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Extended Poisson-Fréchet Distribution: Mathematical Properties and Applications to Survival and Repair Times

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Abstract

In this paper, a new four parameter zero truncated Poisson Fréchet distribution is defined and studied. Various structural mathematical properties of the proposed model including ordinary moments, incomplete moments, generating functions, order statistics, residual and reversed residual life functions are investigated. The maximum likelihood method is used to estimate the model parameters. We assess the performance of the maximum likelihood method by means of a numerical simulation study. The new distribution is applied for modeling two real data sets to illustrate empirically its flexibility.

Keywords: Zero Truncated Poisson Distribution; Fréchet Distribution; Maximum Likelihood Estimation; Simulation; Generating Function; Moments; Order Statistics.

1. Genesis, physical motivation and justification

Assume that X_1, X_2, ... , X_n is a finite sequence of independent and identically distributed random variables (iid rvs) with common cumulative distribution function (CDF). One of the most interesting statistics is the sample maximum

$$M_n = X = \max_{i=1}^{N}\{X_i\}.$$

One is interested in the behavior of M_n as the sample size n increases to infinity, then

$$\begin{aligned}\Pr\{M_n \leq x\} &= \Pr\{X_1 \leq x, ..., X_n \leq x,\} \\ &= \Pr\{X_1 \leq x\} ...p_r\{X_n \leq x\} \\ &= G(x)^n .\end{aligned}$$

Suppose there are sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\Pr\left\{\left[(M_n - b_n)\, a_n^{-1}\right] \leq x\right\} \rightarrow G(x)\,|_{(n\rightarrow\infty)}\ .$$

Then, if $G(x)$ is a non-degenerate CDF, then it will belong to one of the three following fundamental types of classic extreme value family:

1-Gumbel (Gum) model (Type **I** extreme value distribution);

2-Fréchet (Fr) model (Type **II** extreme value distribution);

3-Weibull (W) model (Type **III** extreme value distribution).

The extreme value theory focuses on the behavior of the block maxima or minima. The extreme value theory was firstly introduced by Fréchet (1927) then followed by Von Mises (1936), Gnedenko (1943), Von Mises (1964), Kotz and Johnson (1992), among others. The Fr distribution is one of the important distributions in extreme value theory and has many applications such as accelerated life testing, earthquakes, floods, horse racing, rainfall, queues in supermarkets, wind speeds and sea waves. For more details about the Fr distribution and its applications, see Kotz and Nadarajah (2000). Moreover, applications of this distribution in various fields are given in Harlow (2002). Recently, some extensions of the Fr distribution were considered. The exponentiated Fr by Nadarajah and Kotz (2003), beta Fr by Nadarajah and Gupta (2004), Nadarajah and Kotz (2008) and Zaharim et al. (2009), beta Fr (Barreto-Souza et al. (2011) and Mubarak (2013)) , Marshall-Olkin Fr (Krishna et.al. (2013)), transmuted Fr (Mahmoud and Mandouh (2013)), gamma extended Fr (da Silva et al. (2013)), transmuted exponentiated Fr (Elbatal et al. (2014)), transmuted Marshall-Olkin Fr (Afify et al. (2015)), transmuted exponentiated generalized Fr (Yousof et al. (2015)), beta exponential Fr (Mead et al. (2016)), Weibull Fr (Afify et al. (2016b)), Kumaraswamy Marshall-Olkin Fr (Afify et al. (2016b)), Kumaraswamy transmuted Marshall-Olkin Fr (Yousof et al. (2016)), beta transmuted Fr by Afify et al. (2016c), odd Lindley Fréchet distribution the (Korkmaz et al. (2017)), Transmuted Topp-Leone Fr (Yousof et al. (2017)), Topp Leone Generated Fr (Yousof et al. (2018b)) and Odd log-logistic Féchet (Yousof et al. (2018a)), among others.

The goal of this paper is to propose a new generalization of the Topp Leone Fr (TL-Fr) distribution (Yousof et al. (2018b)) using the zero-truncated Poisson (ZTP) model. the probability density function (PDF) and CDF of TL-Fr distribution are given by

$$\begin{aligned} g(x) &= 2\theta\beta\delta^{\beta}x^{-(\beta+1)}\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\exp\left[-(\theta-1)\left(\delta x^{-1}\right)^{\beta}\right] \\ &\quad\times\left\{1-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}\left\{2-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}^{\theta-1},\ 1 \end{aligned} \tag{1}$$

and

$$G(x) = \left(\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\left\{2-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}\right)^{\theta}, \tag{2}$$

respectively, where $\delta > 0$ is a scale parameter and $\beta, \theta > 0$ is a shape parameter. The ZTP distribution is a discrete probability model whose support is the set of only the positive integers $(\mathbf{I}^{(+)})$ with probability mass function (PMF) of N given by

$$P\left(N = n|_{n\in\mathbf{I}^{(+)}}\right) = \frac{\exp(-\alpha)\alpha^{n}}{\Delta_{(\alpha)}\Gamma(1+n)}. \tag{3}$$

Suppose that a system has N subsystems functioning independently at a given time where N follows the ZTP distribution with parameter α. The expected value ($\mathbf{E}\left(N|\alpha\right)$) and variance ($\mathbf{V}(N|\alpha)$) are, respectively, given by

$$\mathbf{E}\left(N|_{\alpha,n\in\mathbf{I}^{(+)}}\right)=\alpha\Delta_{(\alpha)}^{-1},$$

where $\Delta_{(\alpha)}=1-\exp\left(-\alpha\right)$and

$$\mathbf{V}(N|_{\alpha,n\in\mathbf{I}^{(+)}})=\alpha\left(1+\alpha\right)\Delta_{(\alpha)}^{-1}-\alpha^{2}\Delta_{(\alpha)}^{-2}=\mathbf{E}\left(N|\alpha\right)\left[1+\alpha-\mathbf{E}\left(N|\alpha\right)\right].$$

The ZTP is known also as the positive Poisson distribution or the conditional Poisson distribution. It is the conditional probability distribution of a Poisson distributed rv, given that the value of the rv is not zero. Thus it is impossible for a ZTP rv to be zero.

Suppose that the failure time of each subsystem has the TL-Fr model defned by PDF and CDF in (1) and (2). Let Y_i denote the failure time of the $i^{(th)}$ subsystem and let

$$X=\min_{i=1}^{N}\{Y_i\}\text{ or }X=\max_{i=1}^{N}\{Y_i\},$$

then the conditional CDF of X given N can be written as

$$F\left(x\mid N\right)=1-\Pr\left(X>x\mid N\right)=1-\Pr\left(Y_1>x\right)^{N}=1-\left(1-G_{TL-Fr}^{(\theta,\beta,\delta)}\left(x\right)\right)^{N},\qquad(4)$$

therefore, the marginal CDF of X can be expressed as

$$F\left(x\right)|_{(\alpha\in\mathbf{R})}=\frac{1-\exp\left\{-\alpha\exp\left[-\theta\left(\delta x^{-1}\right)^{\beta}\right]\left\{2-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}^{\theta}\right\}}{\Delta_{(\alpha)}},\qquad(5)$$

equation (5) is called the CDF of the zero truncated Poisson Topp Leone Fr (ZTPTL-Fr) model. The corresponding PDF of (5) reduces to

$$\begin{aligned}f\left(x\right)|_{(\alpha\in\mathbf{R})}&=\frac{2\alpha\theta\beta\delta^{\beta}}{\Delta_{(\alpha)}x^{\beta+1}}\exp\left[-\theta\left(\delta x^{-1}\right)^{\beta}\right]\\&\times\left\{1-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}\left\{2-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}^{\theta-1}\\&\underbrace{\times\exp\left(-\alpha\exp\left[-\theta\left(\delta x^{-1}\right)^{\beta}\right]\left\{2-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}^{\theta}\right)}_{A(x)},6\qquad(2)\end{aligned}$$

