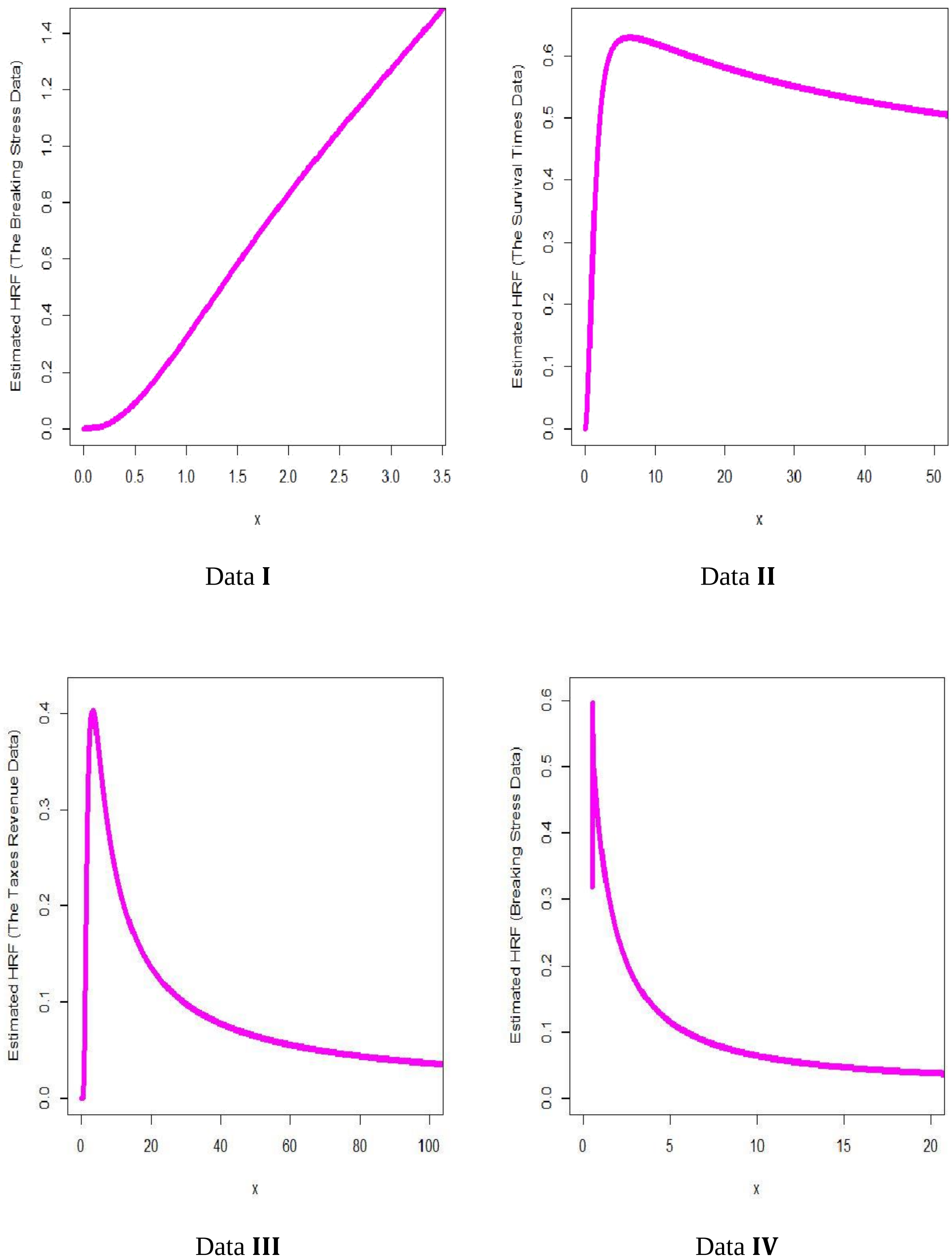


Figure 8: Estimated HRFs.

6 Conclusions

A new version of the Poisson Lomax distributions called the Poisson Burr X generalized Lomax (PBXGL) is proposed and studied. The new density function is expressed as a linear mixture of the Lomax densities. The PBXGL density can be right skewed and unimodal. The failure rate function of the PBXGL model can be increasing-constant or increasing or U shape or decreasing or upside down-increasing. We are also motivated to introduce the PBXGL distribution that contains many new lifetime models. The statistical properties are derived and four applications are provided to illustrate the importance of the new PBXGL density. The method of maximum likelihood is used to estimate the unknown parameters of the PBXGL density. Adequate fitting is provided by the new model.

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