CUBIC RANK TRANSMUTED MODIFIED BURR III DISTRIBUTION: DEVELOPMENT, PROPERTIES, CHARACTERIZATIONS AND APPLICATIONS

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ABSTRACT

We propose a lifetime distribution with flexible hazard rate called cubic rank transmuted modified Burr III (CRTMBIII) distribution. We develop the proposed distribution on the basis of the cubic ranking transmutation map. The density function of CRTMBIII is symmetrical, right-skewed, left-skewed, exponential, arc, J and bimodal shaped. The flexible hazard rate of the proposed model can accommodate almost all types of shapes such as unimodal, bimodal, arc, increasing, decreasing, decreasing-increasing-decreasing, inverted bathtub and modified bathtub. To show the importance of proposed model, we present mathematical properties such as moments, incomplete moments, inequality measures, residual life function and stress strength reliability measure. We characterize the CRTMBIII distribution via techniques. We address the maximum likelihood method for the model parameters. We evaluate the performance of the maximum likelihood estimates (MLEs) via simulation study. We establish empirically that the proposed model is suitable for strengths of glass fibers. We apply goodness of fit statistics and the graphical tools to examine the potentiality and utility of the CRTMBIII distribution.

Keywords: Moments, Reliability, Characterizations, Maximum Likelihood Estimation.

1. INTRODUCTION

In recent decades, many continuous univariate distributions have been developed, however, various data sets from reliability, insurance, finance, climatology, biomedical sciences and other areas do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to problems in these areas is a clear need of day.

The modified, extended and generalized distributions are obtained by the introduction of some transformation or addition of one or more parameters to the well-known baseline distributions. These new developed distributions provide better fit to the data than the sub and competing models.

Shaw and Buckley (2009) proposed ranking quadratic transmutation map to solve financial problems.

1.1 Quadratic Ranking Transmutation Map

Theorem 1.1: Let X_1 and X_2 be independent and identically distributed ($i.i.d.$) random variables with the common cumulative distribution function $G(x)$. Then, the ranking quadratic transmutation map is

$$F_Y(x) = (1+\lambda)\,G(x) - \lambda\,G^2(x), \quad \lambda \in [-1,1]. \tag{1}$$

Proof

Let X_1 and X_2 be $i.i.d.$ random variables with the common cumulative distribution function $G(x)$. Now, consider the following order statistics:

$$X_{1:2} = \min(X_1, X_2) \ \ and \ \ X_{2:2} = \max(X_1, X_2)$$

let $Y \overset{d}{=} X_{1:2}$, with probability π, $Y \overset{d}{=} X_{2:2}$, with probability $1-\pi$, $where\ 0 \le \pi \le 1$. The cumulative distribution function of Y is

$$F_Y(x) = \pi\,\mathrm{P}(X_{1:2} \le x) + (1-\pi)\,\mathrm{Pr}(X_{2:2} \le x),$$

or

$$F_Y(x) = \pi\left[1 - \left[1 - G(x)\right]^2\right] + (1-\pi)\left[G(x)\right]^2,$$

$$F_Y(x) = 2\pi G(x) + (1-2\pi)\left[G(x)\right]^2. \tag{2}$$

If we take $\lambda = 2\pi - 1$, the distribution in equation (2) is known as ranking quadratic transmutation map or transmuted distribution.

1.2 Cubic Ranking Transmutation Map

Theorem 1.1: Let X_1, X_2 and X_3 be i.i.d. random variables with the common cumulative distribution function $G(z)$. Then, the cubic ranking transmutation map is

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1) G^2(x) + (1 - \lambda_2) G^3(x), \ \ \lambda_1 \in [0,1], \ \ \lambda_2 \in [-1,1]. \tag{3}$$

Proof

Consider the following order statistics:

$$X_{1:3} = \min(X_1, X_2, X_3), \quad X_{2:3} \quad and \quad X_{3:3} = \max(X_1, X_2, X_3).$$

Let $Y \overset{d}{=} X_{1:3}$, with probability π_1, $Y \overset{d}{=} X_{2:3}$, with probability π_2, $Y \overset{d}{=} X_{3:3}$, with probability π_3, $where$ $0 \le \pi_i \le 1$, $\pi_3 = 1 - \pi_1 - \pi_2$ and $\sum_{i=1}^{3} \pi_i = 1$.