Then we provide a linear mixture for the ZTPTL-Fr density function in (6). Expanding the quantity $A\left(x\right)$ in power series, we can write

$$A\left(x\right)=_{\tau=0}^{\infty}\frac{\left(-1\right)^{\tau}\alpha^{\tau}\left\{2-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}^{\theta\tau}}{\tau!\exp\left[\theta\tau\left(\delta x^{-1}\right)^{\beta}\right]},$$

then

$$f(x) = \sum_{\tau=0}^{\infty} \frac{(-1)^{\tau}\, \alpha^{\tau+1} \theta 2^{(1+\tau)\theta} \beta \delta^{\beta}}{\tau! \left(1-\Delta_{(\alpha)}\right) x^{\beta+1}} \exp\left[-\left[(1+\tau)\theta\right]\left(\delta x^{-1}\right)^{\beta}\right] \times \left\{1-\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}\left\{1-2^{-1}\exp\left[-\left(\delta x^{-1}\right)^{\beta}\right]\right\}, 7 \tag{3}$$

consider the power series

$$\left(1-\frac{a_1}{a_2}\right)^{a_3} = \sum_{m=0}^{\infty} \frac{\Gamma(1+a_3)}{m!\,\Gamma(1+a_3-m)} \left(-\frac{a_1}{a_2}\right)^{m}, \tag{8}$$

which holds for $\left|\frac{a_1}{a_2}\right| < 1$ and $q > 0$ real non-integer. Using the power series in (8) and after some algebra the PDF of the ZTPTL-Fr in (7) can be expressed as

$$f(x) = \sum_{\tau,\kappa=0}^{\infty} \begin{bmatrix} \mathbf{c}_{\tau,\kappa}\ \pi_{(1+\tau)\theta+\kappa}(x;\beta,\delta) \\ -\mathbf{c}^{\bigstar}_{\tau,\kappa}\ \pi_{1+\kappa+(1+\tau)\theta}(x;\beta,\delta) \end{bmatrix}, \tag{9}$$

where

$$\mathbf{c}_{\tau,\kappa} = \frac{\theta \alpha^{\tau+1} (-1)^{\tau+\kappa}\, 2^{(1+\tau)\theta-\kappa}}{\tau! \Delta_{(\alpha)} \left[(1+\tau)\theta+\kappa\right]} \binom{-1+(1+\tau)\theta}{\kappa},$$

$$\mathbf{c}^{\bigstar}_{\tau,\kappa} = \frac{\theta \alpha^{\tau+1} (-1)^{\tau+\kappa}\, 2^{(1+\tau)\theta-\kappa}}{\tau! \Delta_{(\alpha)} \left[1+\kappa+(1+\tau)\theta\right]} \binom{-1+(1+\tau)\theta}{\kappa},$$

the function $\pi_{(1+\tau)\theta+\kappa}(x;\beta,\delta)$ is the Fr density with scale parameter $\delta\left[(1+\tau)\theta+\kappa\right]^{\frac{1}{\beta}}$ and shape parameter β and $\pi_{1+\kappa+(1+\tau)\theta}(x;\beta,\delta)$ is the Fr density with scale parameter $\delta\left[1+\kappa+(1+\tau)\theta\right]^{\frac{1}{\beta}}$ and shape parameter β. Equation (9) reveals that the density of X can be expressed as a double linear mixture of Fr densities. So, several of its structural properties can be obtained from Equation (9) and those properties of the Fr distribution. By integrating (9), we obtain the same mixture representation

$$F(x) = \sum_{\tau,\kappa=0}^{\infty} \begin{bmatrix} \mathbf{c}_{\tau,\kappa}\ \Pi_{(1+\tau)\theta+\kappa}(x;\beta,\delta) \\ -\mathbf{c}^{\bigstar}_{\tau,\kappa}\ \Pi_{1+\kappa+(1+\tau)\theta}(x;\beta,\delta) \end{bmatrix}, \tag{10}$$

where $\Pi_{(1+\tau)\theta+\kappa}(x;\beta,\delta)$ is the CDF of the Fr model with scale parameter $\delta\left[(1+\tau)\theta+\kappa\right]^{\frac{1}{\beta}}$ and shape parameter β and $\Pi_{1+\kappa+(1+\tau)\theta}(x;\beta,\delta)$ is the CDF of the Fr model with scale parameter $\delta\left[1+\kappa+(1+\tau)\theta\right]^{\frac{1}{\beta}}$ and shape parameter β. The hazard rate function (HRF) can be derived as $f(x)/\left[1-F(x)\right]$. Figure 1 gives some plots of the ZTPTL-Fr PDF and HRF for some parameter values.

The justification for the practicality of the ZTPTL-Fr lifetime model is based on the wider use of the Fr model. As well as we are motivated to introduce the ZTPTL-Fr lifetime model because it exhibits a unimodal hazard rate as illustrated in Figure 1(b). It is shown in above that the ZTPTL-Fr lifetime model can be viewed as a double linear mixture of the Fr distributions. It can be viewed as a suitable model for fitting the unimodal and right skewed data. The ZTPTL-Fr model provide adequate fits as compared to other Fr models in both applications with small values for **AIC** and **BIC**. The proposed ZTPTL-Fr model is much better than the Kumaraswamy-Marshall–Olkin Fr, Marshall–Olkin Kumaraswamy Fr, Marshall–Olkin Fr, Kumaraswamy Fr, beta Fr, Marshall–Olkin inverse exponential, Marshall–Olkin inverse Rayleigh,

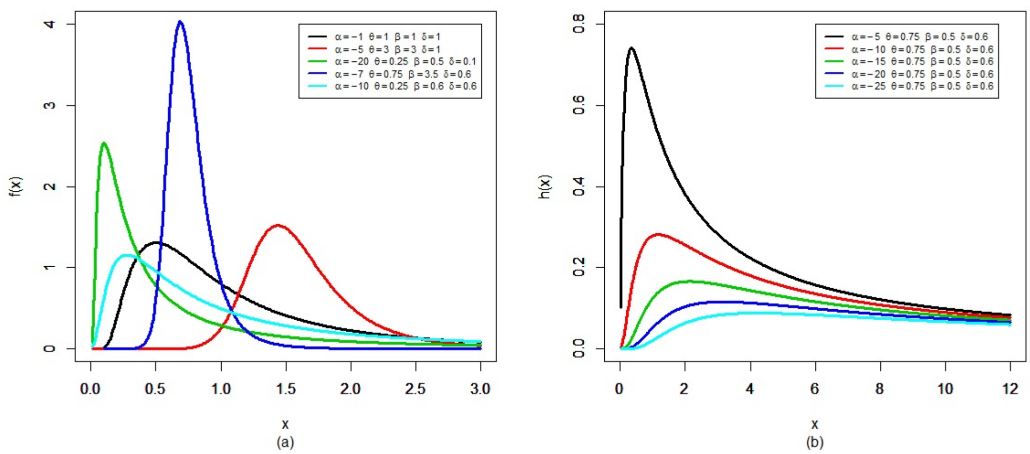

Figure 1: Plots of the ZTPTL-Fr PDF and HRF for some parameter values.

exponentiated Fr and Fr models, so the ZTPTL-Fr model is a suitable alternative to these models for modeling survival times data as illustrated in application 1. As well as the proposed ZTPTL-Fr lifetime model is much better than the Topp Leone Generated Fr, Fr, Kumaraswamy Fr, exponentiated Fr, beta Fr, Transmuted Fr, Marshall–Olkin Fr and Mcdonald Fr models, so the ZTPTL-Fr model is a suitable alternative to these models for modeling repair times data as illustrated in application 2.

The rest of the paper is outlined as follows. In Section 2, we derive some statistical properties for the new model. Maximum likelihood estimation of the model parameters is addressed in Section 3. Simulation results are presented in Section 4. Two applications to real data sets to illustrate the importance of the new family are provided in Section 5. Finally, we offer some concluding remarks in Section 6.

