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Log-Weighted Pareto Distribution And Its Statistical Properties

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ABSTRACT

The Pareto distribution is a power law probability distribution that is used to describe social scientific, geophysical, actuarial, and many other types of observable phenomena. A new weighted Pareto distribution is proposed using a logarithmic weight function. Several statistical properties of the weighted Pareto distribution are studied and derived including cumulative distribution function, location measures such as mode, median and mean, reliability measures such as reliability function, hazard and reversed hazard functions and the mean residual life, moments, shape indices such as skewness and kurtosis coefficients and order statistics. A parametric estimation is performed to obtain estimators for the distribution parameters using three different estimation methods the maximum likelihood method, the L-moments method and the method of moments. Numerical simulation is carried out to validate the robustness of the proposed distribution. The distribution is fitted to a real data set to show its importance in real life applications.

Keywords: Log-weighted Pareto distribution, weighted distributions, survival function, hazard rate function, order statistics, maximum likelihood, moment, L-moment.

1. Introduction

The theory of weighted distributions provides a collective access to the problems of model specification and data interpretation. It provides a technique for fitting models to the unknown weight functions when samples can be taken both from the original distribution and a developed distribution. Fisher [6] introduced the basic idea of the concept of weighted distributions when he studied the effect of methods of ascertainment upon estimation of frequencies. In extending the basic ideas of Fisher, Rao [16] introduced a unified theory of weighted distributions and formulated it in general terms in connection with modeling statistical data where the standard distributions were not found to be appropriate. He identified various situations that can be modeled by weighted distributions.

To introduce the concept of a weighted distribution, suppose X is a non-negative random variable with probability density function (pdf) $f(x;\theta)$, where the parameter is $\theta \in \Omega$ (Ω is the parameter space). Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x,\beta)$ which is a non-negative weight function with parameterβ. So, the recorded x is not an observation on$X$, but on the random variable$Z$, having such pdf

$$g(z;\theta,\beta) = \frac{w(z,\beta)f(z,\theta)}{W} \quad (1)$$

where $W = E[w(x,\beta)]$ is the normalizing factor obtained to make the total probability equal to unity. The random variable Z is called the weighted version of X, and its distribution in relation to that of X is called the weighted distribution with weight function $w(x,\beta)$,Patil [15]. Different models can be obtained depending upon the choice of the weight function $w(x)$.

The concept of weighted distributions has been employed in a wide variety of applications in reliability and survival analysis, analysis of family data, meta-analysis, ecology, medicine, and forestry. The contributions of researchers to the weighted distributions vary between single and double weight distributions. Shaban and Boudrissa [19] presented a length-biased version of the Weibull distribution. Gupta and Kundu [7] developed the weighted exponential (WE) distribution as a lifetime model. Levia, et al. [11] developed a length-biased version of the Birnbaum-Saunders [BS] distribution with applications in water quality. Kersey [10] presented the size-biased inverse Weibull distribution. Shi, et al. [20] presented the theoretical properties of weighted generalized Rayleigh distribution. Hussian [9] presented the weighted inverted exponential distribution as a generalized version of the inverted exponential distribution. Seenoi, et al. [18] developed a length-biased version of the exponentiated inverted Weibull (EIW) distribution. Dey, et al. [5] and Nasiru [14] presented different forms of the Weighted Weibull Distribution. Mahmoud, et al. [12] developed the weighted Quasi-Lindley distribution and weighted Lomax distribution. Bashir and Rasul [3] introduced a new weighted Rayleigh distribution named area-biased Rayleigh distribution and they derived some of its mathematical properties. Al-kadim and Hantoosh [2] presented the double weighted exponential distribution (DWED).Ahmed and Ahmed [1] presented double weighted Rayleigh distribution (DWRD) version. Saghirand Saleem [17] presented a new version of the double weight Inverse Weibull (DWIW) distribution. Ajami and Jahanshahi [21] introduced a weighted model based on the Rayleigh distribution and they derived and studied its statistical and reliability properties. Asgharzadeh, et al. [22] introduced a generalization of Lindley distribution that provides fits for the Lindley distribution and some other distributions as special cases. Fatima and Ahmad [23] introduced weighted inverse Rayleigh distribution and investigated its different statistical properties.

2. Pareto Distribution

The Pareto distribution is named after the well-known Italian-born Swiss sociologist and economist Vilfredo Pareto (1848-1923). The Pareto distribution is a power law probability distribution that is used to describe social, scientific, geophysical, actuarial, and many other types of observable phenomena, Johnson et al. [24]. Pareto in 1896 defined his Law, which can be stated as $N = Cx^{-a}$, where N represents the number of individuals in the population whose income exceeded a given level x,for some real number C and some $a > 0$ [34]. Pareto distribution is commonly used in modeling heavy tailed distributions, including but not limited to income, insurance and city size populations. The Pareto distribution is often described as the basis of 80/20 rule that describes the larger compared to the smaller. A classic example is that 80% of the wealth is owned by 20% of the population. Pareto type I distribution is sometimes called the classical Pareto distribution or the European Pareto distribution. The pdf of a random variable X that follows the Pareto distribution with shape and scale parameters α , β respectively, is given by:

$$f(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}} \quad ,x \geq \beta > 0\,,\alpha > 0 \qquad (2)$$

During the recent decades, several new generalized Pareto distributions have been developed using modern methodologies in order to generate new families of distributions. Examples include the exponentiated Pareto distribution by Gupta et al. [25], the beta-Pareto distribution by Akinsete et al. [26] and the beta generalized Pareto distribution by Mahmoudi [27]. Alzaatreh et al. [28] defined and studied the Weibull-Pareto distribution which is a unimodal distribution and its shape can be skewed to the right or skewed to the left. Sarabia and Prieto [29] proposed Pareto positive stable distribution to study city size data. The Generalized Feller-Pareto (GFP) family was defined by Zandonatti in 2001 as mentioned in Kleiber and Kotz [30]. Odubote and Oluyede [31] defined a six-parameter class of distributions called weighted Feller-Pareto (WFP) and some related family of distributions including several other Pareto-type distributions as special cases. Hamed, et al. [32] defined and studied a generalization of the two-parameter Pareto distribution to the T-Pareto{Y} family using the T-R{Y} framework including six generalized Pareto families. Andrade and Zea [33] defined and studied a three-parameter model called the exponentiated generalized extended Pareto distribution and they provided a comprehensive mathematical treatment. In the next sections, a new distribution is proposed called Log-weighted Pareto distribution. It comes as a weighted version of the Pareto Type I distribution.

3. Log-weighted Pareto Distribution

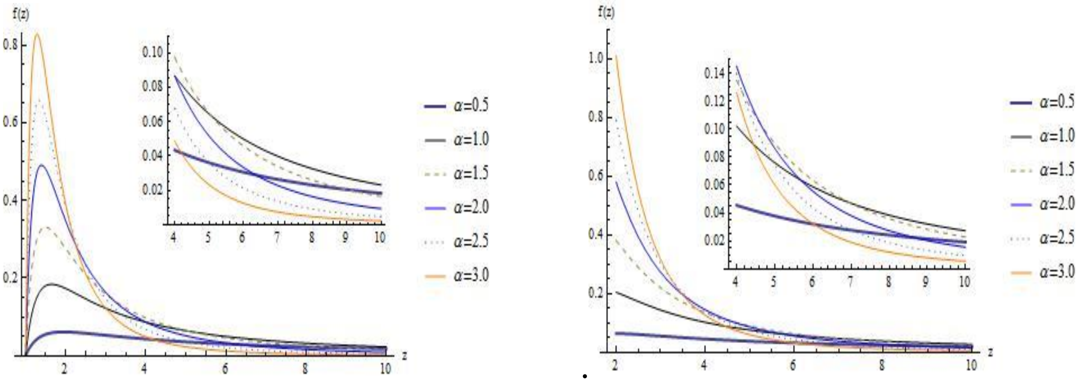
Suppose X is a continuous random variable following the Pareto distribution with shape and scale parameters $\alpha and \beta$ respectively. Its pdf is defined as given in equation (2), as follows:

$$f(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\,, \qquad x \geq \beta > 0, \qquad \alpha > 0$$

Using Pareto as a base distribution, a new weighted distribution is introduced with $w(z) = Log(z)$ as a weight function. A random variable Z is said to have a Log-weighted Pareto (LWP) distribution, if its pdf and cdf are, respectively, given by:

$$f(z;\alpha,\beta) = \frac{\alpha^2\beta^\alpha}{1+\alpha log\beta}\frac{log\, z}{z^{\alpha+1}}, \qquad z \geq \beta > 1, \qquad \alpha > 0 \qquad (3)$$

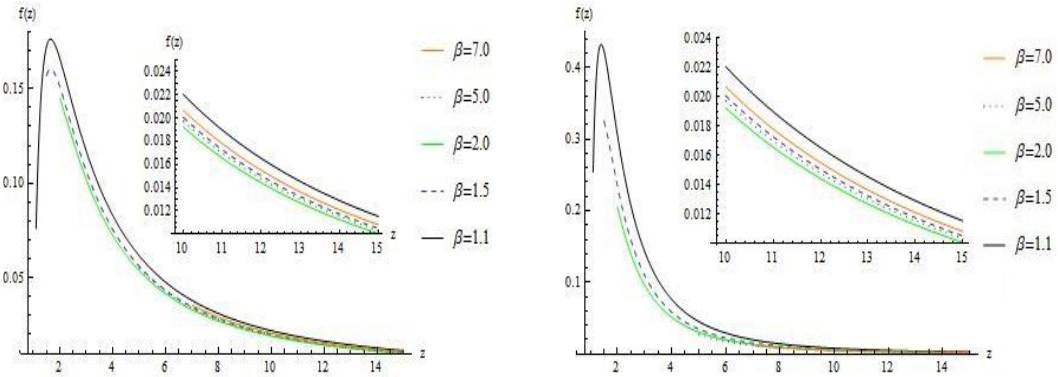
$$F(z;\alpha,\beta) = 1 - \left(\frac{\beta}{z}\right)^\alpha \left(\frac{1+\alpha\, log\, z}{1+\alpha\, log\, \beta}\right), \qquad z \geq \beta > 1, \qquad \alpha > 0 \qquad (4)$$



(a) $\beta = 1.0$ (b) $\beta = 2.0$

Figure (1): The LWP pdf at $\beta = 1.0$ $and \beta = 2.0$ and selected values of α

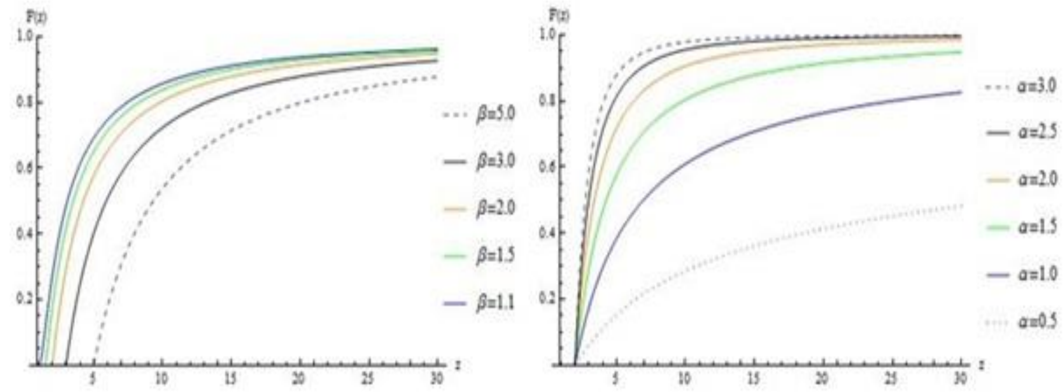
It can be easily noticed from Figure. (1a) and (1b) that at a certain value of scale parameter,β, the peak of the density function becomes higher as the value of α increases. In other words, the modal value is in inverse relation with the value of α holding the value of β constant. Moreover, the greater the value of scale parameterβ, the density function becomes strictly decreasing function.



(a) $\alpha = 1.0$ (b) $\alpha = 2.0$

Figure (2): The LWP pdf at $\alpha = 1.0$ $and \alpha = 2.0$ and selected values of β

Similarly, it can be easily observed from Figure. (2a) and (2b) that at a certain value of the shape parameter,α, as the value of β increases, the shape of the density function is transformed from a unimodal shape to a reversed J-shape.



(a) $\alpha = 1.5$ (b) at $\beta = 2.0$

Figure (3): The LWP cdf at selected values of β and α

Also, it is clear from Figure (3a) that the smaller the value of $\beta$is, the quicker the cumulative distribution function approaches its upper bound. On the contrary to that, one can see from Figure (3b) that the smaller the value of $\alpha$is, the slower the cumulative distribution function approaches its upper bound.

