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Four Parameters Kumaraswamy Reciprocal Family Of Distributions

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ABSTRACT

In this paper, kumaraswamy reciprocal family of distributions is introduced as a new continues model with some of approximation to other probabilistic models as reciprocal, beta, uniform, power function, exponential**,** negative exponential, weibull, rayleigh and pareto distribution. Some fundamental distributional properties, force of mortality, mills ratio, bowley skewness, moors kurtosis, reversed hazard function, integrated hazard function, mean residual life, probability weighted moments, bonferroni and lorenz curves, laplace-stieltjes transform of this new distribution with the maximum likelihood method of the parameter estimation are studied. Finally, four real data sets originally presented are used to illustrate the proposed estimators.

Keywords: Kumaraswamy distribution, Reciprocal distribution, Force of mortality, Mills Ratio, Bowley skewness, Moors kurtosis, Reversed hazard function, Integrated hazard function, Mean residual life, Probability weighted moments, Bonferroni and Lorenz curves, Laplace-Stieltjes transform.

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1. Introduction

The reciprocal distribution is a continuous probability distribution, gets its name because the density function is proportional to x^{-1}. It is a right-skewed distribution bounded between min (a) and max (b), and it is looks like a triangle (min, mode, max), if the min and the max are almost equal. Also, it is an example of an inverse distribution where the reciprocal of a random variable with a reciprocal distribution itself has a reciprocal distribution.

The probability density function (pdf) of the reciprocal distribution is

$$f(x) = A x^{-1} \ , \ a \le x \le b \ , b > a > 0$$

with cumulative distribution function $F(x) = A\ Q(x,a) \quad a \le x \le b \ , b > a > 0$

$$where, \quad A = (\ln b - \ln a)^{-1} \ , Q(x,a) = (\ln x - \ln a) .$$

The reciprocal distribution is of considerable importance in numerical analysis as a computer's arithmetic operations transform mantissas with initial arbitrary distributions to the reciprocal distribution as a limiting distribution. Also, it is widely used as an uninformed prior distribution in Bayesian inference for scale parameters, Hamming (1970) .

In the last few years, many new generated families are formed by adding additional one or more shape parameters to develop new models and increase its flexibility, such as Quasi Lindley distribution (Rama and Mishra, 2013), Kumaraswamy Weibull-generated family of distributions (Hassan, and Elgarhy, 2016), Kumaraswamy sushila distribution (Shawki and Elgarhy, 2017), L-Quadratic distribution (Salma, 2018), TAS distribution (Salma and Arwa, 2018), and others. The Kumaraswamy distribution is one of them, it has the following density function with two additional shape parameters $\alpha \ and \ \beta$

$$g(x) = \alpha\beta\, f(x) F(x)^{\alpha-1} \left[1 - F(x)^{\alpha}\right]^{\beta-1}, \quad \alpha,\beta > 0$$

with distribution function $G(x) = \left[1 - F(x)^{\alpha}\right]^{\beta}$

It is well known in general that, a generalized model is having more flexibility than the base model and it is favored by data analysts in analyzing data. Therefore, the main objective of this paper to propose a new continuous kumaraswamy reciprocal family and explain its flexible behavior, and how it is fitted for various continuous data sets in different applied fields. Section (2) identifies the four parameters kumaraswamy reciprocal family with some of its approximation to other probabilistic models and studding its reliability measurements behavior. Section (3) contains the properties of the kumaraswamy reciprocal family and the Maximum Likelihood (ML) method of the parameter estimation. Section (4) presents the practical applications to illustrate the proposed model.

2. The Four Parameters Kumaraswamy Reciprocal Family

The new distribution will abbreviate by $KRF(a,b,\alpha,\beta)$ with probability function takes the form

$$f(x) = \alpha\beta x^{-1} A^{\alpha} Q(x,a)^{\alpha-1} \left[1 - A^{\alpha}\, Q(x,a)^{\alpha}\right]^{\beta-1} \ , a \le x \le b \ , b > a > 0 \ \rightarrow (1)$$

with cumulative distribution function $F(x) = 1 - \left[1 - A^{\alpha}\, Q(x,a)^{\alpha}\right]^{\beta}, \ \rightarrow (2)$ $where, \ A = (\ln b - \ln a)^{-1}$,

$$Q(x,a) = (\ln x - \ln a).$$

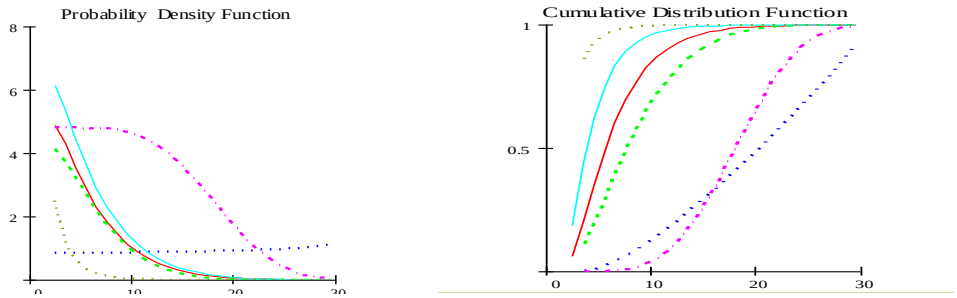

Figure 1: PDF and CDF of KRF(a,b,α,β)

Figure.1 illustrate plot the PDF and the CDF of the $KRF(a,b,\alpha,\beta)$ under different values of the shape parameters (α,β) and the scale parameters (a,b). It is noted that the PDF of the new distribution takes many shapes, and the plot of the CDF indicates increasing cumulative distribution function.

2.1 Approximation to other probabilistic models

Some of approximation to other probabilistic models are presented in Table.1 as some special cases of $KRF(a,b,\alpha,\beta)$.