2. Mathematical properties

2.1 Moments and incomplete moments

The $r^{(th)}$ ordinary moment of X is given by

$$\mu_r' = \mathbf{E}(X^r) = \int_{-\infty}^{\infty} x^r \, f(x) \, dx,$$

then we obtain

$$\mu_r' = \delta^r \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{\tau,\kappa=0}^{\infty} \left[\mathbf{c}_{\tau,\kappa}^{(1)} - \mathbf{c}_{\tau,\kappa}^{(2)}\right] |_{(\beta>r)}, \tag{11}$$

where

$$\mathbf{c}_{\tau,\kappa}^{(1)}=\frac{\mathbf{c}_{\tau,\kappa}}{\left[(1+\tau)\,\theta+\kappa\right]^{-\frac{r}{\beta}}},$$

and

$$\mathbf{c}_{\tau,\kappa}^{(2)}=\frac{\mathbf{c}_{\tau,\kappa}^{\bigstar}}{\left[1+\kappa+(1+\tau)\,\theta\right]^{-\frac{r}{\beta}}}.$$

The constants $c_{\tau,\kappa}$ and $c_{\tau,\kappa}^{\bigstar}$ have been defined before in Section 1, and

$$\Gamma\left(1+\varphi\right)|_{(\varphi\in\mathbb{R}^{+})}=\int_{0}^{\infty}x^{\varphi}\,\exp\left(-t\right)dx=\varphi!=_{w=0}^{\varphi-1}\left(\varphi-w\right).$$

Setting $r=1$ in (11), we have the mean of X (μ'). The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The $r^{(th)}$ incomplete moment, say $\Upsilon_{r}\left(t\right)$, of X can be expressed from (9) as

$$\Upsilon_{r}\left(t\right)=\int_{-\infty}^{t}x^{r}f\left(x\right)dx,$$

then

$$\Upsilon_{r}\left(t\right)=\delta^{r}\sum_{\tau,\kappa=0}^{\infty}\left[\begin{array}{c}\mathbf{c}_{\tau,\kappa}^{(1)}\gamma\left(1-\frac{r}{\beta},\left[(1+\tau)\,\theta+\kappa\right]\left(\frac{\delta}{t}\right)^{\beta}\right)\\-\mathbf{c}_{\tau,\kappa}^{(2)}\gamma\left(1-\frac{r}{\beta},\left[1+\kappa+(1+\tau)\,\theta\right]\left(\frac{\delta}{t}\right)^{\beta}\right)\end{array}\right]|_{(\beta>r)},\tag{12}$$

where

$$\begin{aligned}\gamma\left(\zeta,z\right)&=\int_{0}^{z}t^{\zeta-1}\exp\left(-t\right)dt\\&=\frac{z^{\zeta}}{\zeta}\left\{{}_{1}\mathbf{F}_{1}\left[\zeta;\zeta+1;-z\right]\right\}\\&=\sum_{\nu=0}^{\infty}\frac{\left(-1\right)^{\nu}}{\nu!\left(\zeta+\nu\right)}z^{\zeta+\nu}|_{(\zeta\neq0.-1,-2,\ldots)},\end{aligned}$$

where ${}_{1}\mathbf{F}_{1}\left[\cdot\right]$ is a confluent hypergeometric function, which can be evaluated by statistical software like R software.

2.2 Numerical analysis for the μ', variance ($\mathbf{V}(X)$), skewness ($\mathbf{Sk}(X)$) and kurtosis ($\mathbf{Ku}(X)$) measures

Numerical analysis for the μ', $\mathbf{V}(X)$, $\mathbf{Sk}(X)$ and $\mathbf{Ku}(X)$ are listed in Tables 1 and 2 for the ZTPTL-Fr model and for the Fr model respectively for some selected values of parameter α,θ,β and δ using the R software. Based on Table 1 we note that:

1- The $\mathbf{Sk}(X)$ of the ZTPTL-Fr model is always positive.

2- The $\mathbf{Ku}(X)$ of the ZTPTL-Fr model can be more than 3 or less than 3.

Based on Tables 1 and 2 we note that: The skewness of the ZTPTL-Fr distribution can range in the interval (1.704, 99.032), whereas the skewness of the Fr distribution varies only in the interval (1.001, 3.53). Further, the spread for the ZTPTL-Fr kurtosis is ranging from 0.625235 to 148485.5, whereas the spread for the Fr kurtosis only varies from 1.002 to 98.8 with the above parameter values.

Table 1: $\mathbf{E}(X)$, $\mathbf{V}(X)$, $\mathbf{Sk}(X)$ and $\mathbf{Ku}(X)$ of the ZTPTL-Fr model.

δ	β	θ	α	$\mathbf{E}(X)$	$\mathbf{V}(X)$	$\mathbf{Sk}(X)$	$\mathbf{Ku}(X)$
1	1.5	1.5	-3	2.571567	0.9743642	**99.03197**	**148485.5**
5				12.85782	24.21726	75.966	29355.88
20				51.43077	379.0067	57.35025	7170.361
50				128.5695	2264.166	46.6469	2813.691
200				513.7455	28255.84	37.25156	745.9437
500				1276.935	89460.14	56.69273	511.4524
100	0.5	1.75	-10	2375.807	4580343	1.541674	4.744495
	1			738.8926	453222.2	5.88400	48.87404
	1.25			496.6668	119304	9.412066	130.6124
	1.5			381.6909	32180.45	15.22058	310.0421
	1.75			318.3019	7494.697	37.44547	995.5315
10	0.5	1	5	6.333456	3333.628	86.60225	10002.62
		5		37.02739	17198.3	37.36708	1887.333
		10		74.76474	35530.41	25.50535	889.1923
		50		358.0939	210585.3	9.633289	131.465
		100		689.3368	473059.3	6.105031	53.45829
		200		1303.943	1057407	3.793141	21.14257
		500		2820.337	2758617	1.717923	5.869974
		1000		4204.399	5754855	0.1825467	**0.625235**
5	0.25	10	-10	932.3464	4975511	2.469925	8.055438
			-5	1548.428	6149727	**1.703678**	4.898315
			5	272.3675	525808.9	7.220045	67.53774
			20	31.23634	469.4178	21.24751	2997.731
			30	21.80845	63.51442	18.39078	327.8126

Table 2: $\mathbf{E}(X)$, $\mathbf{V}(X)$, $\mathbf{Sk}(X)$ and $\mathbf{Ku}(X)$ of the Fr model.

δ	β	$\mathbf{E}(X)$	$\mathbf{V}(X)$	$\mathbf{Sk}(X)$	$\mathbf{Ku}(X)$
0.5	5	0.5821149	0.0334404	3.535071	48.0915
	10	0.5343144	0.0055656	1.910339	10.9774
	25	0.5123659	0.0007351	1.400443	6.85310
	50	0.5059737	0.0001736	1.264099	6.04447
2.5	5	2.910574	0.8360089	3.535072	48.0915
	10	2.671572	0.1391401	1.910339	10.97857
	50	2.529868	0.0043398	1.264099	6.045233
	75	2.519686	0.0018938	1.221761	5.760403
4.5	5	5.239034	2.70867	3.53507	48.0915
5	4.5	5.950756	4.60640	4.23885	98.8016
10	7.5	10.97054	4.47131	2.29491	15.5896
50	50	$1.392\times\mathrm{e}^{-6}$	2561.83	1.00104	1.0028
60	20	61.8872	17.03792	1.473884	7.33349
60	50	60.71684	2.499703	1.2641	6.04521

2.3 Moment generating function

The moment generating function (MGF) $M_X(t) = \mathbf{E}(\exp(t\,X))$ of X can be derived from equation (9) as

$$M_X(t) = \delta^r \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{\tau,\kappa,r=0}^{\infty} \left[m^{(1)}_{\tau,\kappa,r} - m^{(2)}_{\tau,\kappa,r}\right] |_{(\beta>r)},$$

where

$$m^{(1)}_{\tau,\kappa,r} = \frac{t^r}{r!}\mathbf{c}_{\tau,\kappa}\left[(1+\tau)\,\theta + \kappa\right]^{\frac{r}{\beta}},$$

and

$$m^{(2)}_{\tau,\kappa,r} = \frac{t^r}{r!}\mathbf{c}^{\star}_{\tau,\kappa}\left[1+\kappa+(1+\tau)\,\theta\right]^{\frac{r}{\beta}}.$$