3.1 The Mode and Quantiles

The Log-weighted Pareto is a unimodal distribution and its modal value is achieved at $z = e^{\frac{1}{(\alpha+1)}}$. On the other hand, no closed form exists for the quantiles, but it can be obtained numerically by using the following relation:

$$p = F(q_p) = 1 - \left(\frac{\beta}{q_p}\right)^{\alpha} \left(\frac{1 + \alpha \log q_p}{1 + \alpha \log \beta}\right) \tag{5}$$

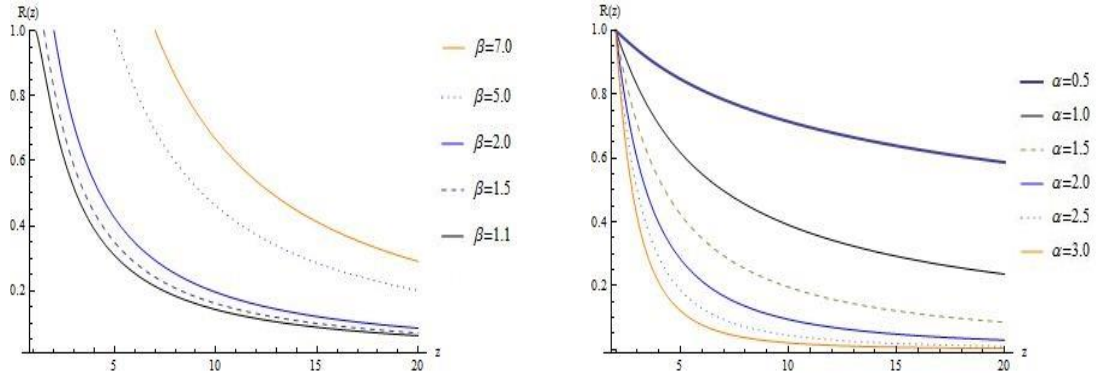
3.2 Reliability measures

The reliability function$R(\cdot)$, the hazard rate function$h\ (\cdot)$, the reversed hazard rate $\lambda(\cdot)$ and the mean residual life $m(\cdot)$ of the LWP distribution are given as follows:

The reliability function:

The reliability function for the Log-weighted Pareto distribution is given by:

$$R(z) = \left(\frac{\beta}{z}\right)^{\alpha} \left(\frac{1 + \alpha\, log\, z}{1 + \alpha\, log\, \beta}\right), \qquad z \geq \beta > 1, \qquad \alpha > 0 \tag{6}$$



(a) $\alpha = 1.5$ (b) $\beta = 2.0$

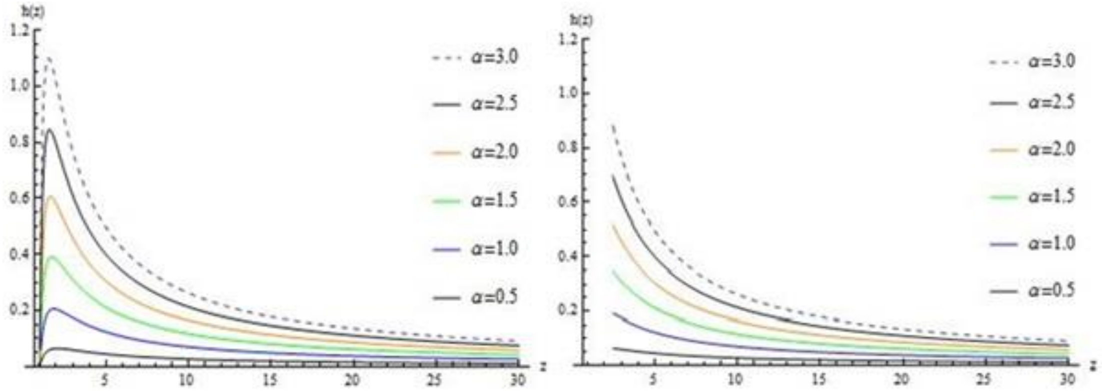
Figure (4): The LWP Reliability function at selected values of β and α

It is obvious from figure (4) that the reliability function is decreasing in z. Moreover, figure (4a) shows that at a certain value of the shape parameter α, as the scale parameter β increases, the curve of the function moves up. To the contrary to that, from figure(4b), if one holds the scale parameter β at a certain value, the curve moves down when the value of the shape parameter α increases. i.e. the reliability function has a direct relation with the scale parameter β and it has an inverse relation with the shape parameter α

The hazard function:

The hazard function for the Log-weighted Pareto distribution takes the following formula:

$$h(z) = \frac{\alpha^2 \log z}{z(1+\alpha \log z)}, \qquad z > \beta > 1, \qquad \alpha > 0 \tag{7}$$



(a) $\beta = 1.1$ (b) $\beta = 2.0$

Figure (5): The hazard rate function at selected values of α

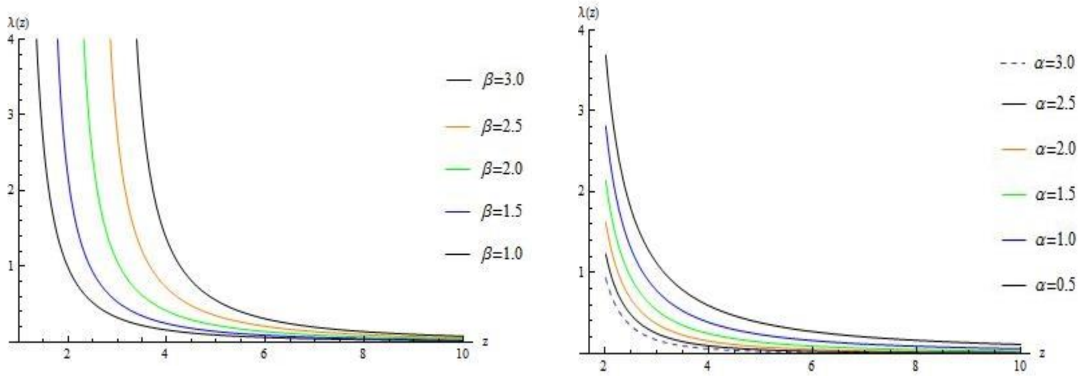
From figure (5a), one can see that, when the value of scale parameter$\beta \to 1^+$, the hazard rate increases first then it decreases, but in figure (5b), when the value of scale parameterβ increases, the hazard rate is monotonically decreasing. This behavior of the hazard function can be interpreted using theorem 1 introduced by Chechile[4].The behavior of the hazard function is distributed in two ways depending on the value of the scale parameter, β. If the value of $\beta$is less than $\exp\left[\frac{\sqrt{1+4\alpha}-1}{2\alpha}\right]$, then the hazard function is a unimodal function and it reaches its peak at$z = \exp\left[\frac{\sqrt{1+4\alpha}-1}{2\alpha}\right]$, and it approaches zero as $z \to \beta^+$ or $z \to \infty$. On the other hand, if the value of β is greater than $\exp\left[\frac{\sqrt{1+4\alpha}-1}{2\alpha}\right]$, then the hazard function is monotonically decreasing and it approaches zero as $z \to \infty$. Moreover, It is clear that the hazard function of the Log-weighted Pareto is increasing in the shape parameter, α.

The reversed hazard rate:

The reversed hazard rate is defined as the ratio of the density function to the distribution function of a random variable. The reversed hazard rate is a useful tool in the area of maintenance management. The Log-weighted Pareto distribution has the following reversed hazard function:

$$\lambda(z) = \frac{\alpha^2 \beta^\alpha z^{-1} \log z}{z^\alpha (1 + \alpha \log \beta) - \beta^\alpha (1 + \alpha \log z)}, \qquad z \geq \beta > 1, \qquad \alpha > 0 \qquad (8)$$

It is clear from figure (6) that the reversed hazard rate is decreasing in Z. Also, one can observe from figure (6a) that, when the value of the scale parameter β increases, the curve of the reversed hazard rate function shifts to the right direction. Moreover, from figure (6b), it is obvious that as the value of the shape parameter α increases, the curve of the reversed hazard rate function moves down.



(a) $\alpha = 1.5$ (b) $\beta = 1.5$

Figure (6): The reversed hazard rate at selected values of β and α

The mean residual life:

The mean residual life function measures the average residual life of a component when it has completed t units of time. It is defined by $m(t) = E(T - t \mid T > t)$. The mean residual life for a variable follows the Log-weighted Pareto distribution has the following form:

$$m(z) = \frac{\alpha^2 z(1 + (\alpha - 1) \log z)}{(\alpha - 1)^2 (1 + \alpha \log z)} - z, \qquad z \geq \beta > 1, \qquad \alpha > 1 \qquad (9)$$

$$m(z) = \frac{\alpha^2 z(1+(\alpha-1) \log z) - z(\alpha-1)^2(1+\alpha \log z)}{(\alpha-1)^2(1+\alpha \log z)}, z \geq \beta > 1, \ \alpha > 1 \qquad (10)$$

3.3 The moments

The rth non-central moment for a random variable Z that follows the Log-weighted Pareto is given by:

$$E(Z^r) = \frac{\alpha^2 \beta^r [1 + (\alpha - r) \log \beta]}{(\alpha - r)^2 [1 + \alpha \log \beta]}, \qquad r < \alpha \qquad (11)$$

Using the previous formula, the expectation and variance, coefficient of variation and coefficient of skewness can be obtained as below:

The expectation

$$E(Z) = \frac{\alpha^2 \beta [1 + (\alpha - 1) \log \beta]}{(\alpha - 1)^2 [1 + \alpha \log \beta]}, \qquad \alpha > 1 \qquad (12)$$

The variance

$$V(Z) = \frac{\alpha^2\beta^2\{1+2\alpha(\alpha-2)+(\alpha-1)\log\beta[2+4\alpha(\alpha-2)+(\alpha-2)(\alpha-1)\alpha\log\beta]\}}{(\alpha-2)^2(\alpha-1)^4(1+\alpha\log\beta)^2}, \ \alpha > 2 \qquad (13)$$

The coefficient of variation

$$CV = \frac{\sqrt{\{1+2\alpha(\alpha-2)+(\alpha-1)\log\beta[2+4\alpha(\alpha-2)+(\alpha-2)(\alpha-1)\alpha\log\beta]\}}}{\alpha(\alpha-2)(1+(\alpha-1)\log\beta)}, \ \alpha > 2 \qquad (14)$$

Table 1 Mean and variance of LWPD for some values of **α** and **β**

β	1.5		2		2.5		5		10	
α	**Mean**	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**
2.5	3.3277	22.524	4.1465	32.080	5.0108	44.128	9.4392	135.43	18.3112	460.71
3	2.7576	5.2367	3.4871	7.6338	4.2502	10.689	8.1434	34.421	15.9484	121.52
4	2.2543	1.1346	2.9023	1.6973	3.5715	2.4238	6.9654	8.2051	13.7686	30.013
7	1.8260	0.1440	2.3998	0.2238	2.9822	0.3284	5.9126	1.1826	11.7803	4.4970

From the previous table, it is obvious that the values of mean and variance are decreasing inα, but they are increasing inβ. If the values of the coefficient of variation are calculated for these combinations of values of the parameters α andβ, one can observe that the value of the coefficient of variation is decreasing in bothα andβ.

Index of Skewness:

The coefficient of skewness is given by:

$$\sqrt{\beta_1} = \frac{\mu_3}{\sqrt{\mu_2^3}}, \qquad \alpha > 3, \qquad \beta > 1 \qquad (15)$$

where $\mu_2$and μ_3 are given by:

$$\mu_2 = \frac{\alpha^2\beta^2\{1+2\alpha(\alpha-2)+(\alpha-1)\log\beta[2+4\alpha(\alpha-2)+(\alpha-2)(\alpha-1)\alpha\log\beta]\}}{(\alpha-2)^2(\alpha-1)^4(1+\alpha\log\beta)^2} \qquad (16)$$

$$\mu_3 = \frac{2\alpha^2\beta^3(2 + A_1 + [(\alpha-1)\log\beta]\{6 + 3A_1 + [\alpha(\alpha-1)\log\beta]A_2\})}{(\alpha-3)^2(\alpha-2)^2(\alpha-1)^6(1+\alpha\log\beta)^3} \qquad (17)$$

where $A_1 and A_2$ are as follow:

$$A_1 = \alpha(\alpha(\alpha(\alpha(2\alpha-5)-10)+29)-14) \qquad (18)$$

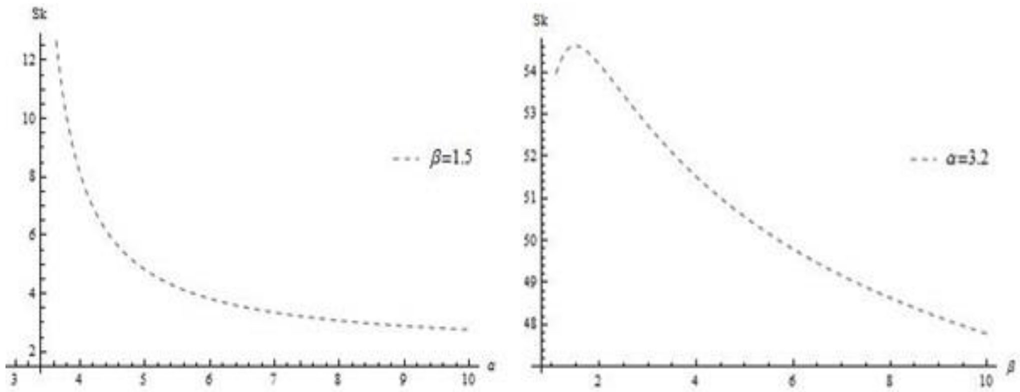
$$A_2 = 3(2\alpha^4 - 7\alpha^3 + 13\alpha - 4) + (\alpha-3)(\alpha-2)(\alpha-1)\alpha(\alpha+1)\log\beta \qquad (19)$$

The value of the coefficient of skewness can be calculated for several values of the parameters α and β which presented in the following table. It is obvious from the values in the table that all the values are positive. Moreover, this coefficient is decreasing inα. For$\alpha \leq 3$, another formula may be used instead of the moment formula.