The cumulative distribution function (cdf) of Y is

$$F_Y(x) = \pi_1 P(X_{1:3} \le x) + \pi_2 P(X_{2:3} \le x) + (1 - \pi_1 - \pi_2) P(X_{3:3} \le x),$$

where $P(X_{1:3} \le x) = 1 - [1 - G(x)]^3$, $\Pr(X_{2;3} \le x) = 3G^2(x) - 2G^3(x)$ and $P(X_{3:3} \le x) = [G(x)]^3$.

Now, the cdf of Y becomes

$$F_Y(x) = 3\pi_1 G(x) + 3(\pi_2 - \pi_1)[G(x)]^2 + (1 - \pi_2)[G(x)]^3. \tag{4}$$

If we take $\lambda_1 = 3\pi_1$ and $\lambda_2 = 3\pi_2$ the distribution in equation (4) is known as cubic ranking transmutation map or transmuted distribution of order 2.

Definition 1.1: The cdf and probability density function (pdf) for cubic rank transmuted distribution are given, respectively, by

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1) G^2(x) + (1 - \lambda_2) G^3(x), \ \lambda_1 \in [0,1], \ \lambda_2 \in [-1,1], \tag{5}$$

and

$$f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1 - \lambda_2)G^2(x)], x \in R \tag{6}$$

Afify et al. (2017) proposed the beta transmuted-H family of distributions. Al-Kadim and Mohammed (2017) presented the cubic transmuted Weibull distribution in terms of basic mathematical properties. Nofal et al. (2017) studied a generalized transmuted-G family of distributions. Alizadeh et al. (2017) developed generalized transmuted family of distributions. Bakouch et al. (2017) also introduced a new family of transmuted distributions. Granzotto et al. (2017) proposed a cubic ranking transmutation map and its studied different properties. They studied properties of cubic rank transmuted Weibull distribution and cubic rank transmuted log logistic distribution. Yilmaz (2018) proposed a new distribution family developed with polynomial rank transmutation. Rahman et al. (2018) developed general family of transmuted distributions. Riffi (2019) presented higher rank transmuted families of distributions. Rahman et al. (2019) studied properties of cubic rank transmuted Weibull distribution.

Burr III (Burr; 1942) has wide range of applications in failure time modeling, reliability, business failure data, modeling finance, insurance data and quality control plans. Burr III (BIII) model accommodates only decreasing and inverted bathtub hazard rate functions (hrf). Transmuted Burr III (TBIII) accommodates only inverted bathtub hazard rate functions (Abdul-Moniem; 2015). The failure rate for modified Burr III (MBIII) can take only increasing, decreasing, inverted bathtub and modified bathtub shapes (Bhatti et al. 2019). Transmuted modified Burr III (TMBIII) accommodates only decreasing and inverted bathtub hazard rate functions (Ali and Ahmad; 2016).The hrf for the CRTMBIII distribution accommodates almost all shapes such as bimodal, arc, increasing, decreasing, decreasing-

increasing- decreasing, inverted bathtub (unimodal) and modified bathtub. Due to its flexible failure rate, it can be applicable to lifetime applications.

The basic motivations for proposing the CRTMBIII distribution are: (i) to generate distributions with symmetrical, right-skewed, left-skewed, exponential, arc, J and bimodal shaped; (ii) to obtain unimodal, bimodal, arc, increasing, decreasing, decreasing-increasing-decreasing, inverted bathtub and modified bathtub hazard rate function; (iii) to serve as the best alternative model for the current models to explore and modeling real data in economics, life testing, reliability, survival analysis manufacturing and other areas of research and (iv) to provide better fits than other sub-models.

This paper is sketched into the following sections. In Section 2, we develop and study the CRTMBIII distribution. We also present the basic structural properties and sub-models. We also study some plots of density and hazard rate functions. In Section 3, we derive mathematical properties such as moments, incomplete moments, inequality measures, residual and reverse residual life function and stress-strength reliability measure. In Section 4, two characterizations of the CRTMBIII distribution are studied. In Section 5, we address the parameters of the CRTMBIII distribution via maximum likelihood method. In Section 6, we evaluate the performance of the maximum likelihood estimates (MLEs) of the modal parameters via simulation study. In Section 7, we establish empirically that the proposed model is suitable for strengths of glass fibers. We apply goodness of fit statistics and graphical tools to examine the potentiality and utility of the CRTMBIII distribution. The concluding remarks are given in Section 8.