Table 1 : Some Probabilistic Models of the Kumaraswamy Reciprocal Family

Parameters	*Random Variable*	*Distributed as*	*Density Functions*
$\alpha=\beta=1$	X	*Reciprocal*	$f(x)=A\,x^{-1}\ ,\ a\le x\le b\ ,\ b>a>0$
β	$y=A^{\alpha}\,Q(x,a)^{\alpha}$	*Beta* $(1,\beta)$	$f(y)=\frac{1}{(1,\beta)}(1-y)^{\beta-1}\ ,\ 0<y<1\ ,\ \beta>0$
$\beta=1$	$y=A^{\alpha}\,Q(x,a)^{\alpha}$	*Uniform*	$f(y)=1\ ,\ 0<y<1$
β	$y=1-A^{\alpha}\,Q(x,a)^{\alpha}$	*Power Function*	$f(y)=\beta\,y^{\beta-1}\ ,\ 0<y<1\ ,\ \beta>0$
$\beta=\frac{1}{\theta}$	$y=-\ln\left\{1-A^{\alpha}\,Q(x,a)^{\alpha}\right\}$	*Exponential*	$f(y)=\frac{1}{\beta}e^{\frac{-1}{\beta}y}\ ,\ y>0\ ,\ \theta>0$
β	$y=-\ln\left\{1-A^{\alpha}\,Q(x,a)^{\alpha}\right\}$	*-ve Exponential*	$f(y)=\beta\,e^{-\beta y}\ ,\ y>0\ ,\ \beta>0$
β,K	$y=k\left\{1-A^{\alpha}\,Q(x,a)^{\alpha}\right\}^{-1}$	*Pareto*	$f(y)=\frac{\beta\,k^{\beta}}{y^{\beta+1}}\ ,\ y\ge k\ ,\ \beta,k>0$
$\alpha=\frac{1}{\lambda}$	$y^{\frac{1}{\alpha}}=-\ln\left\{1-A^{\alpha}\,Q(x,a)^{\alpha}\right\}$	*Weibull*	$f(y)=\frac{\beta}{\alpha}y^{\frac{1}{\alpha}-1}e^{\frac{-\beta}{\alpha}y^{\frac{1}{\alpha}}}\ ,\ y>0\ ,\ \lambda.\beta>0$
$\alpha=\frac{1}{2},\beta=\frac{1}{2\alpha^2}$	$y^{\frac{1}{\alpha}}=-\ln\left\{1-A^{\alpha}\,Q(x,a)^{\alpha}\right\}$	*Rayleigh*	$f(y)=\frac{1}{\alpha^2}y\ e^{\frac{-1}{\alpha^2}y^2}\ ,\ y>0\ ,\ \beta>0$

2.2 Maximum likelihood estimation

The reliability function (RF) of the four parameters $KRF(a,b,\alpha,\beta)$ is

$$R(x)=1-F(x)=\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta}$$

The force of mortality (FM) of the four parameters $KRF(a,b,\alpha,\beta)$ is

$$h(x)=\frac{f(x)}{R(x)}=\frac{\alpha\beta}{x\,Q(x,a)\left[A^{-\alpha}\,Q(x,a)^{-\alpha}-1\right]}$$

The reversed hazard function (RHF) of the four parameters $KRF(a,b,\alpha,\beta)$ is

$$R_h(x)=\frac{f(x)}{F(x)}=\frac{\alpha\beta\left[A^{-\alpha}\,Q(x,a)^{-\alpha}-1\right]^{-1}}{xQ(x,a)^{\alpha-1}\left[\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{-\beta}-1\right]}$$

The integrated hazard function (IHF) of the four parameters $KRF(a,b,\alpha,\beta)$ is

$$H(x)=-\ln R(x)=-\beta\ln\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]$$

The mills ratio (MR) of the four parameters $KRF(a,b,\alpha,\beta)$ is

$$mi(x)=\frac{1}{h(x)}=\frac{x\left[A^{-\alpha}\,Q(x,a)^{-\alpha}-1\right]}{\alpha\beta Q(x,a)}.$$

3. Properties of the Kumaraswamy Reciprocal Family

3.1 Quantile Function

The quantile function $Q(u)$ of the four parameters $KRF(a,b,\alpha,\beta)$ has closed form and is obtained by inverting Eq.(2), therefore

$$Q(u)=a\,exp\left\{A^{-1}\left[1-(1-u)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}}\right\}\rightarrow(3)$$

3.2 Median and Random Numbers

From Eq.(3), if u equal to 0.5 then the median of $KRF(a,b,\alpha,\beta)$ is

$$m=a\,exp\left\{A^{-1}\left[1-0.5^{\beta^{-1}}\right]^{\frac{1}{\alpha}}\right\}.$$

Also, from Eq.(3), if $u\sim U(0,1)$ and the parameters (a,b,α,β) are known, then the random numbers of $KRF(a,b,\alpha,\beta)$ can be generated.

3.3 Bowley Skewness and Moors Kurtosis

The Bowley Skewness (BS), and Moors Kurtosis (MK) of the four parameters $KRF(a,b,\alpha,\beta)$ are obtained as follows

$$BS=\frac{Q(0.75)-2Q(0.5)+Q(0.25)}{Q(0.75)-Q(0.25)},\quad MK=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(0.75)-Q(0.25)}$$

Where $Q(u)$ is the quantile function of the four parameters $KRF(a,b,\alpha,\beta)$.

3.4 Mean Residual Life

Theorem 1: The mean residual life of a r.v X with pdf (1), is given by

$$\mu(t)=\frac{a}{\alpha\,R(t)}\sum_{\zeta=1}^{\infty}\frac{1}{A^{\zeta+\alpha}\,\zeta!}I_{1-T}\left(\beta+1,\frac{\zeta+1}{\alpha}\right),\quad T=A^{\alpha}\left(\ln t-\ln a\right)^{\alpha}$$

Proof: The Mean residual life of a r.v X is defined as

$\mu(t)=\frac{1}{R(t)}\int_{t}^{b}R(x)\,dx$, then

$$\frac{1}{R(t)}\int_t^b R(x)dx = \frac{1}{R(t)}\int_t^b \left[1 - A^{\alpha}\, Q(x,a)^{\alpha}\right]^{\beta} dx$$

$$= \frac{a}{\alpha\, A\, R(t)}\int_T^1 y^{\frac{1}{\alpha}-1}\left[1-y\right]^{\beta} e^{\frac{y^{\frac{1}{\alpha}}}{A}}\, d\,y, \quad such\,that \quad y = A^{\alpha}\, Q(x,a)^{\alpha}\ , \quad T = A^{\alpha}\left(ln\,t - ln\,a\right)^{\alpha}.$$