Table2: Skewness coefficient at several combinations of values of **α** and **β**

α \ β	1.0	1.2	1.5	2.0	2.5	3.0	5.0
3.2	53.7293	54.2818	54.6231	54.1713	53.4397	52.7244	50.5569
3.5	15.7919	16.0756	16.3608	16.4416	16.3796	16.2881	15.9557
3.8	9.5001	9.7335	9.9766	10.0911	10.0964	10.0734	9.9545
4.0	7.6755	7.8967	8.1242	8.2405	8.2590	8.2508	8.1808
4.2	6.5312	6.7470	6.9636	7.0783	7.1028	7.1022	7.0585
4.5	5.4413	5.6555	5.8612	5.9716	5.9997	6.0047	5.9820

4.8	4.7426	4.9595	5.1577	5.2634	5.2923	5.2998	5.2882
5.0	4.4009	4.6208	4.8151	4.9177	4.9466	4.9550	4.9479



(a) $\beta = 1.5$ (b) $\alpha = 3.2$

Figure (7): The coefficient of skewness versus the values of α and β

From figure (7a), one can observe that the value of the coefficient of skewness value decreases sharply for the value ofα, holding the value of β constant. On the other hand, the skewness coefficient value increases and then decreases for the value of β, keeping the value of α at the same level.

Index of Kurtosis:

The coefficient of kurtosis of the Log-weighted Pareto distribution takes the formula:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \tag{20}$$

where

$$\mu_4 = \frac{A}{B}, \quad \alpha > 4 \tag{21}$$

$$A \ = 3\alpha^2\beta^4(A_1 + 4A_2 + 2A_3 + 4A_4 + A_5) \tag{22}$$

$$A_1 = 12 - 116\alpha + 423\alpha^2 - 694\alpha^3 + 503\alpha^4 - 288\alpha^5 + 180\alpha^6 - 64\alpha^7 + 8\alpha^8 \tag{23}$$

$$A_2 = log\,\beta\,(128\alpha - 539\alpha^2 + 1117\alpha^3 - 1197\alpha^4 + 791\alpha^5 - 468\alpha^6 + 244\alpha^7 - 72\alpha^8 + 8\alpha^9 - 12) \tag{24}$$

$$A_3 = \alpha(\alpha-1)^2\,log^2\,\beta\,(460\alpha - 934\alpha^2 + 649\alpha^3 - 500\alpha^4 + 439\alpha^5 - 172\alpha^6 + 22\alpha^7 - 72) \tag{25}$$

$$A_4 = \alpha^2(\alpha-1)^3\,log^3\,\beta\,(-124\alpha + 89\alpha^2 - 118\alpha^3 + 123\alpha^4 - 48\alpha^5 + 6\alpha^6 + 36) \tag{26}$$

$$A_5 = \alpha^3(\alpha-1)^4\,log^4\,\beta\,(28\alpha - 64\alpha^2 + 71\alpha^3 - 26\alpha^4 + 3\alpha^5 - 48) \tag{27}$$

$$B \ = (\alpha-4)^2(\alpha-3)^2(\alpha-2)^2(\alpha-1)^8(1+\alpha\,log\,\beta)^4 \tag{28}$$

The value of the coefficient of kurtosis can be calculated for several values of the two distribution parameters α and β which is presented in the following table. It is obvious from the values in the table that the coefficient is decreasing sharply in α for$\alpha > 4$. For the interval$\alpha \leq 4$, another formula may be used instead of the moment formula.

Table3: Kurtosis coefficient at several combinations of values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

α β	1.0	1.2	1.5	2.0	2.5	3.0	5.0
4.3	444.326	467.836	490.022	497.141	494.035	488.829	469.658
4.5	202.803	215.169	227.256	232.556	232.559	231.268	225.237
4.8	104.021	111.461	118.720	122.375	122.995	122.802	120.895
5.0	77.300	83.332	89.131	92.151	92.801	92.802	91.755
5.5	46.797	51.162	55.143	57.258	57.820	57.949	57.656
7.0	22.641	25.675	27.978	29.122	29.456	29.573	29.605

3.4 Order Statistics

Let Z be a random variable following the Log-weighted Pareto distribution with shape and scale parameters $\alpha and \beta$ respectively. The density function of the k^{th} order statistic, $Z_{(k)}$, from a sample of size n has the following form:

$$f_{k:n}(z) = \frac{n!}{(k-1)!(n-k)!}\frac{\alpha^2\beta^{(n-k+1)\alpha}}{1+\alpha\, log\beta}\left(1-\left(\frac{\beta}{z}\right)^{\alpha}\left(\frac{1+\alpha\, log\, z}{1+\alpha\, log\,\beta}\right)\right)^{k-1}\left(\frac{1+\alpha\, log\, z}{1+\alpha\, log\,\beta}\right)^{n-k}\frac{log\, z}{z^{(n-k+1)\alpha+1}} \quad (29)$$

The minimum and the maximum order statistics probability density functions are as follow:

$$f_{1:n}(z) = \frac{n\alpha^2\beta^{n\alpha}}{(1+\alpha\, log\beta)^n}\frac{log\, z}{z^{n\alpha+1}}(1+\alpha\, log\, z)^{n-1} \;, z \geq \beta > 1 \,, \alpha > 0 \quad (30)$$

$$f_{n:n}(z) = n\frac{\alpha^2\beta^{\alpha}}{1+\alpha\, log\beta}\frac{log\, z}{z^{\alpha+1}}\left(1-\left(\frac{\beta}{z}\right)^{\alpha}\left(\frac{1+\alpha\, log\, z}{1+\alpha\, log\,\beta}\right)\right)^{n-1} \;,\; z \geq \beta > 1 \,, \alpha > 0 \quad (31)$$

A special case

If n=2, the pdf, cdf and the expectation of the first and the second order statistics are:

$$f_{1:2}(z) = \frac{2\alpha^2\beta^{2\alpha}}{(1+\alpha\, log\beta)^2}\frac{log\, z}{z^{2\alpha+1}}(1+\alpha\, log\, z), \qquad z \geq \beta > 1\,, \qquad \alpha > 0 \quad (32)$$

$$F_{1:2}(z) = 1-\frac{\beta^{2\alpha}}{(1+\alpha\, log\,\beta)^2}\frac{(1+\alpha\, log\, z)^2}{z^{2\alpha}}, \qquad z \geq \beta > 1\,, \qquad \alpha > 0 \quad (33)$$

$$E(Z_{1:2}) = \frac{2\alpha^2\beta[\{1-6\alpha+8\alpha^2+(1-2\alpha)^2\alpha\, log\,\beta\}\, log\,\beta+4\alpha-1]}{(2\alpha-1)^3(1+\alpha\, log\,\beta)^2}, \qquad \alpha > \frac{1}{2} \quad (34)$$

$$f_{2:2}(z) = \frac{2\alpha^2\beta^{\alpha}}{1+\alpha\, log\beta}\frac{log\, z}{z^{\alpha+1}}\left(1-\left(\frac{\beta}{z}\right)^{\alpha}\left(\frac{1+\alpha\, log\, z}{1+\alpha\, log\,\beta}\right)\right), z \geq \beta > 1\,, \qquad \alpha > 0 \quad (35)$$

$$F_{2:2}(z) = \left[1+\frac{\left(\frac{\beta}{z}\right)^{\alpha}(1+\alpha\, log\, z)\left[\left(\frac{\beta}{z}\right)^{\alpha}(1+\alpha\, log\, z)-2(1+\alpha\, log\,\beta)\right]}{(1+\alpha\, log\,\beta)^2}\right], z \geq \beta \quad (36)$$

$$E(Z_{2:2}) = \frac{2\alpha^4\beta[log\,\beta\,\{(\alpha-1)(1-2\alpha)^2\, log\,\beta+2\alpha(4\alpha-5)+3\}+4\alpha-3]}{(\alpha-1)^2(2\alpha-1)^3(1+\alpha\, log\,\beta)^2}, \alpha > 1 \quad (37)$$

4. Parameter Estimation

The Log-weighted Pareto distribution has two parameters. In order to obtain estimators for these parameters, three different methods are used here, namely, moments method, L-moments method and the maximum likelihood method.

4.1 The method of moments

The method of moments is a simple technique based on the idea that the sample moments are "natural" estimators of population moments. If $Z_1, \dots, Z_n$ are assumed to be independent and identically distributed then the estimators of the distribution parameters $\theta_1, \dots, \theta_p$ are obtained by solving the set of p equations:

$$\mu'_k = m'_k \quad , \quad k = 1,2, \dots, p \tag{38}$$

where μ'_k is the k^{th} population moment and m'_k is the k^{th} sample moment.

Applying the method of moments on the Log-weighted Pareto distribution, the following equations are obtained:

$$\frac{\sum z_i}{n} = \frac{\alpha^2 \beta [1 + (\alpha - 1) \log \beta]}{(\alpha - 1)^2 [1 + \alpha \log\beta]}, \qquad \alpha > 1 \tag{39}$$

$$\frac{\sum z_i^2}{n} = \frac{\alpha^2 \beta^2 [1 + (\alpha - 2) \log \beta]}{(\alpha - 2)^2 [1 + \alpha \log\beta]}, \qquad \alpha > 2 \tag{40}$$

Solving equations (39) and (40) simultaneously, we obtain $\hat{\beta} and \hat{\alpha}$.

4.2 The method of L-moments

L-moments have some theoretical advantages over conventional moments of being more robust to the presence of outliers in the data. Experience also shows that, compared with conventional moments, L-moments are less subject to bias in estimation and approximate their asymptotic normal distribution more closely in finite samples. Parameter estimates obtained from L-moments are some times more accurate in small.

The first two L-moments are that defined by:

$$l_1 = E[Z_{1:1}] \tag{41}$$

$$l_2 = \frac{1}{2} E[Z_{2:2} - Z_{1:2}] \tag{42}$$

In the Log-weighted Pareto distribution, the previous L-moments are given by:

$$l_1 = \frac{\alpha^2 \beta (1 + (\alpha - 1) \log (\beta))}{(\alpha - 1)^2 (1 + \alpha \log (\beta))} \quad , \quad \alpha > 1 \tag{43}$$

$$l_2 = \frac{\alpha^2 \beta \left(\left(12\alpha^3 - 18\alpha^2 + 8\alpha - 1 \right) \log \beta + (1 - 2\alpha)^2 (\alpha - 1) \alpha \log^2 \beta + 6\alpha^2 - 6\alpha + 1 \right)}{(\alpha - 1)^2 (2\alpha - 1)^3 (1 + \alpha \log \beta)^2} \quad , \quad \alpha > 1 \tag{44}$$

The following two equations can be solved simultaneously to obtain the estimators for the two parameters$\alpha and \beta$.

$$l_1 = \bar{Z} \tag{45}$$

$$l_2 = \frac{1}{2} \binom{n}{2}^{-1} \sum_i \sum_j \left(z_{i:n} - z_{j:n} \right) \ , \ i > j \ , \ \alpha > 1 \tag{46}$$

4.3 The maximum likelihood method

The likelihood function is viewed as a function of the parameters$\alpha and \beta$as follows:

$$L(\boldsymbol{z}; \alpha, \beta) = \prod_{i=1}^{n} \frac{\alpha^2 \beta^\alpha}{1 + \alpha \log\beta} \frac{\log z_i}{z_i^{\alpha+1}} = \frac{\alpha^{2n} \beta^{n\alpha}}{(1 + \alpha \log\beta)^n} \prod_{i=1}^{n} z_i^{-(\alpha+1)} \log z_i \tag{47}$$

The Log-likelihood function is given by:

$$log\, L = log\left[\frac{\alpha^{2n}\beta^{n\alpha}}{(1+\alpha\, log\beta)^n}\right] + \sum_{i=1}^{n} log\left(z_i^{-(\alpha+1)}\, log\, z_i\right) \tag{48}$$

$$log\, L = 2n\, log\, \alpha + n\alpha\, log\, \beta - n\, log(1+\alpha\, log\beta) - (\alpha+1)\sum_{i=1}^{n} log\, z_i + \sum_{i=1}^{n} log(log\, z_i) \tag{49}$$

It is clear that the previous function is increasing inβ, so the value ofβ that maximizes this function is the first order statistic$Z_{(1)}$, then $\tilde{\beta} = Z_{(1)}$is the maximum likelihood estimator of β. Moreover, using the partial derivative of the likelihood function with respect to α, one can obtain the following equation:

$$\frac{2n}{\alpha} + n\, log\, \beta - \frac{n\, log\, \beta}{(1+\alpha\, log\, \beta)} - \sum_{i=1}^{n} log\, z_i = 0 \tag{50}$$

Substitute $\tilde{\beta} = Z_{(1)}$in (50), we obtain the maximum likelihood estimator, $\tilde{\alpha}$ for α.