Using the Wright generalized hypergeometric (WGH) function which defined as

$${}_{(p)}\mathbf{\Psi}_{(q)}\left[\begin{array}{c}(\delta_1, A_1), \ldots, (\delta_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\end{array} ; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{\kappa=1}^{p} \Gamma(\delta_\kappa + A_\kappa\, n)}{\prod_{\kappa=1}^{q} \Gamma(\beta_\kappa + B_\kappa\, n)} \frac{x^n}{n!},$$

Then, we can write $M(t;\delta,\beta)$ as

$$M(t;\delta,\beta) = {}_{(1)}\mathbf{\Psi}_{(0)}\left[\begin{array}{c}\left(1, -\frac{1}{\beta}\right) \\ -\end{array} ; \delta\, t\right].$$

Combining (9) and the last equation, we obtain the MGF of X in terms of WGH function, say $M(t)$, as

$$M(t)=\sum_{\kappa,\nu=0}^{\infty}\left(\begin{array}{c}\mathbf{c}_{\tau,\kappa}\left\{{}_{(1)}\mathbf{\Psi}_{(0)}\left[\begin{array}{c}\left(1,-\frac{1}{\beta}\right)\\ -\end{array};\delta t\left[(1+\tau)\,\theta+\kappa\right]^{\frac{1}{\beta}}\right]\right\}\\ -\mathbf{c}^{\bigstar}_{\tau,\kappa}\left\{{}_{(1)}\mathbf{\Psi}_{(0)}\left[\begin{array}{c}\left(1,-\frac{1}{\beta}\right)\\ -\end{array};\delta t\left[1+\kappa+(1+\tau)\,\theta\right]^{\frac{1}{\beta}}\right]\right\}\end{array}\right).$$

2.4 Probability weighted moments

The $(s,r)^{th}$ PWM of X following the ZTPTL-Fr is formally defined by

$$\rho_{r,s}=\mathbf{E}\left\{X^{s}\,F(X)^{r}\right\}=\int_{-\infty}^{\infty}x^{s}\,F(x)^{r}\,f\left(x\right)\,dx.$$

Using equations (5) and (6), we can write

$$f\left(x\right)\,F(x)^{r}=\sum_{\tau,\kappa=0}^{\infty}\left[\begin{array}{c}d_{\tau,\kappa}\;\pi_{(1+\tau)\theta+\kappa}(x;\beta,\delta)\\ -d^{\bigstar}_{\tau,\kappa}\;\pi_{1+\kappa+(1+\tau)\theta}(x;\beta,\delta)\end{array}\right],$$

where

$$d_{\tau,\kappa}=\sum_{\nu=0}^{\infty}\frac{2^{(1+\tau)\theta-\kappa}\theta\alpha^{\tau+1}\left(-1\right)^{\tau+\kappa+\nu}\left(\nu+1\right)^{\tau}}{\tau!\Delta_{(\alpha)}^{r+1}\left[(1+\tau)\,\theta+\kappa\right]}\binom{r}{\nu}\binom{-1+(1+\tau)\,\theta}{\kappa}$$

and

$$d^{\bigstar}_{\tau,\kappa}=\sum_{\nu=0}^{\infty}\frac{2^{(1+\tau)\theta-\kappa}\theta\alpha^{\tau+1}\left(-1\right)^{\tau+\kappa+\nu}\left(\nu+1\right)^{\tau}}{\tau!\Delta_{(\alpha)}^{r+1}\left[1+\kappa+(1+\tau)\,\theta\right]}\binom{r}{\nu}\binom{-1+(1+\tau)\,\theta}{\kappa},$$

then, the $(s,r)^{th}$ PWM of X can be expressed as

$$\rho_{r,s}=\delta^{s}\Gamma\left(1-\frac{s}{\beta}\right)\sum_{\tau,\kappa=0}^{\infty}\left[d^{(1)}_{\tau,\kappa}-d^{(2)}_{\tau,\kappa}\right]|_{(\;\beta>s)},$$

where

$$d^{(1)}_{\tau,\kappa}=\frac{d_{\tau,\kappa}}{\left[(1+\tau)\,\theta+\kappa\right]^{-\frac{s}{\beta}}},$$

and

$$d^{(2)}_{\tau,\kappa}=\frac{d^{\bigstar}_{\tau,\kappa}}{\left[1+\kappa+(1+\tau)\,\theta\right]^{-\frac{s}{\beta}}}.$$

2.5 Residual life and reversed residual life functions

The $n^{(th)}$ moment of the residual life, say

$$m_{n}(t)=\mathbf{E}[(X-t)^{n}\,|_{(X>t)}^{(n=1,2,\ldots)}],$$

uniquely determine $F(x)$. The $n^{(th)}$ moment of the residual life of X is given by

$$m_{n}(t)=\left[1-F(t)\right]^{-1}\int_{t}^{\infty}(x-t)^{n}dF(x),$$

then

$$m_n(t) = \frac{\delta^n}{1-F(t)} \sum_{r=0}^{n} \sum_{\tau,\kappa=0}^{\infty} \left[\begin{array}{c} b_{\tau,\kappa}^{(1)} \Gamma\left(1-\frac{n}{\beta}, [(1+\tau)\theta+\kappa]\left(\frac{\delta}{t}\right)^{\beta}\right) \\ -b_{\tau,\kappa}^{(2)} \Gamma\left(1-\frac{n}{\beta}, [1+\kappa+(1+\tau)\theta]\left(\frac{\delta}{t}\right)^{\beta}\right) \end{array} \right] |_{(\beta>n)},$$

where

$$\begin{aligned} \Gamma(\zeta, z)\,|_{(z>0)} &= \int_0^z t^{\zeta-1} \exp(-t)\, dt \\ &\sim \frac{z^{\zeta-1}}{\exp(z)} \left[\begin{array}{c} 1 \\ +\frac{\zeta-1}{z} \\ +\frac{(\zeta-1)(\zeta-2)}{z^2} + \ldots \end{array} \right], \end{aligned}$$

$$\Gamma(\zeta, z) = \Gamma(\zeta) - \gamma(\zeta, z),$$

$$b_{\tau,\kappa}^{(1)} = \frac{\mathbf{c}_{\tau,\kappa}}{[(1+\tau)\theta+\kappa]^{\frac{-n}{\beta}}} \binom{n}{r} (-t)^{n-r}$$

and

$$b_{\tau,\kappa}^{(2)} = \frac{\mathbf{c}_{\tau,\kappa}^{\star}}{[1+\kappa+(1+\tau)\theta]^{\frac{-n}{\beta}}} \binom{n}{r} (-t)^{n-r}.$$

The $n^{(th)}$ moment of the reversed residual life, say

$$M_n(t) = \mathbf{E}\left[(t-X)^n\,|_{(X\le t)}^{(n=1,2,\ldots)}\ \forall\,(t>0)\right],$$

or

$$M_n(t) = F(t)^{-1} \int_0^t (t-x)^n dF(x).$$

Then, the $n^{(th)}$ moment of the reversed residual life of X becomes

$$M_n(t) = \delta^n F(t)^{-1} \sum_{r=0}^{n} \sum_{\tau,\kappa=0}^{\infty} \left[\begin{array}{c} \xi_{\tau,\kappa}^{(1)} \gamma\left(1-\frac{n}{\beta}, [(1+\tau)\theta+\kappa]\left(\frac{\delta}{t}\right)^{\beta}\right) \\ -\xi_{\tau,\kappa}^{(2)} \gamma\left(1-\frac{n}{\beta}, [1+\kappa+(1+\tau)\theta]\left(\frac{\delta}{t}\right)^{\beta}\right) \end{array} \right] |_{(\beta>n)},$$

where

$$\xi_{\tau,\kappa}^{(1)} = \frac{\mathbf{c}_{\tau,\kappa}}{[(1+\tau)\theta+\kappa]^{\frac{-n}{\beta}}} (-1)^r t^{n-r},$$

and

$$\xi_{\tau,\kappa}^{(2)} = \frac{\mathbf{c}_{\tau,\kappa}^{\star}}{[1+\kappa+(1+\tau)\theta]^{\frac{-n}{\beta}}} (-1)^r t^{n-r}.$$

The mean waiting time or mean inactivity time (MIT) also called the mean reversed residual life function is given by $M_1(t) = \mathbf{E}[(t-X)\,|_{(X\le t)}]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the ZTPTL-Fr distribution can be obtained easily by setting $n=1$ in the above equation.