For $n,m>0$ integer, it follows that

$$\int_a^1 w^{n-1}\left[1-w\right]^{m-1} e^{w}\, d\,w = \sum_{\zeta=1}^{\infty}\frac{I_{1-a}(n+\zeta,m)}{\zeta!} \quad \rightarrow (4)$$

where $I_{1-a}(a,b) = \int_a^1 w^{a-1}\left[1-w\right]^{b-1} d\,w$ denotes the complementary incomplete beta function which can be evaluated in MathCad 15. Using Eq.(4), the mean residual life can be expressed as

$$\mu(t) = \frac{a}{\alpha\ R(t)}\sum_{\zeta=1}^{\infty}\frac{1}{A^{\zeta+\alpha}\,\zeta!}\, I_{1-T}\left(\beta+1,\frac{\zeta+1}{\alpha}\right), \quad T = A^{\alpha}\left(ln\,t - ln\,a\right)^{\alpha}.$$

3.5 Moments about Origin

Theorem 2: If X is a r.v distributed as $KRF(a,b,\alpha,\beta)$, then r^{th} moments of X is given by

$$\mu'_r = a^r + \frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\,\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)$$

Proof: r^{th} moments is defined as $\mu'_r = \int_a^b x^r\, f(x)dx$, then

$$\int_a^b x^r\, f(x)dx = \beta\int_a^b \frac{\alpha\, x^r}{x} A^{\alpha} Q(x,a)^{\alpha-1}\left[1 - A^{\alpha}\, Q(x,a)^{\alpha}\right]^{\beta-1} dx.$$

By using the partial derivative

$$\int_a^b x^r\, f(x)dx = (-\beta)\left[\frac{x^r\left[1 - A^{\alpha}\, Q(x,a)^{\alpha}\right]^{\beta}}{\beta}\right|_a^b - \frac{r}{\beta}\int_a^b x^{r-1}\left[1 - A^{\alpha}\, Q(x,a)^{\alpha}\right]^{\beta} dx\Bigg] = a^r + \frac{r\,a^r}{\alpha\, A}\int_0^1 y^{\frac{1}{\alpha}-1}\left[1-y\right]^{\beta} e^{\frac{r\,y^{\frac{1}{\alpha}}}{A}}\, d\,y.\ \ y = A^{\alpha}\, Q(x,a)^{\alpha}$$

For $n,m>0$ integer, it follows that

$$\int_0^1 w^{n-1}\left[1-w\right]^{m-1} e^{w}\, d\,w = \sum_{\zeta=1}^{\infty}\frac{\beta(n+\zeta,m)}{\zeta!} \quad \rightarrow (5)$$

where $\beta(a,b) = \int_0^1 w^{a-1}\left[1-w\right]^{b-1} d\,w = \frac{\Gamma a\ \Gamma b}{\Gamma(a+b)}$ denotes the beta function.

Using Eq.(5), the r^{th} moments can be expressed as

$$\mu'_r = a^r + \frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\,\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right), r = 1,2,... \rightarrow (6)\ .$$

From Eq.(6) , the 1^{st} four moments and the variance of the $KRF(a,b,\alpha,\beta)$ are

$$\mu = \mu'_1 = a + \frac{a}{\alpha}\sum_{\zeta=1}^{\infty}\frac{\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)}{A^{\zeta+1}\,\zeta!},\ \ \mu'_2 = a^2 + \frac{a^2}{\alpha}\sum_{\zeta=1}^{\infty}\frac{2^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\,\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)$$

$$\mu'_3 = a^3 + \frac{a^3}{\alpha}\sum_{\zeta=1}^{\infty}\frac{3^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right),\ \mu'_4 = a^4 + \frac{a^4}{\alpha}\sum_{\zeta=1}^{\infty}\frac{4^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)$$

$$\sigma^2 = \left[a^2 + \frac{a^2}{\alpha}\sum_{\zeta=1}^{\infty}\frac{2^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right] - \left[a + \frac{a}{\alpha}\sum_{\zeta=1}^{\infty}\frac{\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)}{A^{\zeta+1}\,\zeta!}\right]^2$$

Put $\alpha=\beta=1$ then $\mu = A(b-a)$ and $\sigma^2 = \frac{A(b-a)}{2}\left[(b+a)-2A(b-a)\right]$ which are the mean and the variance of the base reciprocal distribution **,** respectively.

3.6 Incomplete Moments

Theorem 3: If X is a r.v distributed as $KRF(a,b,\alpha,\beta)$, then r^{th} incomplete moments of X is given by

$$I(\mu'_r) = \psi(x;a,b) + \frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}I_y\left(\frac{\zeta+1}{\alpha},\beta+1\right),$$

Where

$$\psi(x;a,b) = a^r - x^r\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta},\ \ and\ \ y = A^{\alpha}\,Q(x,a)^{\alpha}.$$

Proof: r^{th} incomplete moments is defined as $I(\mu'_r) = \int_a^x x^r\ f(x)dx$, then

$$I(\mu'_r) = \int_a^x x^r\ f(x)dx = \beta\int_a^x \frac{\alpha\,x^r}{x}A^{\alpha}Q(x,a)^{\alpha-1}\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta-1}dx$$

By using the partial derivative

$$\int_a^x x^r\ f(x)dx = (-\beta)\left[\frac{x^r\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta}}{\beta}\Bigg|_a^x - \frac{r}{\beta}\int_a^x x^{r-1}\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta}dx\right] = \psi + \frac{r\,a^r}{\alpha\,A}\int_0^{\nabla} y^{\frac{1}{\alpha}-1}\left[1-y\right]^{\beta}e^{\frac{r\,y^{\frac{1}{\alpha}}}{A}}\,d\,y,\ \ y = A^{\alpha}\,Q(x,a)^{\alpha}$$

For $n,m>0$ integer, it follows that

$$\int_0^b w^{n-1}\left[1-w\right]^{m-1}e^{w}\,d\,w = \sum_{\zeta=1}^{\infty}\frac{I_b(n+\zeta,m)}{\zeta!} \rightarrow (7)$$

Where $I_b(a,b) = \int_0^{\hbar} w^{a-1}\left[1-w\right]^{b-1}d\,w$ denotes the complementary incomplete beta function, which can be evaluated in MathCad 15. Using Eq.(7), the incomplete moments can be expressed as $I(\mu'_r) = \psi(x;a,b) + \frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}I_y\left(\frac{\zeta+1}{\alpha},\beta+1\right)$.