4.4 Simulation study

In this section, a study of the behavior of the estimators for the unknown parameters $\alpha$and $\beta$is considered. The estimation is made when the two parameters are unknown. Three different methods were used: the maximum likelihood method, the L-moments method and method of moments. Several combinations of the values of the two parameters are assumed. The shape parameter α is assumed to take the values 0.2, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0 and 5.0, where the scale parameter β is as 1.2, 1.5, 2.0, 2.5, 3.0 and 5.0. The estimated values, biases, relative biases, mean squared error (MSE) and the scaled mean squared error were computed. This is done by generating samples from the Log-weighted Pareto distribution and considering samples of sizes of 30, 50, 100, 150 and 200. The simulations are based on 10000 replications. There were some constraints on using the estimation methods according to the value of the shape parameter, α. For the interval $0 < \alpha \leq 1$, only the maximum likelihood method is applicable. For the interval $1 < \alpha \leq 2$, both the maximum likelihood and L-moments methods are applicable. For the interval$\alpha > 2$, all of the three considered estimation methods are applied. The following tables present some of the obtained results.

The maximum likelihood method is applied for estimating the two parameters in the interval$\alpha \in (0,1]$. From tables 4, 5 and 6, it can be observed bias, relative bias, MSE and scaled MSE values of both parameters α and β decrease as the sample size increases. Moreover, the values of relative bias and scaled MSE are also decreasing inα. In addition, the values of bias and MSE for α are always less than the corresponding values for β in this interval.

The maximum likelihood method and the L-moments method are applied for estimating the two parameters in the interval, $\alpha \in (1,2]$. From table 7 through table 11, bias, relative bias, MSE and scaled MSE values of both parameters α and β decrease as the sample size increases. Moreover, the smaller the values ofα, the less the values of bias and MSE for α than the corresponding values forβ. The larger the values of α, the less the values of bias and MSE for β than the corresponding values forα. In addition, it is clear that the maximum likelihood estimators for both parameters are less bias and more efficient than those of the L-moments either for small sample sizes or large sample sizes.

From table 12 through table 15, it can be easily observed that the values of bias, relative bias, MSE and scaled root MSE of both parameters α and β decrease as the sample size increases for all estimation methods. Moreover, the values of bias and MSE for β are less than the corresponding values forα. In addition, it is clear that the maximum likelihood estimators for both parameters are less bias and more efficient than the other methods either for small sample sizes or large sample sizes.

To sum up, it is clear that the estimators obtained by the maximum likelihood method are the most efficient among the other used estimation methods. However, the L-moments method gives better estimates than the moment method which appears in the corresponding values of bias and MSE. All the estimates improved in the values of bias, relative bias, MSE and scaled root MSE as the sample size increases, but the maximum likelihood estimates still the best. Moreover, the estimates of the shape parameter α have less bias and MSE values than the estimates of the scale parameterβ for small values of the shape parameterα, but the opposite becomes true for the large values ofα.

5. Application to Real Data

A real data set is used to compare the fits of the Log-weighted Pareto distribution with the Pareto distribution. This data set represents the failure times of 24 Mechanical Components which is mentioned in Murthy et. al, [13]. The data set is reported in the following table:

30.94	18.51	16.62	51.56	22.85	22.38	19.08	49.56
17.12	10.67	25.43	10.24	27.47	14.70	14.10	29.93
27.98	36.02	19.40	14.97	22.57	12.26	18.14	18.84

In the following table, the parameters of the distributions are estimated using the maximum likelihood method. For comparison, the Kolmogorov-Smirnov statistic is considered where the lower value for this statistic indicates a good fit. The computations of this statistic are carried out using mathematica software. The results are listed in the following table

Table: Parameter estimates and K–S statistics for Failure Times of 24 Mechanical Components

Distribution	Pareto	Log-weighted Pareto
Maximum likelihood estimates	$\hat{\alpha} = 1.394$ $\hat{\beta} = 10.240$	$\hat{\alpha} = 1.678$ $\hat{\beta} = 10.240$
K–S statistics	0.24083	0.23276
P-value	0.10397	0.12593

6. Conclusion

In this paper, the Log-weighted Pareto distribution is proposed. A mathematical treatment of the proposed distribution including explicit formulas for the density and hazard functions, moments, order statistics have been provided. The estimation of the parameters has been approached by maximum likelihood method and method of moments and the L-moments method. A simulation study based on 10000 replications was applied for estimating the distribution parameters by several sample sizes.

References

[1] Ahmed, A., and Ahmed, A. (2014). Characterization and Estimation of Double Weighted Rayleigh Distribution. Journal of Agriculture and Life Sciences, 1(2), 21-28.

[2] Al-kadim, K. A., and Hantoosh, A. F. (2013). Double Weighted Distribution and Double Weighted Exponential Distribution. Mathematical Theory and Modeling, 3(5), 124-134.

[3] Bashir, S., and Rasul, M. (2018). A New Weighted Rayleigh Distribution: Properties and Applications on Lifetime Time Data. Open Journal of Statistics, 8, 640-650.

[4] Chechile, R. A. (2003). Mathematical tools for hazard function analysis. Journal of Mathematical Psychology, 47, 478–494.

[5] Dey, S., Dey, T., and Anis, M. Z. (2015). Weighted Weibull Distribution: Properties and Estimation. Journal of Statistical Theory and Practice, 9, 250-265.

[6] Fisher, R. A. (1934). The Effects of Methods of Ascertainment upon the Estimation of Frequencies. The Annals of Eugenics, 6, 13-25.

[7] Gupta, R., and Kundu, D. (2009). A New Class of Weighted Exponential Distributions. Statistics, 43(6), 621-634.

[8] Hosking, J. R. (1990). Analysis and Estimation of Distributions Using Linear Combinations of Order. Journal of the Royal Statistical Society, 52(1), 105-124.

[9] Hussian, M. A. (2013). A Weighted Inverted Exponential Distribution. International Journal of Advanced Statistics and Probability, 1(3), 142-150.

[10] Kersey, J. X., and Oluyede, B. O. (2010). Weighted Inverse Weibull and Beta-Inverse Weibull Distribution. Electronic Theses and Dissertations. 661. https://digitalcommons.georgiasouthern.edu/etd/661.

[11] Levia, V., Sanhueza , A., and Angulo, J. M. (2009). A Length-Biased Version of the Birnbaum-Saunders Distribution with Application in Water Quality. Stoch Environ Res Risk Assess, 23, 299-307.

[12] Mahmoud, A. W., Elbatal, I. I., and Gad, L. S. (2015). About Some New Weighted Distributions. MSc thesis. Institute of Statistical Studies and Research, Cairo University.

[13] Murthy, D. N. P., Xie, M. and Jiang, R. (2004). Weibull Models. John Wiley & Sons, Inc.

[14] Nasiru, S. (2015). Another Weighted Weibull Distribution from Azzalini's Family. European Scientific Journal, 11(9), 134-144.

[15] Patil, G. P. (2002). Weighted Distributions. (A. El-Shaarawi, and W. W. Piegorsch, Eds.) in Encyclopedia of Environmetrics, 4, pp. 2369-2377.

[16] Rao, C. R. (1965). On Discrete Distributions Arising out of Methods of Ascertainment. Sankhya: The Indian Journal of Statistics, Series A (1961-2002), 27(2/4), 311-324.

[17] Saghir, A., and Saleem, M. (2016). Double Weighted Weibull Distribution Properties and Application. Mathematical Theory and Modeling, 6, 28-46.

[18] Seenoi, P., Supapakorn, T., and Bodhisuwan, W. (2014). The Length-Biased Exponentiated Inverted Weibull Distribution. International Journal of Pure and Applied Mathematics, 92(2), 191-206.

[19] Shaban, S. A., and Boudrissa, N. A. (2007). The Weibull Length Biased Distribution-Properties and Estimation, InterStat, January 2007.

[20] Shi, X., Oluyede, B., and Pararai, M. (2012). Theoretical Properties of Weighted Generalized Rayleigh and Related Distributions. Theoretical Mathematics and Applications, 2(2), 45-62.

[21] Ajami, M., and Jahanshahi, S. M. (2017). Parameter Estimation in Weighted Rayleigh Distribution. Journal of Modern Applied Statistical Methods, 16(2), 256-276.

[22] Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., andSharafi, F. (2016). A New Weighted Lindley Distribution with Application. Brazilian Journal of Probability and Statistics, 30(1), 1-27.

[23] Fatima, K., and Ahmad, S. (2017). Weighted Inverse Rayleigh Distribution. International Journal of Statistics and Systems, 12(1), 119-137.

[24] Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). Continuous Univariate Distributions 2nd ed., vol. 1. New York: John Wiley & Sons.

[25] Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and Methods, 27(4), 887-904.

[26] Akinsete, A., Famoye, F. & Lee, C. (2008). The beta-Pareto distribution. Statistics, 42(6), 547-563.

[27] Mahmoudi, E. (2011). The beta generalized Pareto distribution with application to lifetime data. Mathematics and Computers in Simulation, 81, 2414-2430.

[28] Alzaatreh, A., Famoye, F., and Lee, C. (2013). Weibull-Pareto Distribution and Its Applications. Communications in Statistics—Theory and Methods, 42, 1673–1691.

[29] Sarabia, J.M. and Prieto, F. (2009). The Pareto-positive stable distribution: A new descriptive model for city size data. Physica A: Statistical Mechanics and its Applications. 388, 4179-4191.

[30] Kleiber, C., Kotz, S. (2003). Statistical Size Distributions in Economics and Actuarial Sciences. Hoboken, NJ: John Wiley.

[31] Odubote, O., andOluyede, B. O. (2014). Theoretical Properties of the Weighted Feller-Pareto and Related Distributions. Asian Journal of Mathematics and Applications.

[32] Hamed, D., Famoye, F., and Lee, C. (2018). On Families of Generalized Pareto Distributions: Properties and Applications. Journal of Data Science, 16(2), 377-396.

[33] Andrade, T. A. and Zea, L. M. (2018). The Exponentiated Generalized Extended Pareto Distribution. Journal of Data Science, 16(4), 781-800.

[34] Pareto, V. (1896). Coursd'Économie Politique, Reprinted as a volume of Oeuvres Complétes. Droz, Geneva, 1965.