2.6 Order statistics and quantile spread ordering

Let $X_1, \ldots, X_n$ be a random sample (RS) from the ZTPTL-Fr distribution and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The PDF of $i^{(th)}$ order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{\kappa=0}^{n-i} (-1)^{\kappa} \binom{n-i}{\kappa} F^{\kappa+i-1}(x), \tag{13}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in equation (13) and using a power series expansion,we get

$$f(x)\ F(x)^{\kappa+i-1} = \sum_{w,m=0}^{\infty} \left[\begin{array}{c} t_{w,m}\ \pi_{(1+w)\theta+m}(x;\beta,\delta) \\ -t^{\star}_{w,m}\ \pi_{1+m+(1+w)\theta}(x;\beta,\delta) \end{array} \right],$$

where

$$t_{w,m} = \sum_{\nu=0}^{\infty} \frac{2^{\theta(w+1)-m}\theta\alpha^{w+1}(-1)^{w+m+\nu}(\nu+1)^{w}}{w!\Delta_{(\alpha)}^{\kappa+i}\left[\theta(w+1)+m\right]} \times \binom{-1+\kappa+i}{\nu}\binom{-1+(1+w)\theta}{m},$$

and

$$t^{\star}_{w,m} = \sum_{\nu=0}^{\infty} \frac{2^{\theta(w+1)-m}\theta\alpha^{w+1}(-1)^{w+m+\nu}(\nu+1)^{w}}{w!\Delta_{(\alpha)}^{\kappa+i}\left[\theta(w+1)+m+1\right]} \times \binom{-1+\kappa+i}{\nu}\binom{-1+(1+w)\theta}{m}.$$

The PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{\kappa=0}^{n-i} \frac{(-1)^{\kappa}\binom{n-i}{\kappa}}{\mathrm{B}(i,n-i+1)} \sum_{w,m=0}^{\infty} \left[t_{w,m}\ \pi_{(1+w)\theta+m}(x) - t^{\star}_{w,m}\ \pi_{1+m+(1+w)\theta}(x)\right].$$

The $q^{(th)}$ ordinary moment of $X_{i:n}$ can be expressed as

$$\mathbf{E}\left(X_{i:n}^{q}\right) = \delta^{q}\Gamma\left(1-\frac{q}{\beta}\right)\sum_{\kappa=0}^{n-i}\sum_{w,m=0}^{\infty}\frac{(-1)^{\kappa}\binom{n-i}{\kappa}}{\mathrm{B}(i,n-i+1)}\left[t^{(1)}_{w,m,h} - t^{(2)}_{w,m,h}\right]|_{(\beta>q)},$$

where

$$t^{(1)}_{w,m,h} = \frac{t_{w,m,h}}{\left[(1+w)\theta+m\right]^{-\frac{q}{\beta}}},$$

and

$$t^{(2)}_{w,m,h} = \frac{t^{\star}_{w,m,h}}{\left[1+m+(1+w)\theta\right]^{-\frac{q}{\beta}}}.$$

The quantile spread (QS) of the rv $T \sim$ZTPTL-Fr$(\alpha,\theta,\beta,\delta)$ is given by

$$\{\mathbf{QS}\}_T(\eta) = \left[F^{-1}(\eta) - F^{-1}(1-\eta)\right] \mid \left(\eta \in \left(\frac{1}{2}, 1\right)\right),$$

which implies

$$\{\mathbf{QS}\}_T(\eta) = \left[S^{-1}(1-\eta)\right] - \left[S^{-1}(\eta)\right],$$

where

$$F^{-1}(\eta) = S^{-1}(1-\eta) \text{ and } S(T) = 1 - F(T),$$

is the survival function. The QS of a any probability distribution describes how the probability mass is placed symmetrically about its median and hence it can be used to formalize concepts such as peakedness and tail weight traditionally associated with the kurtosis. So, it allows us to separate concepts of the kurtosis and peakedness for asymmetric models. Let T_1 and T_2 be two rvs following the ZTPTL-Fr model with $\{\mathbf{QS}\}_{T_1}$ and $\{\mathbf{QS}\}_{T_2}$, respectively. Then T_1 is called smaller than T_2 in quantile spread order, denoted as $T_1 \leq_{\{\mathbf{QS}\}} T_2$, if

$$\{\mathbf{QS}\}_{T_1}(\xi)\,|_{\left(\eta\in\left(\frac{1}{2},1\right)\right)} \leq \{\mathbf{QS}\}_{T_2}(\eta),$$

then we have the following results:

1−The order $\leq_{\{\mathbf{QS}\}}$ is *a* location-free, where $T_1 \leq_{\{\mathbf{QS}\}} T_2$ if

$$(T_1 + \kappa) \leq_{\{\mathbf{QS}\}} T_2|_{(\kappa\in\mathbb{R})}.$$

2−The order $\leq_{\{\mathbf{QS}\}}$ is dilative, where

$$T_1 \leq_{\{\mathbf{QS}\}} \kappa T_1|_{(\kappa\geq 1)},$$

and

$$T_2 \leq_{\{\mathbf{QS}\}} \kappa T_2\,|_{(\kappa\geq 1)}.$$

3−Let F_{T_1} and F_{T_2} be symmetric, then $T_1 \leq_{\{\mathbf{QS}\}} T_2$ if, and only if

$$F_{T_1}^{-1}(\eta) \leq F_{T_2}^{-1}(\eta)\,|_{\left(\eta\in\left(\frac{1}{2},1\right)\right)}.$$

4−The order $\leq_{\{\mathbf{QS}\}}$ implies ordering of the mean absolute deviation around the median, say $\xi(T_i)|_{(i=1,2)}$, the we have

$$\xi(T_i) = \mathbf{E}\left[|T_i - Median(T_i)|\right],$$

where

$$T_1 \leq_{\{\mathbf{QS}\}} T_2 \ \Rightarrow \xi(T_1) \leq_{\{\mathbf{QS}\}} \xi(T_2),$$

finally $T_1 \leq_{\{\mathbf{QS}\}} T_2$ if, and only if

$$-T_1 \leq_{\{\mathbf{QS}\}} -T_2.$$

3. Estimation

Let $x_1, \ldots, x_n$ be a RS from the ZTPTL-Fr distribution with parameters α, θ, β and δ. Let $\mathbf{\Theta} = (\alpha, \theta, \beta, \delta)^\intercal$ be the 4×1 parameter vector. For determining the maximum likelihood estimators (MLEs) of $\mathbf{\Theta}$, we have the log-likelihood function

$$\begin{aligned}\ell &= \ell(\Theta) = n\log 2 + n\log\alpha + n\log\theta + n\log\beta + n\beta\log\delta \\ &\quad -n\log\left(\Delta_{(\alpha)}\right) - (\beta+1)\sum_{i=1}^{n}\log x_i - \theta\delta^{\beta}\sum_{i=1}^{n}x_i^{-\beta} \\ &\quad -\alpha\sum_{i=1}^{n}\exp\left[-\theta\left(\delta x_i^{-1}\right)^{\beta}\right]\left\{2-\exp\left[-\left(\delta x_i^{-1}\right)^{\beta}\right]\right\}^{\theta} \\ &\quad +\sum_{i=1}^{n}\log\left\{1-\exp\left[-\left(\delta x_i^{-1}\right)^{\beta}\right]\right\} + (\theta-1)\sum_{i=1}^{n}\log\left\{2-\exp\left[-\left(\delta x_i^{-1}\right)^{\beta}\right]\right\}.\end{aligned}$$