3.7 Probability Weighted Moments

Theorem 4: The probability weighted moments ($\mu'_{r,s}$) of a r.v X distributed as $KRF(a,b,\alpha,\beta)$ is

$$\mu'_{r,s} = \frac{1}{s+1}\left[b^r - \frac{a^r}{\alpha}\sum_{i,j=0}^{\infty}\frac{(-1)^j\ r^{i+1}}{A^{i+1}\,i!\ j!}\binom{s+1}{j}\beta\left(\frac{i+1}{\alpha},j\beta+1\right)\right]$$

Proof: The probability weighted moments of a r.v X is defined a $\mu'_{r,s} = \int_a^b x^r\ f(x)\left[F(x)\right]^s dx$

Therefore,

$$\int_a^b x^r\ f(x)\left[F(x)\right]^s dx = \beta \int_a^b \frac{\alpha x^r}{x} A^{\alpha} Q(x,a)^{\alpha-1}\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta-1}\left[1-\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta}\right]^s dx$$

By using the partial derivative

$$\int_a^b x^r\ f(x)\left[F(x)\right]^s dx = \frac{1}{s+1}\left[b^r - r\int_a^b x^{r-1}\left\{1-\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta}\right\}^{s+1} dx\right]$$

Put $y = A^{\alpha}\ Q(x,a)^{\alpha}$ and using Eq.(5)

$$r\int_a^b x^{r-1}\left\{1-\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta}\right\}^{s+1} dx = \frac{a^r}{\alpha}\sum_{i,j=0}^{\infty}\frac{(-1)^j\ r^{i+1}}{A^{i+1}\ i!}\binom{s+1}{j}\beta\left(\frac{i+1}{\alpha}, j\beta+1\right)$$

Therefore,

$$\mu'_{r,s} = \frac{1}{s+1}\left[b^r - \frac{a^r}{\alpha}\sum_{i,j=0}^{\infty}\frac{(-1)^j\ r^{i+1}}{A^{i+1}\ i!}\binom{s+1}{j}\ \beta\left(\frac{i+1}{\alpha}, j\beta+1\right)\right].$$

3.8 Bonferroni and Lorenz Curves

Suppose that Y is a count response variable that follows the EWGD in Equation (2.2) and Y is associated with a set of predictors. We wish to fit the response variable Y by using the predictors. Suppose we have a $k-1$ row vector of predictors $x_i = (x_{i0}=1, x_{i1}, x_{i2}, \ldots, x_{i,(k-1)})$.

Theorem 5: If X is a r.v distributed as $KRF(a,b,\alpha,\beta)$, then the Bonferroni and Lorenz curves (BLC) of X is

$$B_F\left[F(x)\right] = \frac{1}{\mu F(x)}\left\{a - x\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta} + \frac{a}{\alpha}\sum_{\zeta=1}^{\infty}\frac{I_y\left(\frac{\zeta+1}{\alpha}, \beta+1\right)}{A^{\zeta+1}\ \zeta!}\right\}$$

Proof: The Bonferroni and Lorenz curves $B_F\left[F(x)\right]$ is defined by the following formula**:** $B_F\left[F(x)\right] = \frac{1}{\mu F(x)}\int_a^x w\ f(w)\,dw$. Therefore,

$$\frac{1}{\mu F(x)}\int_a^x w\ f(w)\,dw = \beta\int_a^x \frac{\alpha w}{w} A^{\alpha} Q(w,a)^{\alpha-1}\left[1-A^{\alpha}\ Q(w,a)^{\alpha}\right]^{\beta-1} d\,w$$

By using the partial derivative

$$\frac{1}{\mu F(x)}\int_a^x w\ f(w)\,dw = a - x\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta} + \int_a^x \left[1-A^{\alpha}\ Q(w,a)^{\alpha}\right]^{\beta} d\,w$$

Put $y = A^{\alpha}\ Q(x,a)^{\alpha}$, and using Eq.(7)

$$\int_a^x \left[1-A^{\alpha}\ Q(w,a)^{\alpha}\right]^{\beta} d\,w = \frac{a}{\alpha}\sum_{\zeta=1}^{\infty}\frac{I_y\left(\frac{\zeta+1}{\alpha}, \beta+1\right)}{A^{\zeta+1}\ \zeta!}.$$

Therefore

$$B_F\left[F(x)\right] = \frac{1}{\mu F(x)}\left\{a - x\left[1-A^{\alpha}\ Q(x,a)^{\alpha}\right]^{\beta} + \frac{a}{\alpha}\sum_{\zeta=1}^{\infty}\frac{I_y\left(\frac{\zeta+1}{\alpha}, \beta+1\right)}{A^{\zeta+1}\ \zeta!}\right\}.$$