Appendix 1: Tables

Table 4: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML method at $\beta = 1.2$ $and \beta = 1.5$ and multiple α values

$\beta = 1.2$									
		$\hat{\alpha}$				$\hat{\beta}$			
α	n	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE
0.2	30	0.06949	0.34745	0.00567	0.37650	2.99495	2.49579	26.9198	4.32369
	50	0.06350	0.31750	0.00448	0.33466	1.61560	1.34633	5.70443	1.99033
	100	0.05957	0.29785	0.00376	0.30659	0.82954	0.69128	1.26013	0.93546
	150	0.05802	0.29010	0.00350	0.29580	0.56479	0.47066	0.55729	0.62210
	200	0.05737	0.28685	0.00339	0.29112	0.43197	0.35998	0.31937	0.47094
0.5	30	0.02629	0.05258	0.00577	0.15192	0.59976	0.49980	0.68319	0.68879
	50	0.01606	0.03212	0.00306	0.11063	0.37941	0.31618	0.25322	0.41934
	100	0.00776	0.01552	0.00136	0.07376	0.20706	0.17255	0.07581	0.22945
	150	0.00538	0.01076	0.00087	0.05899	0.12130	0.10108	0.03983	0.16631
	200	0.00469	0.00938	0.00066	0.05138	0.11403	0.09503	0.02308	0.12660
1.0	30	0.05052	0.05052	0.02304	0.15179	0.20036	0.16697	0.07304	0.22522
	50	0.02842	0.02842	0.01203	0.10968	0.12603	0.10503	0.02919	0.14238
	100	0.01353	0.01353	0.00558	0.07470	0.06749	0.05624	0.00832	0.07601
	150	0.00847	0.00847	0.00361	0.06008	0.04666	0.03888	0.00405	0.05303
	200	0.00679	0.00679	0.00271	0.05206	0.03505	0.02921	0.00230	0.03997
$\beta = 1.5$									
α	N	$\hat{\alpha}$				$\hat{\beta}$			
0.2	30	0.06224	0.31120	0.00469	0.34242	2.86391	1.90927	22.4779	3.16072
	50	0.05628	0.28140	0.00362	0.30083	1.61731	1.07821	6.62394	1.71580
	100	0.05241	0.26205	0.00296	0.27203	0.76376	0.50917	1.18160	0.72468
	150	0.05143	0.25715	0.00279	0.26410	0.50830	0.33887	0.49767	0.47030
	200	0.05069	0.25345	0.00267	0.25836	0.38419	0.25613	0.28715	0.35724
0.5	30	0.02613	0.05226	0.00575	0.15166	0.53792	0.35861	0.57776	0.50674
	50	0.01638	0.03276	0.00313	0.11189	0.33119	0.22079	0.21343	0.30799
	100	0.00775	0.01550	0.00139	0.07457	0.16398	0.10932	0.05139	0.15113
	150	0.00612	0.01224	0.00091	0.06033	0.11247	0.07498	0.02467	0.10471
	200	0.00439	0.00878	0.00066	0.05138	0.08489	0.05659	0.01400	0.07888
1.0	30	0.04871	0.04871	0.02281	0.15103	0.16743	0.11162	0.05485	0.15613
	50	0.03181	0.03181	0.01274	0.11287	0.10129	0.06753	0.02028	0.09494
	100	0.01468	0.01468	0.00569	0.07543	0.04941	0.03294	0.00484	0.04638
	150	0.00899	0.00899	0.00367	0.06058	0.03398	0.02265	0.00226	0.03169
	200	0.00699	0.00699	0.00275	0.05244	0.02611	0.01741	0.00136	0.02459

Table 5: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML method at $\beta = 2.0$ $and \beta = 2.5$ and multiple α values

		$\beta = 2.0$							
		$\hat{\alpha}$				$\hat{\beta}$			
α	n	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE
0.2	30	0.06514	0.32570	0.00507	0.35602	2.92396	1.46198	29.6099	2.72075
	50	0.05958	0.29790	0.00399	0.31583	1.54829	0.77415	5.83030	1.20730
	100	0.05588	0.27940	0.00334	0.28896	0.72460	0.36230	1.10789	0.52628
	150	0.05462	0.27310	0.00312	0.27928	0.47500	0.23750	0.46014	0.33917
	200	0.05399	0.26995	0.00302	0.27477	0.35646	0.17823	0.25864	0.25428
0.5	30	0.02531	0.05062	0.00581	0.15245	0.55363	0.27682	0.68106	0.41263
	50	0.01509	0.03018	0.00307	0.11082	0.31571	0.15786	0.20757	0.22780
	100	0.00787	0.01574	0.00140	0.07483	0.15626	0.07813	0.04938	0.11111
	150	0.00534	0.01068	0.00091	0.06033	0.10225	0.05113	0.02100	0.07246
	200	0.00428	0.00856	0.00069	0.05254	0.07839	0.03920	0.01232	0.05550
1.0	30	0.05012	0.05012	0.02522	0.15881	0.16559	0.08280	0.05617	0.11850
	50	0.02983	0.02983	0.01350	0.11619	0.09939	0.04970	0.01992	0.07057
	100	0.01326	0.01326	0.00591	0.07688	0.04932	0.02466	0.00485	0.03482
	150	0.00951	0.00951	0.00382	0.06181	0.03247	0.01624	0.00208	0.02280
	200	0.00556	0.00556	0.00291	0.05394	0.02457	0.01229	0.00121	0.01739

		$\beta = 2.5$							
α	n	$\hat{\alpha}$				$\hat{\beta}$			
0.2	30	0.04361	0.21805	0.00267	0.25836	3.43619	1.37448	37.7775	2.45854
	50	0.03855	0.19275	0.00189	0.21737	1.81090	0.72436	8.42220	1.16084
	100	0.03476	0.17380	0.00141	0.18775	0.81352	0.32541	1.45446	0.48240
	150	0.03381	0.16905	0.00127	0.17819	0.51999	0.20800	0.57840	0.30421
	200	0.03325	0.16625	0.00120	0.17321	0.39199	0.15680	0.32089	0.22659
0.5	30	0.02564	0.05128	0.00597	0.15453	0.57422	0.22969	0.72549	0.34070
	50	0.01538	0.03076	0.00313	0.11189	0.33574	0.13430	0.23723	0.19483
	100	0.00765	0.01530	0.00145	0.07616	0.16379	0.06552	0.05527	0.09404
	150	0.00532	0.01064	0.00092	0.06066	0.10823	0.04329	0.02373	0.06162
	200	0.00443	0.00886	0.00069	0.05254	0.08023	0.03209	0.01274	0.04515
1.0	30	0.05037	0.05037	0.02525	0.15890	0.17811	0.07124	0.06706	0.10358
	50	0.03114	0.03114	0.01355	0.11640	0.10599	0.04240	0.02361	0.06146
	100	0.01402	0.01402	0.00594	0.07707	0.05262	0.02105	0.00557	0.02985
	150	0.00845	0.00845	0.00396	0.06293	0.03536	0.01414	0.00250	0.02000
	200	0.00691	0.00691	0.00295	0.05431	0.02601	0.01040	0.00136	0.01475

Table 6: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML method at $\beta = 3.0$ $and \beta = 5.0$ and multiple α values

		$\boldsymbol{\beta = 3}$							
		$\hat{\alpha}$				$\hat{\beta}$			
α	**n**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**
0.2	30	0.03639	0.18195	0.00207	0.22749	3.93984	1.31328	60.1469	2.58515
	50	0.03148	0.15740	0.00139	0.18641	1.98353	0.66118	10.86760	1.09887
	100	0.02839	0.14195	0.00099	0.15732	0.87230	0.29077	1.75114	0.44110
	150	0.02729	0.13645	0.00087	0.14748	0.55955	0.18652	0.67272	0.27340
	200	0.02664	0.13320	0.00080	0.14142	0.41179	0.13726	0.34929	0.19700
0.5	30	0.02658	0.05316	0.00595	0.15427	0.60954	0.20318	0.84532	0.30647
	50	0.01549	0.03098	0.00323	0.11367	0.35288	0.11763	0.26177	0.17054
	100	0.00744	0.01488	0.00141	0.07510	0.17670	0.05890	0.06319	0.08379
	150	0.00569	0.01138	0.00095	0.06164	0.11586	0.03862	0.02734	0.05512
	200	0.00394	0.00788	0.00069	0.05254	0.08689	0.02896	0.01516	0.04104
1.0	30	0.05203	0.05203	0.02646	0.16267	0.20155	0.06718	0.08443	0.09686
	50	0.02876	0.02876	0.01378	0.11739	0.11754	0.03918	0.02835	0.05612
	100	0.01335	0.01335	0.00629	0.07931	0.05794	0.01931	0.00689	0.02767
	150	0.00879	0.00879	0.00401	0.06332	0.03822	0.01274	0.00293	0.01804
	200	0.00702	0.00702	0.00302	0.05495	0.02881	0.00960	0.00168	0.01366
		$\boldsymbol{\beta = 5}$							
α	**n**	$\hat{\alpha}$				$\hat{\beta}$			
0.2	30	0.08477	0.42385	0.00813	0.45083	3.98026	0.79605	53.1158	1.45761
	50	0.07919	0.39595	0.00678	0.41170	2.02662	0.40532	10.80270	0.65735
	100	0.07477	0.37385	0.00584	0.38210	0.93435	0.18687	1.88987	0.27495
	150	0.07312	0.36560	0.00550	0.37081	0.59808	0.11962	0.75709	0.17402
	200	0.07255	0.36275	0.00538	0.36674	0.43429	0.08686	0.38947	0.12482
0.5	30	0.03249	0.06498	0.00652	0.16149	0.83524	0.16705	1.57224	0.25078
	50	0.02092	0.04184	0.00339	0.11645	0.47226	0.09445	0.47052	0.13719
	100	0.01382	0.02764	0.00160	0.08000	0.22844	0.04569	0.10825	0.06580
	150	0.01156	0.02312	0.00106	0.06512	0.15193	0.03039	0.04723	0.04346
	200	0.01036	0.02072	0.00078	0.05586	0.11292	0.02258	0.02622	0.03239
1.0	30	0.05243	0.05243	0.02855	0.16897	0.28071	0.05614	0.16475	0.08118
	50	0.03069	0.03069	0.01506	0.12272	0.16979	0.03396	0.05854	0.04839
	100	0.01597	0.01597	0.00686	0.08283	0.08303	0.01661	0.01404	0.02370
	150	0.01017	0.01017	0.00447	0.06686	0.05431	0.01086	0.00603	0.01553
	200	0.00586	0.00586	0.00321	0.05666	0.04086	0.00817	0.00337	0.01161

Table 7: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML and L-M methods at $\beta = 1.2$ and multiple values of α

		$\beta = 1.2$															
		Maximum likelihood								L-moments							
		$\hat{\alpha}$				$\hat{\beta}$				$\hat{\alpha}$				$\hat{\beta}$			
α	n	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE
1.5	30	0.0759	0.0506	0.0522	0.1522	0.1045	0.0871	0.0202	0.1185	0.2509	0.1673	0.1506	0.2587	0.3121	0.2601	0.2342	0.4033
	50	0.0441	0.0294	0.0273	0.1101	0.0659	0.0549	0.0080	0.0746	0.1948	0.1298	0.0972	0.2079	0.2686	0.2238	0.1716	0.3452
	100	0.0198	0.0132	0.0125	0.0744	0.0348	0.0290	0.0023	0.0400	0.1322	0.0882	0.0537	0.1544	0.2069	0.1724	0.1134	0.2806
	150	0.0140	0.0093	0.0082	0.0604	0.0236	0.0196	0.0011	0.0271	0.1128	0.0752	0.0404	0.1341	0.1842	0.1535	0.0918	0.2525
	200	0.0094	0.0063	0.0060	0.0515	0.0178	0.0148	0.0006	0.0207	0.0980	0.0654	0.0332	0.1214	0.1681	0.1401	0.0805	0.2364
1.6	30	0.0762	0.0476	0.0597	0.1527	0.0938	0.0781	0.0160	0.1053	0.2360	0.1475	0.1580	0.2484	0.2485	0.2071	0.1718	0.3454
	50	0.0463	0.0290	0.0325	0.1126	0.0603	0.0502	0.0068	0.0687	0.1848	0.1155	0.1013	0.1989	0.2105	0.1755	0.1240	0.2934
	100	0.0238	0.0149	0.0143	0.0747	0.0305	0.0254	0.0018	0.0351	0.1259	0.0787	0.0551	0.1467	0.1578	0.1315	0.0799	0.2355
	150	0.0160	0.0100	0.0095	0.0609	0.0209	0.0174	0.0008	0.0242	0.1038	0.0649	0.0411	0.1267	0.1352	0.1127	0.0650	0.2125
	200	0.0125	0.0078	0.0071	0.0526	0.0158	0.0132	0.0005	0.0184	0.0917	0.0573	0.0340	0.1152	0.1241	0.1034	0.0557	0.1967
1.7	30	0.0869	0.0511	0.0658	0.1509	0.0864	0.0720	0.0138	0.0978	0.2356	0.1386	0.1656	0.2394	0.1895	0.1579	0.1285	0.2988
	50	0.0484	0.0284	0.0350	0.1101	0.0538	0.0449	0.0055	0.0616	0.1761	0.1036	0.1066	0.1921	0.1704	0.1420	0.0995	0.2628
	100	0.0251	0.0148	0.0158	0.0739	0.0278	0.0232	0.0015	0.0321	0.1197	0.0704	0.0570	0.1404	0.1252	0.1043	0.0629	0.2090
	150	0.0159	0.0094	0.0105	0.0602	0.0191	0.0159	0.0007	0.0222	0.0904	0.0532	0.0406	0.1186	0.0992	0.0827	0.0497	0.1857
	200	0.0114	0.0067	0.0078	0.0521	0.0143	0.0119	0.0004	0.0167	0.0799	0.0470	0.0326	0.1063	0.0916	0.0763	0.0423	0.1713
2.0	30	0.0966	0.0483	0.0931	0.1526	0.0667	0.0556	0.0082	0.0756	0.2200	0.1100	0.1996	0.2234	0.1136	0.0947	0.0654	0.2132
	50	0.0628	0.0314	0.0508	0.1127	0.0419	0.0349	0.0033	0.0482	0.1613	0.0807	0.1204	0.1735	0.0850	0.0708	0.0467	0.1800
	100	0.0299	0.0150	0.0230	0.0758	0.0218	0.0182	0.0009	0.0250	0.0986	0.0493	0.0665	0.1290	0.0556	0.0463	0.0305	0.1454
	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1