The above log-likelihood function can be maximized numerically by using `R` (`optim`), `SAS` (`PROC NLMIXED`) or `Ox` program (sub-routine `MaxBFGS`), among others. The components of the score vector, $\mathbf{U}(\mathbf{\Theta}) = \frac{\partial\ell}{\partial\mathbf{\Theta}} = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}\right)^{\intercal}$ are availabe if needed. Setting the nonlinear system of equations $\mathbf{U}_\alpha = \mathbf{U}_\theta = \mathbf{U}_\beta = \mathbf{U}_\delta = 0$ and solving them simultaneously yields the MLE $\widehat{\mathbf{\Theta}} = (\widehat{\alpha}, \widehat{\theta}, \widehat{\beta}, \widehat{\delta})^{\intercal}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 4×4 observed information matrix $\mathbf{J}(\mathbf{\Theta}) = \{\frac{\partial^2\ell}{\partial r\,\partial s}\}$ (for $r, s = \alpha, \theta, \beta, \delta$), whose elements can be computed numerically. Under standard regularity conditions when $n\to\infty$, the distribution of $\widehat{\mathbf{\Theta}}$ can be approximated by a multivariate normal $N_4(0, \mathbf{J}(\widehat{\mathbf{\Theta}})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $\mathbf{J}(\widehat{\mathbf{\Theta}})$ is the total observed information matrix evaluated at $\widehat{\mathbf{\Theta}}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

4. Simulation results

We present some simulation experiments for some different sample sizes in order to assess the accuracy of the MLEs. Simulating rvs from well defined probability distributions has been discussed in the literature of computational statistics, e.g. the inverse transformation method, the rejection and acceptance sampling technique, etc. The ideal technique for simulating from the ZTPTL-Fr distribution is the inversion method, we can simulate rv X by

$$X = \delta\left(-\ln\left\{1-\left[1-\left(\frac{-\ln\left\{1-\left[U\left(\Delta_{(\alpha)}\right)\right]\right\}}{\alpha}\right)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right\}\right)^{\frac{1}{\beta}},$$

where U is a uniform random number in $(0,1)$. For selected combinations of α, θ, β and δ we generate samples of sizes $n = 50, 100, 200, 300, 500$ and 1000 from the ZTPTL-Fr distribution. We repeat the simulations $N = 1000$ times,we use two combinations for the parameter values (α=2.5, θ =1.5, $\beta = 1$ and δ=2) in order to obtain average estimates (AEs) and mean square errors (MSEs) of the parameters. The empirical results obtained via using the well-known R package are given in Table 3.

Table 3: Empirical AEs and (MSEs) for α=2.5, θ =1.5, $\beta = 1$ and δ=2.

	AEs and its corresponding (MSEs)			
Sample size(n)	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\beta}$	$\widehat{\delta}$
50	2.61195	1.68512	1.47851	2.61178
	(1.76198)	(0.981681)	(1.342916)	(2.98781)
100	2.59812	1.63221	1.31917	2.52245
	(1.3225)	(0.822465)	(1.15650)	(1.15671)
200	2.60112	1.57613	1.11875	2.38751
	(1.21391)	(0.587918)	(0.89794)	(0.32292)
300	2.51686	1.54572	1.03218	2.31510
	(1.00293)	(0.44371)	(.1988569)	(0.0901)
500	2.50224	1.50319	1.00166	2.0691
	(0.08571)	(0.299718)	(0.09829)	(0.00972)
1000	2.50021	1.50027	1.00071	2.00651
	(0.000163)	(0.019226)	(0.011985)	(0.000567)

We observe that our estimates are pretty stable especially when $n \geq 300$ and as n increases the MSEs and biases decreases. So, the maximum likelihood method works very well to estimate the model parameters.

5. Data analysis

In this section we provide applications of the ZTPTL-Fr distribution using two real data sets. In order to compare the distributions, we consider some criteria like Akaike Information Criterion (**AIC**) and Bayesian Information Criterion (**BIC**) value is chosen as the best model to fit the data.

The first data set consists of 72 observations of survival times for Guinea pigs injected with different doses of tubercle bacilli: 12, 15, 43, 44, 263, 297, 341, 34148, 76, 76, 81, 83, 84, 85, 87, 58, 52, 53, 73, 75, 59, 60, 54, 4, 24, 175, 22, 234, 38, 38, 70, 70, 72, 175, 211, 32, 62, 63, 65, 65, 67, 68, 60, 32, 33, 54, 55, 56, 146, 233, 258, 57, 58, 60, 60, 61, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 258 and 376.These data were previously studied by Krishna et al. (2013). We compare the proposed ZTPTL-Fr distribution with other related models namely: the Kumaraswamy Marshall–Olkin Fr (KwMO-Fr), MOKw-Fr, Kw-Fr, MO-Fr, beta -Fr (B-Fr), Exponentiated Fr (E-Fr), Marshall–Olkin Invere Exponential (MOIE), Marshall–Olkin invere Rayleigh (MOIR) and Fr distributions. The second data set (repair times data) represents an active repair times (hours) for an airborne communication transceiver, to be self-contained, this data set is reproduced as follows: 0.2, 0.5, 0.5, 0.3, 0.5, 1.0, 1.0, 1.3, 11.5,1.5, 0.7, 0.7, 1.0, 1.0, 1.1, 10.3, 22.02.0, 2.0, 2.2, .5, 1.5, 2.5, 2.7, 4.0, 4.0, 4.5, 3.0, 3.0, 3.3, 3.3, 4.7, 7.0, 7.5, 5.0, 5.4, 5.4, 8.8, 9.0, 0.5, 0.6, 0.6, 0.8, 0.8, 0.7 and 24.5. Many other useful real data sets are available in Brito et al. (2017), Alizadeh et al. (2018), Korkmaz et al. (2018 and 2019), Cordeiro et al. (2019), Abouelmagd et al. (2019a,b,c), Goual et al. (2019), Goual and Yousof (2019) Yadav etl al. (2019) and Al-Babtain et al. (2020a and b).

The total time test (TTT) plot is an important graphical approach to verify whether the data set can be applied to a specific model or not. Due to Aarset (1987), the empirical version of the TTT plot is given by plotting

$$T(rn^{-1}) = \left(\sum_{j=1}^{n} y_{j:n}\right)^{-1} \sum_{j=1}^{r} y_{j:n} + (n-r)y_{j:n},$$

against rn^{-1}, where $r = 1, \ldots, n$ and $y_{j:n}|_{(j=1,\ldots,n)}$ are the order statistics of the sample. Aarset (1987) showed that the HRF is constant if the TTT plot is graphically presented as a straight diagonal. The HRF is increasing (or decreasing) if the TTT plot is concave (or convex). The HRF is **U**-shaped (bathtub) if the TTT plot is firstly convex and then concave, if not, the HRF is unimodal. The TTT plots the three real data sets is presented in Figure 2. Plots in Figure 2 indicates that the empirical HRFs of the two data sets are "upside down then bathtub" and upside down respectively. We compare the proposed ZTPTL-Fr distribution with other related

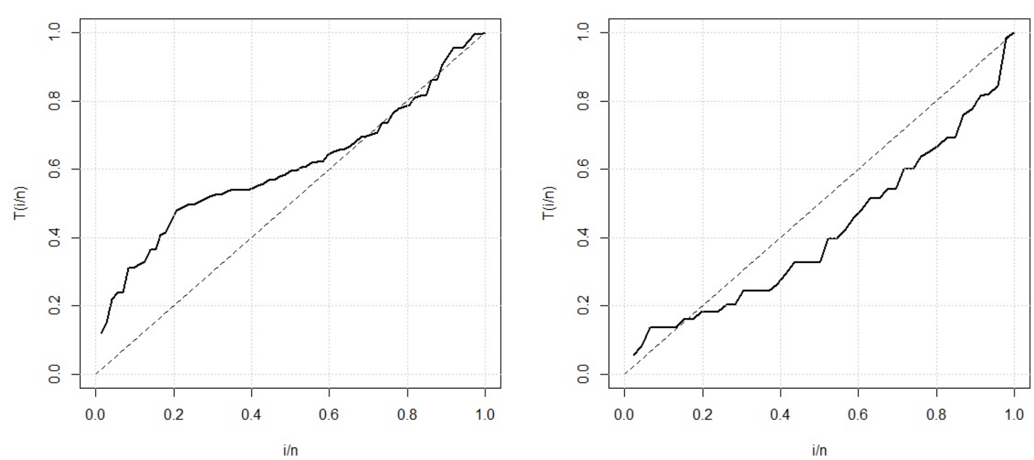

Figure 2: TTT plots for the data set I (left) and data set II (right).

models namely: the Topp Leone Generated Fr (TLG-Fr) , Fr, Kw-Fr, E-Fr, B-Fr, transmuted Fr (T-Fr), MO-Fr and Mcdonald Fr (Mc-Fr) distributions.