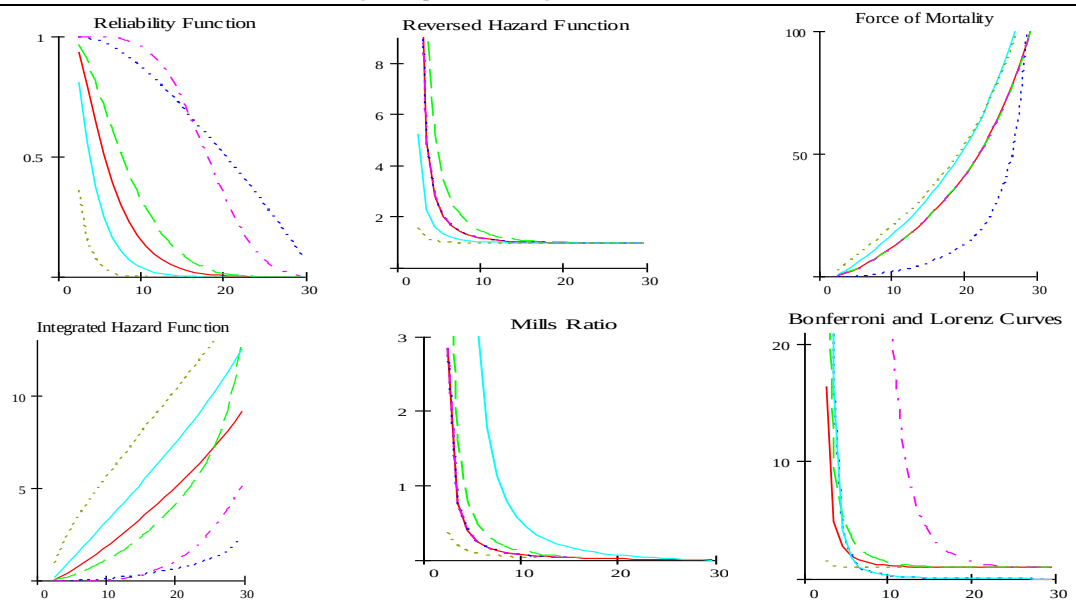

Figure 2: Reliability Measurements and Bonferroni and Lorenz Curves *of KRF(a,b,α,β)*

Figure.2 illustrate plot the RF, FM, RHF, IHF, MR and the BLC of the $KRF(a,b,\alpha,\beta)$ under different values of the shape parameters (α,β) and the scale parameters (a,b). The graphs show that, the integrated hazard function and the force of mortality are increasing functions, while the reliability function and mills ratio are decreasing as the time increasing. On the other hand, the reversed hazard function and the Bonferroni and Lorenz curves are beginning decreasing functions until reached the point $(x=1)$ where became constant functions.

3.9 Moment Generating Function

Theorem 6: The moment generating function of a r.v X distributed as $KRF(a,b,\alpha,\beta)$ is

$$m_X(t)=\sum_{r=1}^{\infty}\frac{t^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}.$$

Proof: The moment generating function of a r.v X is defined as $m_X(t)=\int_a^b e^{tx}\,f(x)dx$, then

$$\int_a^b e^{tx}\,f(x)dx=\beta\int_a^b\frac{\alpha\,e^{tx}}{x}A^{\alpha}Q(x,a)^{\alpha-1}\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta-1}dx$$

$$=\alpha\beta\sum_{r=1}^{\infty}\frac{t^r}{r!}\int_a^b x^{r-1}A^{\alpha}Q(x,a)^{\alpha-1}\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta-1}dx$$

$$=\sum_{r=1}^{\infty}\frac{t^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}$$

Therefore $m_X(t)=\sum_{r=1}^{\infty}\frac{t^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}$, and the Characteristic function of a r.v X distributed as $KRF(a,b,\alpha,\beta)$ is

$$\varphi_X(t)=\sum_{r=1}^{\infty}\frac{(it)^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}=m_X(it).$$

3.10 Laplace-Stieltjes Transform

Theorem 7: The Laplace-Stieltjes transform of a r.v X distributed as $KRF(a,b,\alpha,\beta)$ is given by

$$f^*(s)=\sum_{r=1}^{\infty}\frac{(-s)^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}.$$

Proof: Laplace-Stieltjes transform is defined by

$f^*(s)=\int_a^b e^{-sx}f(x)\,dx$. Then

$$\int_a^b e^{-sx}\,f(x)dx=\beta\int_a^b\frac{\alpha\,e^{-sx}}{x}A^{\alpha}Q(x,a)^{\alpha-1}\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta-1}dx=\alpha\beta\sum_{r=1}^{\infty}\frac{(-s)^r}{r!}\int_a^b x^{r-1}A^{\alpha}Q(x,a)^{\alpha-1}\left[1-A^{\alpha}\,Q(x,a)^{\alpha}\right]^{\beta-1}dx$$

$$=\sum_{r=1}^{\infty}\frac{(-s)^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}$$

Therefore $$f^*(s)=\sum_{r=1}^{\infty}\frac{(-s)^r}{r!}\left\{a^r+\frac{a^r}{\alpha}\sum_{\zeta=1}^{\infty}\frac{r^{\zeta+1}}{A^{\zeta+1}\,\zeta!}\beta\left(\frac{\zeta+1}{\alpha},\beta+1\right)\right\}=m_X(-s)$$.

3.11 Estimation of Parameters

Let $X_1,X_2,\ldots,X_n$ be a random sample of size n, distributed as $KRF(a,b,\alpha,\beta)$ with likelihood function

$$L=\left(\alpha\,\beta\,A^{\alpha}\right)^n\prod_{i-1}^{n}\left\{x_i^{-1}\,Q(x_i,a)^{\alpha-1}\left[1-A^{\alpha}\,Q(x_i,a)^{\alpha}\right]^{\beta}\right\}$$

Then the log likelihood function (l) is given by

$$l=n\ln(\alpha\beta)+n\alpha\ln A+\sum_{i=1}^{n}\left\{(\alpha-1)\ln Q(x_i,a)+(\beta-1)\ln\left[1-\varsigma(x_i,a,b)^{\alpha}\right]-\ln x_i\right\}\quad\rightarrow(8)$$

where, $\varsigma(x_i,a,b)=AQ(x_i,a)$. By taking the partial derivatives of Eq.8 with respect to the four unknown parameters (a,b,α,β), the four normal equations are obtained as

$$\frac{\partial l}{\partial a}=\frac{1}{a}\left\{n\alpha-A\sum_{i=1}^{n}\left\{(\alpha-1)Q(x_i,a)^{-1}+\alpha(\beta-1)A\left(1-\varsigma(x_i,a,b)^{-1}\right)\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-1`}\right\}\right\}$$

$$\frac{\partial l}{\partial b}=\frac{-1}{b}\left\{n\alpha\,A+\alpha(\beta-1)\sum_{i=1}^{n}A\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-1`}\right\}$$

$$\frac{\partial l}{\partial\alpha}=\left\{\frac{n}{\alpha}+n\ln A+\sum_{i=1}^{n}\left\{\ln Q(x_i,a)+(\beta-1)\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-1`}\ln\varsigma(x_i,a,b)\right\}\right\}$$

$$\frac{\partial l}{\partial\beta}=\left\{\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)\right\}$$