50	186	093	147	606	143	119	004	165	732	366	459	071	419	349	241	293
200	0.0125	0.0062	0.0107	0.0517	0.0110	0.0092	0.0002	0.0129	0.0590	0.0295	0.0363	0.0952	0.0344	0.0287	0.0208	0.1202

Table 8: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML and L-M methods at $\beta = 1.5$ and multiple values of α

$\boldsymbol{\beta = 1.5}$																	
		Maximum likelihood								**L-moments**							
		$\hat{\alpha}$				$\hat{\beta}$				$\hat{\alpha}$				$\hat{\beta}$			
α	**n**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**
1.5	30	0.0751	0.0501	0.0552	0.1566	0.0857	0.0571	0.0146	0.0804	0.2546	0.1697	0.1576	0.2646	0.2750	0.1834	0.2448	0.3299
	50	0.0447	0.0298	0.0292	0.1140	0.0520	0.0346	0.0054	0.0489	0.1927	0.1285	0.0991	0.2098	0.2204	0.1469	0.1698	0.2747
	100	0.0219	0.0146	0.0132	0.0766	0.0260	0.0173	0.0014	0.0247	0.1386	0.0924	0.0591	0.1621	0.1707	0.1138	0.1155	0.2266
	150	0.0167	0.0111	0.0085	0.0615	0.0178	0.0119	0.0006	0.0167	0.1166	0.0777	0.0431	0.1384	0.1461	0.0974	0.0925	0.2028
	200	0.0107	0.0071	0.0064	0.0532	0.0130	0.0087	0.0003	0.0123	0.0976	0.0650	0.0347	0.1243	0.1259	0.0839	0.0791	0.1875
1.6	30	0.0830	0.0519	0.0636	0.1576	0.0788	0.0525	0.0122	0.0738	0.2488	0.1555	0.1696	0.2574	0.2056	0.1370	0.1790	0.2821
	50	0.0470	0.0294	0.0337	0.1147	0.0468	0.0312	0.0044	0.0443	0.1829	0.1143	0.1056	0.2031	0.1641	0.1094	0.1430	0.2521
	100	0.0243	0.0152	0.0153	0.0773	0.0240	0.0160	0.0011	0.0225	0.1245	0.0778	0.0596	0.1525	0.1170	0.0780	0.0880	0.1978
	150	0.0147	0.0092	0.0096	0.0614	0.0159	0.0106	0.0005	0.0149	0.1024	0.0640	0.0435	0.1303	0.1014	0.0676	0.0714	0.1782
	200	0.0113	0.0070	0.0073	0.0533	0.0119	0.0079	0.0003	0.0112	0.0861	0.0538	0.0365	0.1193	0.0879	0.0586	0.0622	0.1662
1.7	30	0.0827	0.0487	0.0716	0.1574	0.0708	0.0472	0.0102	0.0672	0.2301	0.1354	0.1725	0.2443	0.1536	0.1024	0.1394	0.2489
	50	0.0506	0.0297	0.0379	0.1145	0.0432	0.0288	0.0037	0.0408	0.1720	0.1012	0.1067	0.1921	0.1207	0.0805	0.1008	0.2116
	100	0.0226	0.0133	0.0171	0.0770	0.0213	0.0142	0.0009	0.0198	0.1138	0.0670	0.0602	0.1443	0.0857	0.0571	0.0662	0.1715
	150	0.0164	0.0097	0.0111	0.0618	0.0144	0.0096	0.0004	0.0135	0.0934	0.0549	0.0435	0.1227	0.0717	0.0478	0.0526	0.1529
	200	0.0129	0.0076	0.0084	0.0539	0.0107	0.0071	0.0002	0.0101	0.0802	0.0472	0.0354	0.1107	0.0616	0.0411	0.0459	0.1429
2.	30	0.0986	0.0493	0.0976	0.1562	0.0541	0.0360	0.0059	0.0510	0.2179	0.1090	0.2098	0.2290	0.0756	0.0504	0.0716	0.1784

0	50	0.0593	0.0297	0.0527	0.1148	0.0336	0.0224	0.0023	0.0317	0.1603	0.0801	0.1290	0.1796	0.0612	0.0408	0.0514	0.1512
	100	0.0276	0.0138	0.0246	0.0785	0.0165	0.0110	0.0006	0.0156	0.0947	0.0474	0.0705	0.1328	0.0345	0.0230	0.0352	0.1250
	150	0.0182	0.0091	0.0159	0.0630	0.0112	0.0075	0.0003	0.0105	0.0733	0.0366	0.0510	0.1129	0.0267	0.0178	0.0282	0.1119
	200	0.0144	0.0072	0.0118	0.0542	0.0083	0.0056	0.0001	0.0079	0.0611	0.0306	0.0395	0.0994	0.0225	0.0150	0.0229	0.1009

Table 9: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML and L-M methods at $\beta = 2.0$ and multiple values of α

$\boldsymbol{\beta = 2.0}$																	
		Maximum likelihood								**L-moments**							
		$\hat{\alpha}$				$\hat{\beta}$				$\hat{\alpha}$				$\hat{\beta}$			
α	**n**	**Bias**	R.Bias	MSE	SMSE	**Bias**	R.Bias	MSE	SMSE	**Bias**	R.Bias	MSE	SMSE	**Bias**	R.Bias	MSE	SMSE
1.5	30	0.0767	0.0511	0.0580	0.1605	0.0885	0.0442	0.0160	0.0633	0.2491	0.1660	0.1701	0.2749	0.2687	0.1343	0.3013	0.2744
	50	0.0464	0.0309	0.0315	0.1182	0.0527	0.0264	0.0057	0.0376	0.1890	0.1260	0.1060	0.2171	0.2182	0.1091	0.2148	0.2317
	100	0.0211	0.0141	0.0140	0.0789	0.0266	0.0133	0.0014	0.0186	0.1320	0.0880	0.0626	0.1668	0.1663	0.0832	0.1457	0.1909
	150	0.0139	0.0093	0.0091	0.0635	0.0176	0.0088	0.0006	0.0124	0.1104	0.0736	0.0472	0.1449	0.1462	0.0731	0.1180	0.1718
	200	0.0106	0.0071	0.0067	0.0547	0.0133	0.0066	0.0004	0.0095	0.0938	0.0625	0.0392	0.1320	0.1255	0.0628	0.1044	0.1615
1.6	30	0.0846	0.0529	0.0680	0.1629	0.0810	0.0405	0.0133	0.0576	0.2457	0.1536	0.1809	0.2658	0.2075	0.1037	0.2234	0.2363
	50	0.0512	0.0320	0.0366	0.1195	0.0477	0.0239	0.0046	0.0338	0.1849	0.1156	0.1132	0.2103	0.1679	0.0840	0.1603	0.2002
	100	0.0250	0.0156	0.0163	0.0797	0.0240	0.0120	0.0012	0.0170	0.1255	0.0785	0.0647	0.1589	0.1230	0.0615	0.1086	0.1648
	150	0.0156	0.0097	0.0107	0.0647	0.0159	0.0080	0.0051	0.0356	0.1002	0.0626	0.0485	0.1376	0.1009	0.0504	0.0862	0.1468
	200	0.0114	0.0071	0.0077	0.0547	0.0119	0.0060	0.0003	0.0084	0.0878	0.0548	0.0393	0.1239	0.0919	0.0459	0.0746	0.1366
1.7	30	0.0931	0.0547	0.0768	0.1630	0.0735	0.0367	0.0108	0.0520	0.2447	0.1440	0.1865	0.2541	0.1661	0.0830	0.1713	0.2069
	50	0.0518	0.0305	0.0412	0.1194	0.0440	0.0220	0.0039	0.0313	0.1780	0.1047	0.1170	0.2012	0.1321	0.0660	0.1222	0.1748
	100	0.0258	0.0152	0.0186	0.0803	0.0216	0.0108	0.0009	0.0152	0.1175	0.0691	0.0646	0.1495	0.0900	0.0450	0.0814	0.1426
	150	0.0178	0.0105	0.0119	0.0642	0.0145	0.0072	0.0004	0.0103	0.0930	0.0547	0.0470	0.1275	0.0695	0.0348	0.0645	0.1270

α	n	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE	Bias	R.Bias	MSE	SMSE
	200	0.0134	0.0079	0.0088	0.0553	0.0108	0.0054	0.0002	0.0077	0.0803	0.0472	0.0371	0.1133	0.0639	0.0319	0.0529	0.1150
2.0	30	0.1085	0.0543	0.1100	0.1658	0.0580	0.0290	0.0068	0.0411	0.2268	0.1134	0.2244	0.2368	0.0830	0.0415	0.0915	0.1512
	50	0.0607	0.0303	0.0574	0.1198	0.0350	0.0175	0.0025	0.0248	0.1582	0.0791	0.1379	0.1856	0.0620	0.0310	0.0658	0.1283
	100	0.0300	0.0150	0.0258	0.0802	0.0171	0.0086	0.0006	0.0120	0.0997	0.0498	0.0748	0.1367	0.0401	0.0200	0.0413	0.1016
	150	0.0228	0.0114	0.0179	0.0669	0.0112	0.0056	0.0003	0.0081	0.0794	0.0397	0.0547	0.1169	0.0316	0.0158	0.0314	0.0885
	200	0.0145	0.0072	0.0124	0.0557	0.0088	0.0044	0.0002	0.0061	0.0624	0.0312	0.0429	0.1035	0.0267	0.0133	0.0259	0.0805

Table 10: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML and L-M methods at $\beta = 2.5$ and multiple values of α

		$\boldsymbol{\beta = 2.5}$															
		Maximum likelihood								**L-moments**							
		$\hat{\boldsymbol{\alpha}}$				$\hat{\boldsymbol{\beta}}$				$\hat{\boldsymbol{\alpha}}$				$\hat{\boldsymbol{\beta}}$			
$\boldsymbol{\alpha}$	**n**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**
1.5	30	0.0825	0.0550	0.0642	0.1689	0.0983	0.0393	0.0198	0.0562	0.2577	0.1718	0.1776	0.2809	0.3001	0.1200	0.3988	0.2526
	50	0.0462	0.0308	0.0323	0.1198	0.0578	0.0231	0.0068	0.0329	0.1948	0.1299	0.1103	0.2214	0.2482	0.0993	0.2812	0.2121
	100	0.0219	0.0146	0.0146	0.0805	0.0292	0.0117	0.0017	0.0165	0.1367	0.0911	0.0641	0.1687	0.1884	0.0754	0.1915	0.1750
	150	0.0160	0.0107	0.0098	0.0658	0.0194	0.0078	0.0007	0.0109	0.1088	0.0725	0.0480	0.1461	0.1490	0.0596	0.1552	0.1576
	200	0.0117	0.0078	0.0071	0.0562	0.0143	0.0057	0.0004	0.0081	0.0990	0.0660	0.0400	0.1334	0.1417	0.0567	0.1317	0.1452
1.6	30	0.0803	0.0502	0.0703	0.1657	0.0891	0.0356	0.0161	0.0507	0.2409	0.1506	0.1795	0.2648	0.2304	0.0922	0.2893	0.2151
	50	0.0510	0.0319	0.0380	0.1218	0.0533	0.0213	0.0058	0.0303	0.1865	0.1165	0.1177	0.2144	0.1892	0.0757	0.2488	0.1995
	100	0.0253	0.0158	0.0171	0.0818	0.0262	0.0105	0.0014	0.0148	0.1262	0.0789	0.0658	0.1603	0.1339	0.0535	0.1398	0.1495
	150	0.0160	0.0100	0.0110	0.0654	0.0176	0.0070	0.0006	0.0100	0.0994	0.0621	0.0492	0.1386	0.1090	0.0436	0.1145	0.1353
	200	0.0132	0.0083	0.0083	0.0568	0.0132	0.0053	0.0003	0.0075	0.0860	0.0538	0.0407	0.1261	0.0945	0.0378	0.0973	0.1248
1.7	30	0.0974	0.0573	0.0829	0.1693	0.0818	0.0327	0.0135	0.0465	0.2540	0.1494	0.2025	0.2647	0.1876	0.0750	0.2240	0.1893
	50	0.0542	0.0319	0.0433	0.1224	0.0492	0.0197	0.0048	0.0278	0.1821	0.1071	0.1200	0.2038	0.1492	0.0597	0.1503	0.1551