Tables 4 and 6 list the values of **AIC** and **BIC** however the MLEs and their corresponding standard errors (in parentheses) of the model parameters are listed in Tables 5 and 7 respectively.

Table 4: **AIC** and **BIC** for data **I**.

Model	**AIC**	**BIC**
ZTPTL-Fr	**696.9**	**706.1**
KwMO-Fr	751.6	762.9
E-Fr	786.5	793.3
Kw-Fr	788.5	797.6
ZB-Fr	787.2	794.1
B-Fr	788.6	797.7
KwMOIE	790.7	799.8
MOKw-Fr	794.2	805.6
Fr	795.3	799.9
MO-Fr	796.1	802.9
KwMOIR	808.2	817.3

All values are obtained using the R program. Figure 3 give the fitted PDF, CDF, HRF, P-P plot and Kaplan-Meier survival plot for data **I**. Figure 4 give the fitted PDF, CDF, HRF, P-P plot and Kaplan-Meier survival plot for data **II**.

Table 5: MLEs and their standard errors (in parentheses) for the survival times for Guinea pigs.

Model	Estimates				
ZTPTL-Fr$(\alpha,\theta,\beta,\delta)$		-3.796	0.0437	1.654	206.008
		(1.1799)	(0.000)	(0.098)	(14.38)
KwMO-Fr$(a,b,\alpha,\beta,\delta)$	0.068	54.638	308.470	0.087	69.693
	(0.984)	(0.063)	(0.229)	(0.038)	(0.082)
E-Fr(b,β,δ)			8.2723	0.6207	336.3679
			(7.953)	(0.208)	(374.803)
Kw-Fr(a,b,β,δ)		45.7326	8.2723	0.6207	0.7111
		(0.092)	(0.979)	(0.003)	(0.013)
ZB-Fr(a,β,δ)		26.048		1.537	6.638
		(0.597)		(0.008)	(0.007)
B-Fr(a,b,β,δ)		19.9786	20.1331	0.322	24.5032
		(7.246)	(7.26)	(0.00115)	(0.087)
KwMOIE(a,b,α,δ)	8.8727	68.1393	2.6258		0.1758
	(1.174)	(0.020)	(0.512)		(0.000)
MOKw-Fr$(\alpha,a,b,\beta,\delta)$	0.449	22.880	1.376	2.666	0.449
	(0.021)	(3.338)	(0.087)	(0.869)	(0.021)
Fr(β,δ)				1.4148	54.1888
				(0.00271)	(0.111)
MO-Fr(α,β,δ)	14.9816			1.7855	13.991
	(4.6305)			(0.193)	(2.964)
Kw-MOIR(a,b,α,δ)	9.993	58.4697	0.6389		1.6788
	(1.972)	(0.105)	(0.098)		(0.001)

Based on the figures in Tables 4 and 6, we conclude that the ZTPTL-Fr model provide adequate fits as compared to other Fr models in both applications with small values for **AIC** and **BIC**. In Application 1, the proposed ZTPTL-Fr model is much better than the B-Fr, E-Fr, MOKw-Fr, MOIE, KwMO-Fr, MO-Fr, Kw-Fr, MOIR and Fr models, so the ZTPTL-Fr model is a good alternative to these models. In Application 2, the proposed ZTPTL-Fr lifetime model is much better than the Fr, T-Fr, Kw-Fr, MO-Fr, TLG-Fr , E-Fr, B-Fr and Mc-Fr models, so the ZTPTL-Fr model a good alternative to these models.

Table 6: **AIC** and **BIC** for data **II**.

Model	**AIC**	**BIC**
ZTPTL-Fr	**144.3**	**151.6**
TLG-Fr	207.2	214.5
Fr	207.4	215.0
Kw-Fr	207.4	214.6
E-Fr	207.4	214.9
B-Fr	207.4	214.7
T-Fr	207.8	215.3
MO-Fr	207.9	214.7
Mc-Fr	207.8	216.9

6. Conclusions

In this paper, a new four parameter zero truncated Poisson Fr distribution called the zero-truncated Poisson Topp Leone Fr (ZTPTL-Fr) model is defined and studied. Various structural mathematical properties of the proposed extreme value model including ordinary and incomplete moments, residual and reversed residual life functions generating functions and order statistics are investigated. The maximum likelihood method is used to estimate the model parameters. The new distribution is applied for modeling two real data sets to illustrate empirically its flexibility. The ZTPTL-Fr model provide adequate fits as compared to other Fr models in both applications with small values for **AIC** and **BIC**. The proposed ZTPTL-Fr model is much better than Marshall–Olkin Kumaraswamy Fr, beta Fr, Marshall–Olkin Fr, Kumaraswamy Fr, the Kumaraswamy-Marshall–Olkin -Fr, Marshall–Olkin inverse exponential, Marshall–Olkin inverse Rayleigh, exponentiated Fr and Fr models, so the ZTPTL-Fr model is a good alternative to these models for modeling survival times data. As well as the proposed ZTPTL-Fr lifetime model is much better than the Fr, Transmuted Fr, Kumaraswamy Fr, Topp Leone Generated Fr, exponentiated Fr, beta Fr, Marshall–Olkin Fr and Mcdonald Fr models, so the ZTPTL-Fr model a good alternative to these models for modeling repair times data. We assess the performance of the maximum likelihood method by means of a numerical simulation study, We observe that our estimates are pretty stable especially when $n \geq 300$ and as n increases the MSEs decreases. So, the maximum likelihood method works very well to estimate the model parameters.

Table 7: MLEs and their standard errors (in parentheses) for the repair times data.

Model	Estimates				
ZTPTL-Fr($\alpha,\theta,\beta,\delta$)	-7.3631	0.0026		1.0887	48.2580
	$1.566\times e^{-1}$	$5.586\times e^{-4}$		$3.163\times e^{-1}$	$5.8795\times e^{1}$
TLG-Fr(a,b,β,δ)	0.1405	2.1672		0.8958	4.9552
	(0.2299)	(20.072)		(0.1675)	(51.257)
Fr(β,δ)				1.0128	1.1297
				(0.1129)	(0.1740)
Kw-Fr(a,b,β,δ)	1.1619	3.8034		0.5401	4.0226
	(7.452)	(4.604)		(0.2753)	(47.459)
E-Fr(a,β,δ)	0.9881			1.0125	1.1433
	(23.679)			(0.1129)	(27.057)
B-Fr(a,b,β,δ)	2.3521	5.8362		0.4147	3.4905
	(8.581)	(14.877)		(0.5619)	(13.461)
T-Fr(a,β,δ)	-0.6364			1.0853	0.7747
	(0.1173)			(0.1226)	(0.3633)
MO-Fr(a,β,δ)	4.9168			1.3384	0.5066
	(6.1834)			(0.2574)	(0.3068)
Mc-Fr(a,b,α,β,δ)	0.0125	96.427	0.8957	12.281	10.570
	(0.0108)	(354.85)	(0.1297)	(45.502)	(35.124)