The solution of the above normal equations cannot be obtained in closed form, so MathCad 15 package can be used to get the MLE of the unknown parameters. Therefore, the elements of the Fisher information matrix (*FI*) for the MLE can be obtained as the expectations of the negative of the second partial derivatives, and the asymptotic variance-covariance matrix $v(\underline{\eta})$ for the MLE is defined as the inverse of the Fisher's information matrix i.e.,

$$FI=-E\begin{bmatrix} \frac{\partial^2 l}{\partial a^2} & \frac{\partial^2 l}{\partial a\partial b} & \frac{\partial^2 l}{\partial a\partial \alpha} & \frac{\partial^2 l}{\partial a\partial \beta} \\ & \frac{\partial^2 l}{\partial b^2} & \frac{\partial^2 l}{\partial b\partial \alpha} & \frac{\partial^2 l}{\partial b\partial \beta} \\ symetric & & \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha\partial \beta} \\ & & & \frac{\partial^2 l}{\partial \beta^2} \end{bmatrix}, \quad V(\eta)=\hat{F}^{-1} \quad \underline{\eta}=(a,b,\alpha,\beta)$$

Where,

$$\frac{\partial^2 l}{\partial a^2}=\frac{1}{a^2}\left\{n\alpha A(A-1)-\sum_{i=1}^{n}\left\{(\alpha-1)Q(x_i,a)^{-2}(1-Q(x_i,a))+\alpha A(\beta-1)\left(1-\varsigma(x_i,a,b)^{-1}\right)\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-2`}\right\}\right.$$
$$\left.\left[\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)\left[A\left(\varsigma(x_i,a,b)^{-1}+1\right)-1\right]-\alpha A\varsigma(x_i,a,b)^{-\alpha}\left(1-\varsigma(x_i,a,b)^{-1}\right)\right]\right\}$$

$$\frac{\partial^2 l}{\partial b^2}=\frac{1}{b^2}\left\{n\alpha A(A+1)+\sum_{i=1}^{n}\left\{\alpha A(\beta-1)\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-2`}\right\}.\left[\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)[A+1]-\alpha A\varsigma(x_i,a,b)^{-\alpha}\right]\right\}$$

$$\frac{\partial^2 l}{\partial \alpha^2}=\left\{\frac{-n}{\alpha^2}+(\beta-1)\sum_{i=1}^{n}\left\{\varsigma(x_i,a,b)^{-\alpha}\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-2`}(ln\,\varsigma(x_i,a,b))^2\right\}\right\}$$

$$\frac{\partial^2 l}{\partial \beta^2}=\frac{-n}{\beta^2}$$

$$\frac{\partial^2 l}{\partial a\partial b}=\frac{-1}{ab}\left\{n\alpha A^2+\alpha A^2(\beta-1)\sum_{i=1}^{n}\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-2`}\left\{1-\varsigma(x_i,a,b)^{-1}\left[1+\alpha\left(1-\varsigma(x_i,a,b)^{-1}\right)\right]\right\}\right\}$$

$$\frac{\partial^2 l}{\partial a\partial \alpha}=\frac{1}{a}\left\{nA+\sum_{i=1}^{n}\left[A(\beta-1)\left(1-\varsigma(x_i,a,b)^{-1}\right)\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-2`}\left\{1-\varsigma(x_i,a,b)^{-1}[1+\alpha\,ln\,\varsigma(x_i,a,b)]\right\}-Q(x_i,a)^{-1}\right]\right\}$$

$$\frac{\partial^2 l}{\partial a\partial \beta}=\frac{\alpha A}{a}\sum_{i=1}^{n}\left(1-\varsigma(x_i,a,b)^{-1}\right)\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-1`}$$

$$\frac{\partial^2 l}{\partial b\partial \alpha}=\frac{-A}{b}\left\{n+(\beta-1)\sum_{i=1}^{n}\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-2`}\left(1-\varsigma(x_i,a,b)^{-\alpha}(1-\alpha\,ln\,\varsigma(x_i,a,b))\right)\right\}$$

$$\frac{\partial^2 l}{\partial b\partial \beta}=\frac{-\alpha A}{b}\sum_{i=1}^{n}\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-1`}$$

$$\frac{\partial^2 l}{\partial \alpha\partial \beta}=\sum_{i=1}^{n}\left(1-\varsigma(x_i,a,b)^{-\alpha}\right)^{-1`}\,ln\,\varsigma(x_i,a,b)$$

.

4. Application

Suppose that Y is a count response variable that follows the EWGD in Equation (2.2) and Y is associated with a set of predictors. We wish to fit the response variable Y by using the predictors. Suppose we have a $k-1$ row vector of predictors $x_i=(x_{i0}=1,x_{i1},x_{i2},\ldots,x_{i,(k-1)})$.

In this section, four applications to real data sets are provided to illustrate the flexibility and the usefulness of the new distribution. The first data represents the rainfall of the city of Zlitan-Libya for the period from 1980 to 2009 from the records of the five meteorological center, the data set is: (0.122, 0.122, 0.139, 0.146, 0.168, 0.17, 0.178, 0.18, 0.183, 0.184, 0.192, 0.199, 0.208, 0.21, 0.212, 0.241). The second data of the air conditioning system of an airplane originally presented by Linhart and Zucchini (1986), the data set is: (0.0417, 0.0655, 0.0655, 0.0655, 0.0714, 0.0833, 0.0833, 0.0833, 0.0952, 0.0952,0.125 , 0.1190, 0.1369, 0.2500, 0.2798, 0.3095, 0.3690, 0.4226, 0.4226, 0.5179, 0.5655, 0.5357, 0.7143 , 0.7143) is the failure times (week) of the air conditioning system of an airplane.