2. THE CRTMBIII DISTRIBUTION

Ali et al. (2015) studied modified Burr III (MBIII) distribution with its properties. Ali and Ahmad (2015) studied transmuted MBIII (TMBIII) distribution and its properties. The cdf and pdf of MBIII distribution are given, respectively, by

$$F(x) = \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}}, \quad x \geq 0,$$

(7)

and

$$f(x) = \alpha\beta x^{-\beta-1}\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} \quad x > 0, \alpha > 0, \beta > 0, \gamma > 0.$$

(8)

Here, the CRTMBIII distribution is introduced with the help of (7) and (8). The cdf and pdf of the CRTMBIII distribution are given, respectively, by

$$F(x) = \lambda_1\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + \left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}} + \left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}}, \quad x \geq 0,$$

(9)

and

$$f(x) = \alpha\beta x^{-\beta-1}\left[\lambda_1\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2\left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1} + 3\left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1}\right], x > 0,$$

(10)

with $\alpha > 0, \beta > 0, \gamma > 0, \lambda_1 \in [0,1], \lambda_2 \in [-1,1]$.

In future, the pdf in (11) is denoted by X~CRTMBIII$(\alpha, \beta, \gamma, \lambda_1, \lambda_2)$.

2.1. Structural Properties

For X~CRTMBIII$(\alpha, \beta, \gamma, \lambda_1, \lambda_2)$, the survival, hazard, cumulative hazard, reverse hazard functions and the Mills ratio are given, respectively, by

$$S(x) = 1 - \left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + \left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}} + \left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}} \right], \quad x \geq 0,$$

(11)

$$h(x) = \frac{\alpha \beta x^{-\beta-1} \left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2\left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1} + 3\left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1} \right]}{1 - \left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + \left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}} + \left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}} \right]},$$

(12)

$$r(x) = \alpha \beta x^{-\beta-1} \left(1 + \gamma x^{-\beta}\right)^{-1} \left[\frac{\lambda_1 + 2\left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + 3\left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}{\lambda_1 + \left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + \left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}} \right],$$

(13)

$$H(x) = -\ln\left\{ 1 - \left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + \left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}} + \left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}} \right] \right\},$$

(14)

and

$$m(x) = \frac{1 - \left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + \left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}} + \left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}} \right]}{\alpha \beta x^{-\beta-1} \left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2\left(\lambda_2 - \lambda_1\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1} + 3\left(1 - \lambda_2\right)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1} \right]}.$$

(15)

The elasticity $e(x) = \dfrac{d\ln F(x)}{d\ln x} = x r(x)$ for the CRTMBIII distribution is

The elasticity of the CRTMBIII distribution shows the behavior of the accumulation of probability in the domain of the random variable.

The quantile function of the CRTMBIII distribution is the solution of the following

$$x_q = \left[\frac{1}{\gamma} \left(-\frac{B}{3A} - \frac{2^{1/3}(-B^2 + 3AC)}{3AM} + \frac{M}{3 2^{1/3} A} \right)^{-\frac{\gamma}{\alpha}} - \frac{1}{\gamma} \right]^{-\frac{1}{\beta}},$$

(17)

where $A = (1 - \lambda_2)$, $B = (\lambda_2 - \lambda_1)$, $C = \lambda_1$ and

$$M = \left[-2B^3 + 9ABC + 27A^2 q + \sqrt{4(-B^2 + 3AC)^3 + (-2B^3 + 9ABC + 27A^2 q)^2} \right]^{1/3}.$$

The random number generator of the CRTMBIII distribution is the solution of the following

$$X = \left[\frac{1}{\gamma} \left(-\frac{B}{3A} - \frac{2^{1/3}(-B^2 + 3AC)}{3A\,\mathrm{M}_Z} + \frac{\mathrm{M}_Z}{32^{1/3}A} \right)^{-\frac{\gamma}{\alpha}} - \frac{1}{\gamma} \right]^{-\frac{1}{\beta}},$$

(18)

where $\mathrm{M}_Z = \left[-2B^3 + 9ABC + 27A^2Z + \sqrt{4(-B^2 + 3AC)^3 + (-2B^3 + 9ABC + 27A^2Z)^2} \right]^{1/3}$ and the

random variable Z has the uniform distribution on $(0,1)$.