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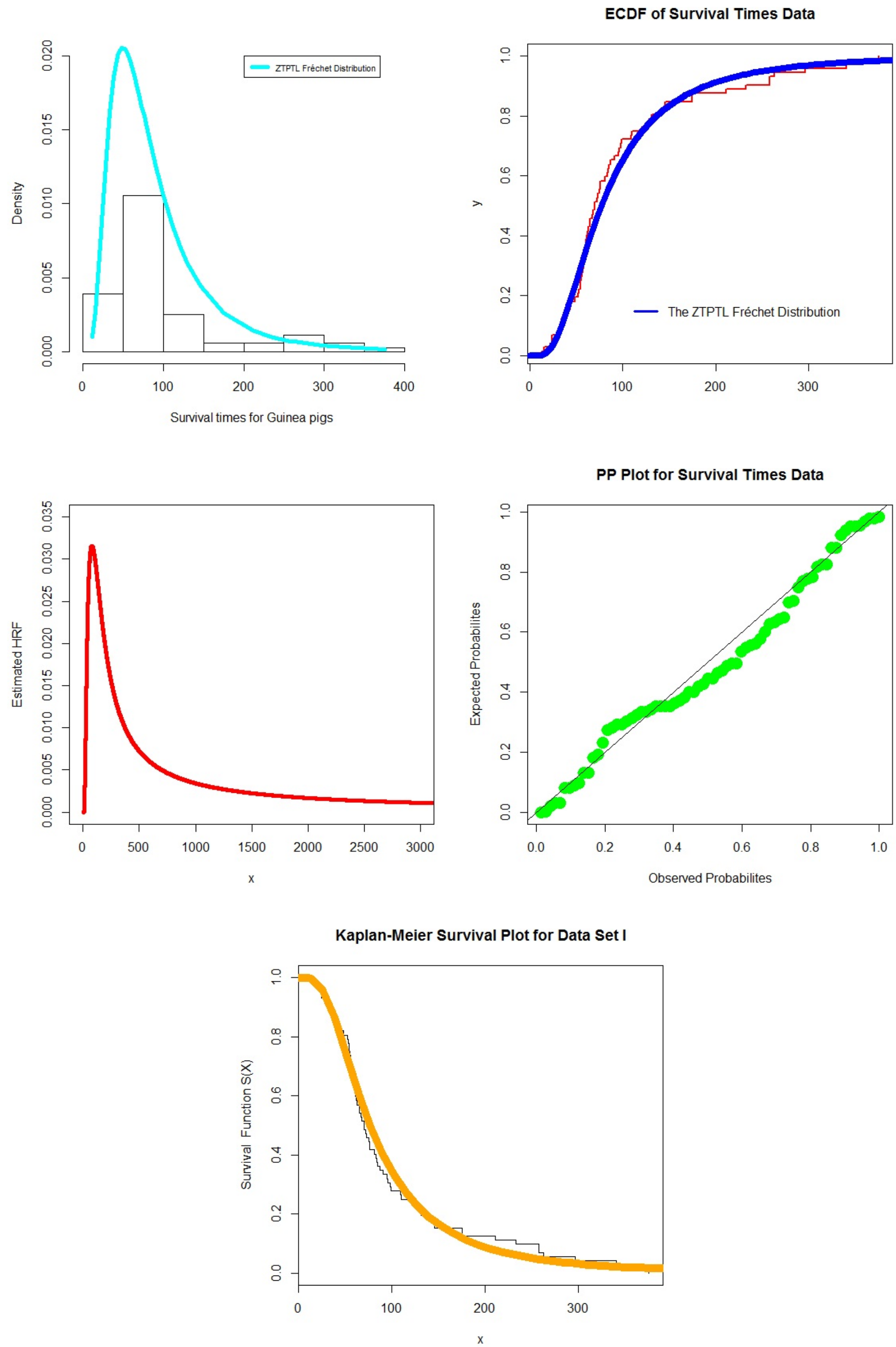

Figure 3: The fitted PDF, CDF, HRF, P-P plot and Kaplan-Meier survival plot for the first data set.

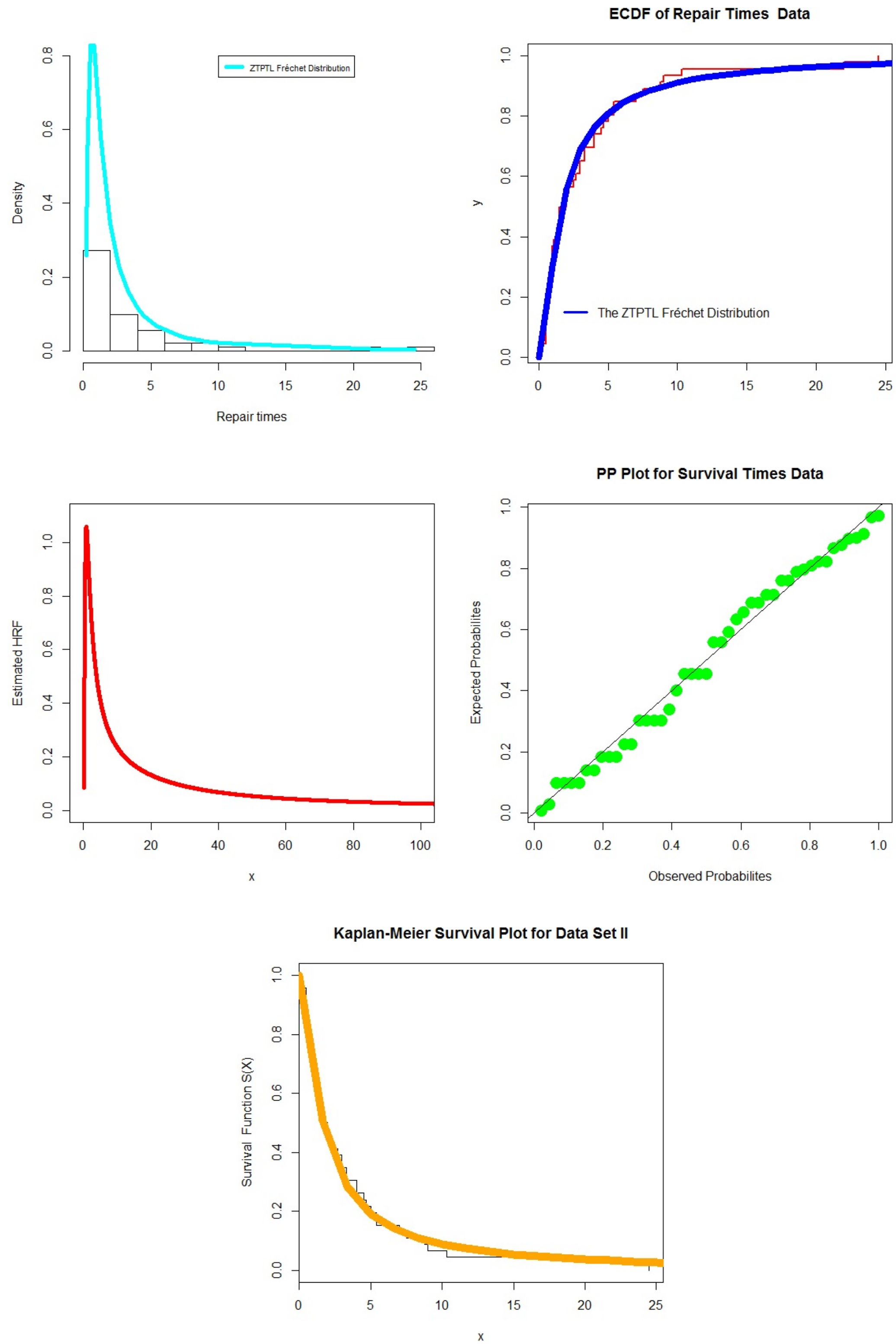

Figure 4: The fitted PDF, CDF, HRF, P-P plot and Kaplan-Meier survival plot for the second data set.