The third data of the Survival time of infected with virulent tubercle bacilli was reported by Haq 2016, the data set is: (1.02, 1.05, 1.07, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.20,1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.60, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96,1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.30, 2.31, 2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02,4.32, 4.58, 5.55). The fourth data set was provided by Ed Fuller of the NICT Ceramics Division in December 1993. It contains polished window strength data (*proprotions*), the maximum value is 45.38, the data set is: (0.41, 0.46, 0.48, 0.51, 0.51, 0.53, 0.54, 0.56, 0.56 , 0.57 0.59, 0.59, 0.59, 0.6, 0.61, 0.66, 0.69, 0.73, 0.74, 0.74, 0.75, 0.77, 0.79, 0.79, 0.81, 0.82, 0.82, 0.87, 0.97, 1.00).

Some measures of goodness of fit are used, including Kolomgrof-Smirnof test (***K_S***), Akaike information criterion (***AIC***), consistent Akaike information criterion (***CAIC***), Bayesian information criterion (***BIC***), Anderson Darling (***AD***) and Cramrvon Mises (***CVM***) statistics for assessing that the data sets follow the $^{KRF(a,b,\alpha,\beta)}$ and they are:

$$AIC = -2L + 2K,\ BIC = -2L + K\log(n),\ CAIC = AIC + \frac{2K(K+1)}{n-K-1},$$

$$AD = -n - \left(\frac{1}{n}\right)\sum_{i=1}^{n}\left\{(2i-1)\left[\log\left(F_i(x)\right) + \log\left(F_{n-i+1}(x)\right)\right]\right\},\ CVM = \frac{1}{12n} + \sum_{i=1}^{n}\left\{F_i(x) - \frac{2i-1}{2n}\right\}^2$$

where n is the sample size, k is the number of parameters of each statistical model, F(x) is the CDF of each statistical model and L is the maximized value of the log-likelihood function under the considered model. The values of the *K_S*, *AIC*, *CAIC*, *BIC*, *AD* and *CVM* are represented in Table.2 for the new proposed model compared with the other sub models: Reciprocal (***RD***), Power Function (***PFD***), exponential (***ExpD***), Weibul (***WD***), Rayleigh (***RayD***) and Pareto (***PD***) distribution. Moreover, ML estimates with confidence intervals for (a,b,α,β) based on the data sets are calculated. Mean Square Error (MSE) with variance of these estimates are represented in Table.3. Also, moments about origin, incomplete moments, probability weighted moments, Laplace-Stieltjes transform and mean residual life at the mission time ***t*** of the 4-kumaraswamy reciprocal family are calculated and represented in Table.4. In addition, mean, median, 1st quartile, 3rd quartile, bowley skewness (BS), moors kurtosis (MK) and mean residual life (MRL) of the four kumaraswamy reciprocal family are calculated and represented in Table.5.

Table 2 : The ML Estimates, *AD, CVM, CAIC, BIC, AIC, K_S* for the four Data Sets

DATA I: $a = 0.01(0.015), b = 0.08(0.326), \alpha = 0.41(1.343), B = 0.90(0.384)$						
Model	*AD*	*CVM*	*CAIC*	*BIC*	*AIC*	*K_S*
KRD	7.088-	0.938	5.110E+9-	5.110E+9-	5.110E+9-	0.389
RD	4.498-	2.256	4.1810E+4-	4.1810E+4-	4.1810E+4-	0.616
PFD	6.898−	1.029	10.503−	11.585−	−10.789	0.421
ExpD	2.698	4.242	2.286	1.204	2.000	0.912
WD	4.570	4.499	4.923	2.408	4.000	0.933
RayD	11.006	4.995	2.286	1.204	2.000	0.968
PD	4.827-	2.128	4.923	2.408	4.000	0.645
DATA II: $a = 0.01(0.183), b = 0.224(1.07), \alpha = 1.35(1.922), B = 0.7(0.774)$						
Model	*AD*	*CVM*	*CAIC*	*BIC*	*AIC*	*K_S*
KRD	15.641-	0.523	1.088	2.603	0.512	0.115
RD	14.655−	35.306	12.935	11.444	12.49	2.156
PFD	12.346−	2.419	11.74	11.074	11.597	0.482
ExpD	13.242−	1.786	48.578	47.913	48.436	0.461
WD	9.237−	3.905	62.906	61.416	62.462	0.669
RayD	2.874−	6.141	113.806	113.141	113.664	0.763
PD	3.631−	6.018	113.46	111.97	113.015	0.77
DATA III: $a = 0.011(0.025), b = 0.08(0.164), \alpha = 0.91(1.323), B = 0.091(0.526)$						
Model	*AD*	*CVM*	*CAIC*	*BIC*	*AIC*	*K_S*
KRD	29.1149-	1.2508	7.1375	5.3917	6.3682	0.1489
RD	26.0807−	2.963	50.546	49.8355	50.3237	0.4631
PFD	−16.8794	6.9113	31.8239	31.5071	31.7512	0.5678
ExpD	28.4839−	1.5806	11.6515	11.3346	11.5787	0.2751
WD	19.3828−	6.4188	62.9427	62.2322	62.7205	0.544

RayD	28.2062	16.5944	323.0255	322.7087	322.9528	0.8638
PD	15.0782	15.5837	191.7351	191.0247	191.5129	0.877
DATA IV: $a = 0.1(0.183), b = 0.224(1.07), \alpha = 1.35(1.922), B = 0.7(0.774)$						
Model	*AD*	*CVM*	*CAIC*	*BIC*	*AIC*	*K_S*
KRD	15.641−	0.523	1.088	2.603−	0.512−	0.213
RD	15.001−	35.306	4.416	2.926	3.971	2.156
PFD	12.346−	2.419	2.126	1.461	1.984	0.482
ExpD	13.242−	1.786	2.143	1.477	2.000	0.461
WD	9.237−	3.905	4.444	2.954	4.000	0.669
RayD	2.874−	6.141	2.143	1.477	2.000	0.763
PD	3.631−	6.018	4.444	2.954	4.000	0.770

The results in Table.2 indicates that, the values of the *K-S* is very small for the $KRF(a,b,\alpha,\beta)$ distribution, and according to the other criteria *AIC*, *CAIC*, *BIC*, *AD* and *CVM*, it is noted that the $KRF(a,b,\alpha,\beta)$ yields a better fit to all the four data sets than the other distributions. So, the $KRF(a,b,\alpha,\beta)$ could be chosen as the best model for the four data sets.