	100	0.0284	0.0167	0.0201	0.0834	0.0240	0.0096	0.0012	0.0138	0.1212	0.0713	0.0685	0.1539	0.1014	0.0406	0.1049	0.1296
	150	0.0173	0.0102	0.0125	0.0659	0.0162	0.0065	0.0005	0.0092	0.0908	0.0534	0.0486	0.1297	0.0774	0.0310	0.0836	0.1156
	200	0.0130	0.0077	0.0092	0.0564	0.0120	0.0048	0.0003	0.0068	0.0794	0.0467	0.0388	0.1159	0.0706	0.0283	0.0678	0.1042
2.0	30	0.1031	0.0515	0.1125	0.1677	0.0655	0.0262	0.0087	0.0372	0.2207	0.1103	0.2304	0.2400	0.0918	0.0367	0.1204	0.1388
	50	0.0671	0.0336	0.0645	0.1270	0.0389	0.0156	0.0031	0.0222	0.1635	0.0818	0.1444	0.1900	0.0702	0.0281	0.0805	0.1135
	100	0.0295	0.0148	0.0278	0.0834	0.0196	0.0078	0.0008	0.0110	0.0972	0.0486	0.0759	0.1378	0.0448	0.0179	0.0513	0.0906
	150	0.0212	0.0106	0.0177	0.0665	0.0130	0.0052	0.0003	0.0074	0.0750	0.0375	0.0536	0.1157	0.0342	0.0137	0.0368	0.0767
	200	0.0157	0.0079	0.0133	0.0577	0.0098	0.0039	0.0002	0.0055	0.0639	0.0319	0.0437	0.1045	0.0309	0.0123	0.0310	0.0704

Table 11: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using ML and L-M methods at $\beta = 3.0$, $\beta = 5.0$ and $\alpha = 1.5$, $\alpha = 2.0$

		Maximum likelihood								L-moments							
		$\hat{\alpha}$				$\hat{\beta}$				$\hat{\alpha}$				$\hat{\beta}$			
$\beta = 3.0$																	
α	**n**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**	**Bias**	**R.Bias**	**MSE**	**SMSE**
1.5	30	0.0832	0.0555	0.0629	0.1671	0.1092	0.0364	0.0244	0.0521	0.2608	0.1739	0.1746	0.2786	0.3327	0.1109	0.4817	0.2314
	50	0.0486	0.0324	0.0342	0.1232	0.0654	0.0218	0.0087	0.0311	0.1983	0.1322	0.1103	0.2214	0.2747	0.0916	0.3510	0.1975
	100	0.0243	0.0162	0.0156	0.0833	0.0325	0.0108	0.0021	0.0153	0.1411	0.0940	0.0631	0.1675	0.2111	0.0704	0.2315	0.1604
	150	0.0144	0.0096	0.0096	0.0654	0.0215	0.0072	0.0009	0.0102	0.1138	0.0759	0.0466	0.1439	0.1787	0.0596	0.1906	0.1455
	200	0.0133	0.0089	0.0074	0.0573	0.0164	0.0055	0.0005	0.0077	0.1033	0.0688	0.0389	0.1315	0.1607	0.0536	0.1631	0.1346
2.0	30	0.1058	0.0529	0.1171	0.1711	0.0741	0.0247	0.0112	0.0352	0.2272	0.1136	0.2300	0.2398	0.1133	0.0378	0.1417	0.1255
	50	0.0710	0.0355	0.0663	0.1287	0.0443	0.0148	0.0040	0.0209	0.1751	0.0876	0.1435	0.1894	0.0916	0.0305	0.0935	0.1019
	100	0.0322	0.0161	0.0291	0.0853	0.0219	0.0073	0.0010	0.0103	0.1093	0.0547	0.0725	0.1346	0.0642	0.0214	0.0571	0.0796
	150	0.0187	0.0094	0.0183	0.0677	0.0144	0.0048	0.0004	0.0067	0.0860	0.0430	0.0512	0.1131	0.0550	0.0183	0.0441	0.0700
	200	0.0154	0.0077	0.0138	0.0588	0.0110	0.0037	0.0002	0.0052	0.0740	0.0370	0.0429	0.1036	0.0465	0.0155	0.0404	0.0670
α	**n**	**$\beta = 5.0$**															
1.5	30	0.0838	0.0559	0.0686	0.1747	0.1625	0.0325	0.0504	0.0449	0.2679	0.1786	0.1829	0.2851	0.4992	0.0998	1.0850	0.2083
	50	0.0508	0.0338	0.0360	0.1265	0.0973	0.0195	0.0192	0.0277	0.1997	0.1331	0.1159	0.2269	0.3729	0.0746	0.8598	0.1854
	100	0.0235	0.0157	0.0170	0.0868	0.0478	0.0096	0.0046	0.0136	0.1398	0.0932	0.0635	0.1680	0.2959	0.0592	0.5290	0.1455
	150	0.0163	0.0109	0.0106	0.0687	0.0315	0.0063	0.0020	0.0090	0.1137	0.0758	0.0491	0.1477	0.2350	0.0470	0.4828	0.1390
	200	0.0137	0.0091	0.0078	0.0588	0.0237	0.0047	0.0011	0.0067	0.1007	0.0671	0.0400	0.1333	0.2100	0.0420	0.4091	0.1279
2.0	30	0.1237	0.0618	0.1329	0.1823	0.1095	0.0219	0.0241	0.0311	0.2496	0.1248	0.2602	0.2550	0.1647	0.0329	0.3374	0.1162
	50	0.0714	0.0357	0.0690	0.1313	0.0671	0.0134	0.0092	0.0192	0.1698	0.0849	0.1538	0.1961	0.1178	0.0236	0.2425	0.0985
	100	0.0307	0.0153	0.0299	0.0864	0.0326	0.0065	0.0021	0.0091	0.1036	0.0518	0.0806	0.1420	0.0811	0.0162	0.1435	0.0758

150	0.0210	0.0105	0.0200	0.0708	0.0218	0.0044	0.0010	0.0062	0.0742	0.0371	0.0575	0.1199	0.0543	0.0109	0.1199	0.0692
200	0.0174	0.0087	0.0152	0.0616	0.0162	0.0032	0.0005	0.0046	0.0641	0.0321	0.0447	0.1058	0.0490	0.0098	0.0865	0.0588

Table 12: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using the three estimation methods at $\beta = 1.5$,2.0 and $\alpha =$ 2.5 , 3.0, 5.0

		Maximum likelihood				L-moments				Moments			
		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\beta}$	
α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
						$\beta = 1.5$							
2.5	30	0.1277	0.1573	0.0396	0.0031	0.2064	0.2879	0.0259	0.0356	0.7650	0.9387	0.4133	0.2326
	50	0.0727	0.0848	0.0241	0.0011	0.1367	0.1787	0.0173	0.0260	0.6109	0.6183	0.3597	0.1797
	100	0.0344	0.0392	0.0118	0.0003	0.0793	0.0906	0.0109	0.0148	0.4722	0.3569	0.3187	0.1288
	150	0.0252	0.0258	0.0081	0.0001	0.0567	0.0648	0.0066	0.0110	0.4069	0.2713	0.2853	0.1048
	200	0.0165	0.0185	0.0060	0.0001	0.0422	0.0498	0.0046	0.0086	0.3690	0.2255	0.2681	0.0935
3.0	30	0.1488	0.2340	0.0306	0.0019	0.2039	0.3939	0.0122	0.0195	0.7312	1.0614	0.2332	0.0850
	50	0.0975	0.1304	0.0182	0.0007	0.1400	0.2404	0.0068	0.0131	0.5693	0.6736	0.1912	0.0614
	100	0.0469	0.0585	0.0091	0.0002	0.0772	0.1210	0.0045	0.0072	0.4030	0.3774	0.1489	0.0428
	150	0.0318	0.0377	0.0060	0.0001	0.0526	0.0820	0.0024	0.0050	0.3324	0.2710	0.1280	0.0349
	200	0.0234	0.0282	0.0045	0.0000	0.0423	0.0656	0.0022	0.0039	0.2926	0.2326	0.1136	0.0324
5.0	30	0.2852	0.7512	0.0149	0.0004	0.2689	1.0441	0.0019	0.0036	0.7831	2.0364	0.0520	0.0110
	50	0.1716	0.4021	0.0090	0.0002	0.1778	0.6150	0.0021	0.0021	0.5533	1.2363	0.0382	0.0082
	100	0.0812	0.1731	0.0045	0.0000	0.0842	0.2831	0.0008	0.0011	0.3261	0.6231	0.0234	0.0057
	150	0.0549	0.1150	0.0030	0.0000	0.0601	0.1908	0.0007	0.0001	0.2488	0.4372	0.0184	0.0043
	200	0.0434	0.0828	0.0023	0.0000	0.0485	0.1422	0.0006	0.0005	0.2034	0.3410	0.0149	0.0037
						$\beta = 2.0$							
2.5	30	0.1408	0.1825	0.0424	0.0036	0.2226	0.3221	0.0330	0.0422	0.7986	1.0225	0.4732	0.3032
	50	0.0794	0.0958	0.0257	0.0014	0.1472	0.1918	0.0236	0.0287	0.6366	0.6555	0.4143	0.2325
	100	0.0369	0.0421	0.0128	0.0003	0.0784	0.0963	0.0119	0.0172	0.4754	0.3783	0.3395	0.1650
	150	0.0258	0.0274	0.0085	0.0001	0.0574	0.0683	0.0083	0.0128	0.4129	0.2887	0.3060	0.1377
	200	0.0200	0.0202	0.0062	0.0001	0.0506	0.0521	0.0091	0.0092	0.3783	0.2416	0.2859	0.1219
3.0	30	0.1763	0.2785	0.0329	0.0022	0.2362	0.4536	0.0171	0.0218	0.7828	1.2046	0.2590	0.1148
	50	0.0990	0.1458	0.0193	0.0008	0.1468	0.2577	0.0112	0.0147	0.5893	0.7210	0.2156	0.0868
	100	0.0477	0.0634	0.0098	0.0002	0.0812	0.1294	0.0069	0.0083	0.4181	0.4012	0.1655	0.0622
	150	0.0317	0.0407	0.0065	0.0001	0.0582	0.0874	0.0055	0.0055	0.3410	0.2976	0.1393	0.0531
	200	0.0244	0.0308	0.0049	0.0001	0.0453	0.0682	0.0041	0.0045	0.2992	0.2390	0.1264	0.0449
5.0	30	0.2799	0.8257	0.0173	0.0006	0.2636	1.1457	0.0027	0.0049	0.8171	2.3183	0.0647	0.0146
	50	0.1691	0.4313	0.0103	0.0002	0.1683	0.6419	0.0020	0.0028	0.5673	1.3297	0.0463	0.0107
	100	0.0835	0.2037	0.0052	0.0001	0.0882	0.3272	0.0012	0.0015	0.3388	0.7019	0.0282	0.0072
	150	0.0524	0.1257	0.0035	0.0000	0.0552	0.2031	0.0007	0.0009	0.2512	0.4690	0.0220	0.0055
	200	0.0431	0.0952	0.0026	0.0000	0.0507	0.1567	0.0011	0.0007	0.2095	0.3759	0.0177	0.0050

Table 13: Biases and MSEs of $\hat{\alpha}$ and $\hat{\beta}$ using the three estimation methods at $\beta = 2.5$,3.0 and $\alpha = 2.5$, 3.0, 5.0