Table 3: MLE, MSE and C.I of the unknown parameters of the $KRF(a,b,\alpha,\beta)$

DATA I: Rainfall of the city of Zlitan						
Para.	*Init. values*	*MLE*	*MSE*	*Variance*	*LBCI*	*UBCI*
a	0.015	0.01818	1.014E-5	1.255E-5	0.01816	0.01821
b	0.1	0.100004	2.14E-12	2.535E-5	0.09995	0.10005
α	4.75	4.65942	8.205E-3	0.54564	3.58996	5.72888
β	1.002	1.00444	5.976E-6	0.01283	0.97931	1.02958
DATA II: Air conditioning system of an airplane						
Para.	*Init. values*	*MLE*	*MSE*	*Variance*	*LBCI*	*UBCI*
a	0.0101	0.0102	110E-8	2.7517E-	0.01015	0.01025
b	0.2	0.02001	0.0324	0.02191	0.02290	0.06296
α	0.625	0.6334	7.05E-5	0.01161	0.61064	0.65616
β	0.4	0.4159	2.52E-4	5.845E-3	0.40444	0.42736
DATA III: Survival time of infected with virulent tubercle bacilli						
Para.	*Init. values*	*MLE*	*MSE*	*Variance*	*LBCI*	*UBCI*
a	0.211	0.21448	1.209E-5	4.925E-3	0.20483	0.22413
b	0.335	0.33521	4.387E-8	8.609E-3	0.31833	0.35208
α	0.4	0.39456	2.958E-5	2.875E-3	0.38893	0.4002
β	0.5	0.48743	1.578E-4	4.529E-3	0.47856	0.49631
DATA IV: Polished window strength data						
Para.	*Init. values*	*MLE*	*MSE*	*Variance*	*LBCI*	*UBCI*
a	0.113	0.21155	9.711E-3	1.383E-3	0.20884	0.21426
b	0.219	0.46465	0.06034	1.539E-3	0.46163	0.46767
α	1.75	1.83947	8.005E-3	0.11677	1.6106	2.06834
β	0.739	0.75661	3.099E-4	0.01299	0.73114	0.78207

Table 5: Measurements of central tendency and dispersion of $KRF(a,b,\alpha,\beta)$

DATA I: Rainfall of the city of Zlitan-Libya		*DATA II: Air conditioning system of an airplane*		*DATA III: Survival time of infected with virulent tubercle bacilli*		*DATA IV: Polished window strength data*	
measue	*Value*	*measure*	*Value*	*measue*	*Value*	*measue*	*Value*
Mean	0.0761	*Mean*	0.09729	*Mean*	0.27111	*Mean*	0.37366
$Q(0.25)$	0.06442	$Q(0.25)$	0.02758	$Q(0.25)$	0.2272	$Q(0.25)$	0.32224
$Q(0.5)$	0.07893	$Q(0.5)$	0.08658	$Q(0.5)$	0.26775	$Q(0.5)$	0.38393
$Q(0.75)$	0.09023	$Q(0.75)$	0.16948	$Q(0.75)$	0.31476	$Q(0.75)$	0.43273
Variance	2.83E-4	*Variance*	4.82E-3	*Variance*	1.605E-3	*Variance*	4.44E-3
Skewnes	-0.5971	*Skewness*	0.06296	*Skewness*	-2.06955	*Skewness*	-0.4428
Bowley Skewness	-0.1243	*Bowley Skewness*	0.16838	*Bowley Skewness*	0.07386	*Bowley Skewness*	-0.1167
kurtosis	2.52304	*kurtosis*	1.3105	*kurtosis*	95.51749	*kurtosis*	1.83106
Moors Kurtosis	1.10851	*Moors Kurtosis*	0.70378	*Moors Kurtosis*	0.74458	*Moors Kurtosis*	1.02842

Based on the simulation study, can be concluded that the maximum likelihood estimators are appropriate for estimating the $KRF(a,b,\alpha,\beta)$. All the real data provides a better fit to the $KRF(a,b,\alpha,\beta)$ and estimate the true parameters (a,b,α,β) well with relatively small *MSE* and reduce towards zero, with short confidence intervals. Moreover, the mean residual life

decreases when the mission time (t) increases as shown from the results in Table.3. In general, the proposed model and the asymptotic approximation work well under the situation and provides a better fit for the real data. So, the $^{KRF(a,b,\alpha,\beta)}$ very useful in many fields and has many benefits especially in practical life. Therefore, it can be applied to several realistic data and chosen as a suitable model for lifetime data.

5. Conclusions

This paper presented a new distribution named the four parameters kumaraswamy reciprocal family abbreviate by $^{KRF(a,b,\alpha,\beta)}$. The maximum likelihood method is applied for estimating the model parameters. Nine special models namely, reciprocal, beta, uniform, power function, exponential**,** negative exponential, weibull, rayleigh and pareto distribution are provided. Further, the derived properties of the $^{KRF(a,b,\alpha,\beta)}$ are valid to the selected model, such as, quantile function, median, random numbers, bowley skewness, moors kurtosis, reliability function, force of mortality, reversed hazard function, integrated hazard function, mills ratio, mean residual life, moments about origin, incomplete moments, probability weighted moments, bonferroni and lorenz curves, moment generating function, characteristic function, laplace-stieltjes transform with estimation of parameters. It also explained the behavior of the reliability function, force of mortality, mills ratio, reversed hazard function, integrated hazard function, bonferroni and lorenz curves. Four real life applications have also presented for explaining the better fit and the benefit of the observed model in many practical life. The better fit and the benefit of the new model is illustrated by four real data sets and the results of the applications nicely exhibit the fact that the $^{KRF(a,b,\alpha,\beta)}$ provides a better fit than others sub models in many. And it performs better and a suitable in many situations and can be applied to several realistic data.

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