		Maximum likelihood				L-moments				Moments			
		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\beta}$	
α	n	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
						$\beta = 2.5$							
2.5	30	0.1429	0.1949	0.0488	0.0048	0.2272	0.3275	0.0419	0.0536	0.8137	1.0359	0.5586	0.4134
	50	0.0867	0.1018	0.0291	0.0017	0.1544	0.1979	0.0283	0.0364	0.6479	0.6653	0.4813	0.2995
	100	0.0379	0.0460	0.0143	0.0004	0.0821	0.1000	0.0161	0.0209	0.4864	0.3817	0.4044	0.2098
	150	0.0271	0.0296	0.0095	0.0002	0.0608	0.0721	0.0107	0.0155	0.4198	0.2911	0.3606	0.1709
	200	0.0196	0.0217	0.0072	0.0001	0.0483	0.0552	0.0092	0.0119	0.3826	0.2434	0.3374	0.1521
3.0	30	0.1726	0.2770	0.0385	0.0030	0.2321	0.4553	0.0206	0.0294	0.7888	1.2008	0.3186	0.1464
	50	0.1039	0.1512	0.0226	0.0010	0.1519	0.2692	0.0133	0.0194	0.6050	0.7469	0.2575	0.1068
	100	0.0490	0.0690	0.0114	0.0003	0.0824	0.1362	0.0083	0.0108	0.4239	0.4052	0.1969	0.0744
	150	0.0322	0.0444	0.0076	0.0001	0.0534	0.0930	0.0042	0.0076	0.3439	0.2920	0.1662	0.0618
	200	0.0259	0.0322	0.0057	0.0006	0.0450	0.0706	0.0038	0.0056	0.3058	0.2382	0.1492	0.0548
5.0	30	0.3187	0.9226	0.0207	0.0009	0.3110	1.2703	0.0045	0.0067	0.8810	2.5868	0.0770	0.0223
	50	0.1763	0.4697	0.0125	0.0003	0.1765	0.6932	0.0028	0.0038	0.5828	1.4232	0.0553	0.0150
	100	0.0885	0.2040	0.0062	0.0008	0.0930	0.3248	0.0016	0.0020	0.3506	0.0347	0.7033	0.0095
	150	0.0670	0.1400	0.0040	0.0000	0.0722	0.2282	0.0009	0.0013	0.2741	0.5207	0.0260	0.0079
	200	0.0404	0.0994	0.0030	0.0000	0.0433	0.1621	0.0007	0.0010	0.2057	0.3838	0.0209	0.0067
						$\beta = 3.0$							
2.5	30	0.1515	0.1982	0.0556	0.0064	0.2334	0.3433	0.0446	0.0691	0.8185	1.0708	0.6231	0.5238
	50	0.0786	0.1025	0.0332	0.0023	0.1482	0.1994	0.0330	0.0462	0.6480	0.6759	0.5514	0.3957
	100	0.0411	0.0474	0.0164	0.0005	0.0868	0.1056	0.0182	0.0270	0.4933	0.3955	0.4574	0.2718
	150	0.0310	0.0309	0.0109	0.0002	0.0634	0.0734	0.0116	0.0197	0.4233	0.2984	0.4048	0.2212
	200	0.0204	0.0226	0.0082	0.0001	0.0493	0.0577	0.0105	0.0163	0.3847	0.2498	0.3799	0.1996
3.0	30	0.1798	0.2964	0.0431	0.0038	0.2462	0.4729	0.0272	0.0379	0.8073	1.2550	0.3548	0.1964
	50	0.1034	0.1539	0.0264	0.0014	0.1546	0.2715	0.0179	0.0246	0.6101	0.7577	0.2952	0.1429
	100	0.0558	0.0720	0.0132	0.0004	0.0874	0.1433	0.0086	0.0144	0.4304	0.4213	0.2216	0.0967
	150	0.0327	0.0457	0.0087	0.0002	0.0574	0.0960	0.0063	0.0096	0.3508	0.3032	0.1905	0.0831
	200	0.0263	0.0337	0.0066	0.0001	0.0472	0.0717	0.0057	0.0072	0.3081	0.2427	0.1699	0.0724
5.0	30	0.3036	0.9192	0.0237	0.0011	0.2990	1.2940	0.0050	0.0091	0.8409	2.5206	0.0838	0.0331
	50	0.1812	0.4716	0.0144	0.0004	0.1830	0.7025	0.0032	0.0053	0.5943	1.4610	0.0645	0.0202
	100	0.0898	0.2190	0.0071	0.0001	0.0955	0.3458	0.0017	0.0027	0.3583	0.7400	0.0403	0.0140
	150	0.0536	0.1371	0.0048	0.0001	0.0593	0.2253	0.0012	0.0017	0.2635	0.5071	0.0312	0.0104
	200	0.0441	0.1030	0.0035	0.0000	0.0469	0.1694	0.0008	0.0014	0.1990	0.3856	0.0218	0.0097

Table 14: R.Biases and SMSEs of $\hat{\alpha}$ and $\hat{\beta}$ using the three estimation methods at $\beta = 1.5$,2.0 and $\alpha =$ 2.5 , 3.0, 5.0

		Maximum likelihood				L-moments				Moments			
		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\beta}$	
α	n	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE
$\beta = 1.5$													
2.5	30	0.0511	0.1587	0.0264	0.0371	0.0825	0.2146	0.0173	0.1258	0.3060	0.3876	0.2755	0.3215
	50	0.0291	0.1165	0.0161	0.0224	0.0547	0.1691	0.0115	0.1076	0.2444	0.3145	0.2398	0.2826
	100	0.0138	0.0792	0.0079	0.0112	0.0317	0.1204	0.0073	0.0810	0.1889	0.2389	0.2125	0.2393
	150	0.0101	0.0643	0.0054	0.0076	0.0227	0.1018	0.0044	0.0700	0.1628	0.2083	0.1902	0.2158
	200	0.0066	0.0544	0.0040	0.0056	0.0169	0.0892	0.0030	0.0618	0.1476	0.1900	0.1787	0.2038
3.0	30	0.0496	0.1612	0.0204	0.0289	0.0680	0.2092	0.0081	0.0932	0.2437	0.3434	0.1555	0.1943
	50	0.0325	0.1204	0.0121	0.0171	0.0467	0.1634	0.0045	0.0762	0.1898	0.2736	0.1275	0.1652
	100	0.0156	0.0806	0.0060	0.0084	0.0257	0.1160	0.0030	0.0564	0.1343	0.2048	0.0993	0.1379
	150	0.0106	0.0647	0.0040	0.0056	0.0175	0.0954	0.0016	0.0471	0.1108	0.1735	0.0853	0.1245
	200	0.0078	0.0559	0.0030	0.0042	0.0141	0.0854	0.0015	0.0418	0.0975	0.1608	0.0757	0.1200
5.0	30	0.0570	0.1733	0.0099	0.0140	0.0538	0.2044	0.0013	0.0399	0.1566	0.2854	0.0347	0.0698
	50	0.0343	0.1268	0.0060	0.0084	0.0356	0.1568	0.0014	0.0308	0.1107	0.2224	0.0254	0.0603
	100	0.0162	0.0832	0.0030	0.0042	0.0168	0.1064	0.0005	0.0219	0.0652	0.1579	0.0156	0.0503
	150	0.0110	0.0678	0.0020	0.0030	0.0120	0.0874	0.0005	0.0056	0.0498	0.1322	0.0123	0.0439
	200	0.0087	0.0576	0.0015	0.0021	0.0097	0.0754	0.0004	0.0153	0.0407	0.1168	0.0099	0.0404
$\beta = 2.0$													
2.5	30	0.0563	0.1709	0.0212	0.0301	0.0891	0.2270	0.0165	0.1028	0.3194	0.4045	0.2366	0.2753
	50	0.0317	0.1238	0.0128	0.0184	0.0589	0.1752	0.0118	0.0847	0.2546	0.3238	0.2072	0.2411
	100	0.0148	0.0820	0.0064	0.0089	0.0314	0.1241	0.0059	0.0655	0.1901	0.2460	0.1698	0.2031
	150	0.0103	0.0662	0.0043	0.0059	0.0230	0.1046	0.0041	0.0565	0.1652	0.2149	0.1530	0.1855
	200	0.0080	0.0569	0.0031	0.0045	0.0202	0.0913	0.0046	0.0479	0.1513	0.1966	0.1430	0.1746
3.0	30	0.0588	0.1759	0.0165	0.0233	0.0787	0.2245	0.0085	0.0738	0.2609	0.3658	0.1295	0.1694
	50	0.0330	0.1273	0.0097	0.0138	0.0489	0.1692	0.0056	0.0607	0.1964	0.2830	0.1078	0.1473
	100	0.0159	0.0839	0.0049	0.0069	0.0271	0.1199	0.0034	0.0456	0.1394	0.2111	0.0828	0.1247
	150	0.0106	0.0672	0.0033	0.0047	0.0194	0.0985	0.0028	0.0371	0.1137	0.1818	0.0696	0.1152
	200	0.0081	0.0585	0.0025	0.0035	0.0151	0.0870	0.0020	0.0334	0.0997	0.1630	0.0632	0.1060
5.0	30	0.0560	0.1817	0.0087	0.0123	0.0527	0.2141	0.0014	0.0348	0.1634	0.3045	0.0323	0.0603
	50	0.0338	0.1313	0.0052	0.0072	0.0337	0.1602	0.0010	0.0264	0.1135	0.2306	0.0232	0.0516
	100	0.0167	0.0903	0.0026	0.0039	0.0176	0.1144	0.0006	0.0191	0.0678	0.1676	0.0141	0.0425
	150	0.0105	0.0709	0.0017	0.0022	0.0110	0.0901	0.0004	0.0153	0.0502	0.1370	0.0110	0.0371
	200	0.0086	0.0617	0.0013	0.0016	0.0101	0.0792	0.0005	0.0133	0.0419	0.1226	0.0088	0.0354

Table 15: R.Biases and SMSEs of $\hat{\alpha}$ and $\hat{\beta}$ using the three estimation methods at $\beta = 2.5$,3.0 and $\alpha =$ 2.5 , 3.0, 5.0

		Maximum likelihood				L-moments				Moments			
		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\alpha}$		$\hat{\hat{\beta}}$		$\hat{\alpha}$		$\hat{\hat{\beta}}$	
α	n	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE	R.Bias	SMSE
$\beta = 2.5$													
2.5	30	0.0572	0.1766	0.0195	0.0277	0.0909	0.2289	0.0167	0.0926	0.3255	0.4071	0.2234	0.2572
	50	0.0347	0.1276	0.0116	0.0165	0.0618	0.1780	0.0113	0.0763	0.2592	0.3263	0.1925	0.2189
	100	0.0152	0.0858	0.0057	0.0081	0.0328	0.1265	0.0064	0.0578	0.1946	0.2471	0.1617	0.1832
	150	0.0108	0.0688	0.0038	0.0054	0.0243	0.1074	0.0043	0.0498	0.1679	0.2158	0.1442	0.1654
	200	0.0078	0.0589	0.0029	0.0040	0.0193	0.0940	0.0037	0.0437	0.1530	0.1974	0.1350	0.1560
3.0	30	0.0575	0.1754	0.0154	0.0219	0.0774	0.2249	0.0082	0.0686	0.2629	0.3653	0.1274	0.1531
	50	0.0346	0.1296	0.0090	0.0128	0.0506	0.1729	0.0053	0.0558	0.2017	0.2881	0.1030	0.1307
	100	0.0163	0.0875	0.0046	0.0064	0.0275	0.1230	0.0033	0.0415	0.1413	0.2122	0.0788	0.1091
	150	0.0107	0.0702	0.0030	0.0042	0.0178	0.1017	0.0017	0.0348	0.1146	0.1801	0.0665	0.0994
	200	0.0086	0.0598	0.0023	0.0100	0.0150	0.0886	0.0015	0.0300	0.1019	0.1627	0.0597	0.0936
5.0	30	0.0637	0.1921	0.0083	0.0119	0.0622	0.2254	0.0018	0.0328	0.1762	0.3217	0.0308	0.0597
	50	0.0353	0.1371	0.0050	0.0072	0.0353	0.1665	0.0011	0.0248	0.1166	0.2386	0.0221	0.0489
	100	0.0177	0.0903	0.0025	0.0113	0.0186	0.1140	0.0006	0.0177	0.0701	0.0372	0.2813	0.0390
	150	0.0134	0.0748	0.0016	0.0022	0.0144	0.0955	0.0004	0.0145	0.0548	0.1443	0.0104	0.0356
	200	0.0081	0.0631	0.0012	0.0018	0.0087	0.0805	0.0003	0.0125	0.0411	0.1239	0.0083	0.0327
$\beta = 3.0$													
2.5	30	0.0606	0.1781	0.0185	0.0266	0.0934	0.2344	0.0149	0.0876	0.3274	0.4139	0.2077	0.2413
	50	0.0315	0.1281	0.0111	0.0158	0.0593	0.1786	0.0110	0.0717	0.2592	0.3289	0.1838	0.2097
	100	0.0164	0.0871	0.0055	0.0077	0.0347	0.1300	0.0061	0.0548	0.1973	0.2516	0.1525	0.1738
	150	0.0124	0.0703	0.0036	0.0052	0.0254	0.1084	0.0039	0.0468	0.1693	0.2185	0.1349	0.1568
	200	0.0081	0.0601	0.0027	0.0038	0.0197	0.0961	0.0035	0.0426	0.1539	0.1999	0.1266	0.1489
3.0	30	0.0599	0.1815	0.0144	0.0204	0.0821	0.2292	0.0091	0.0649	0.2691	0.3734	0.1183	0.1477
	50	0.0345	0.1308	0.0088	0.0125	0.0515	0.1737	0.0060	0.0522	0.2034	0.2902	0.0984	0.1260
	100	0.0186	0.0894	0.0044	0.0062	0.0291	0.1262	0.0029	0.0400	0.1435	0.2164	0.0739	0.1037
	150	0.0109	0.0713	0.0029	0.0041	0.0191	0.1033	0.0021	0.0327	0.1169	0.1835	0.0635	0.0961
	200	0.0088	0.0612	0.0022	0.0032	0.0157	0.0892	0.0019	0.0282	0.1027	0.1642	0.0566	0.0897
5.0	30	0.0607	0.1917	0.0079	0.0112	0.0598	0.2275	0.0017	0.0317	0.1682	0.3175	0.0279	0.0606
	50	0.0362	0.1373	0.0048	0.0067	0.0366	0.1676	0.0011	0.0243	0.1189	0.2417	0.0215	0.0473
	100	0.0180	0.0936	0.0024	0.0033	0.0191	0.1176	0.0006	0.0172	0.0717	0.1720	0.0134	0.0394
	150	0.0107	0.0741	0.0016	0.0024	0.0119	0.0949	0.0004	0.0139	0.0527	0.1424	0.0104	0.0340
	200	0.0088	0.0642	0.0012	0.0015	0.0094	0.0823	0.0003	0.0124	0.0398	0.1242	0.0073	0.0329