2.2 Shapes of the CRTMBIII Density and Hazard Rate Functions

The following graphs show that shapes of CRTMBIII density are arc, exponential, positively skewed, negatively skewed and symmetrical (Fig.1). The CRTMBIII distribution has unimodal, bimodal, arc, increasing, decreasing, decreasing-increasing- decreasing, inverted bathtub and modified bathtub hazard rate function (Fig. 2).
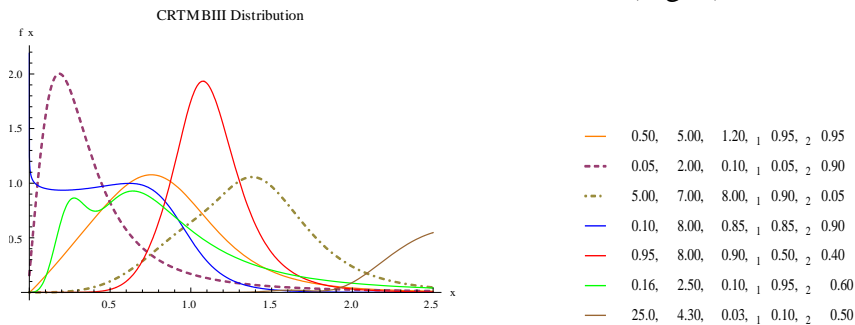

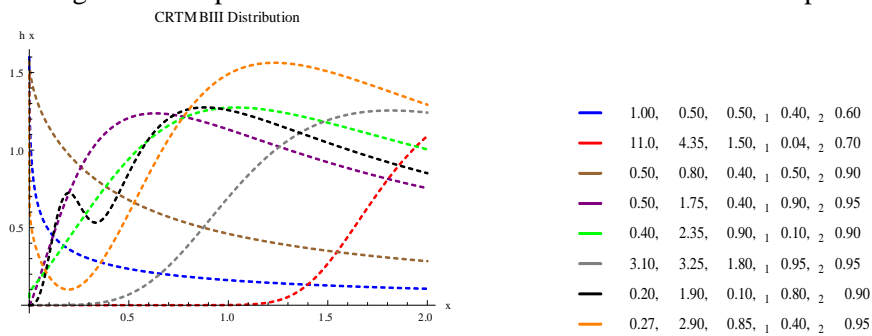Fig.1 Plots of pdf of the CRTMBIII distribution for the selected parameter values

Fig:2 Plots of hrf of the CRTMBIII distribution for the selected parameter values

2.3 Sub-Models

The CRTMBIII distribution has the following sub models (Table 1).

Table 1: Sub-Models of the CRTMBIII Distribution

Sr.No.	X	α	β	γ	λ_1	λ_2	Name of Distribution
1	X	α	β	γ	λ_1	λ_2	CRTMBIII distribution
2	X	α	β	1	λ_1	λ_2	CRTBIII distribution
3	X	1	β	1	λ_1	λ_2	CRTLL distribution
4	X	α	β	γ	λ_1	1	TMBIII distribution
5	X	α	β	1	λ_1	1	TBIII distribution
6	X	1	β	1	λ_1	1	TLL distribution
7	X	α	β	$\gamma \to \infty$	λ_1	λ_2	CRTGIW distribution
8	X	α	β	$\gamma \to \infty$	λ_1	1	TGIW distribution
9	X	α	β	$\gamma \to \infty$	1	1	GIW distribution
10	X	α	β	γ	1	1	MBIII distribution
11	X	α	β	1	1	1	BIII distribution
12	X	1	β	1	1	1	LL distribution

3. MATHEMATICAL PROPERTIES

We derive theoretically some mathematical properties such as the rth ordinary moments, s^{th} incomplete moments, and inequality measures, residual and reverse residual life function and reliability measures in this section.

3.1 Ordinary Moments

The moments are significant tools for statistical analysis in pragmatic sciences. The descriptive measures such as central tendency (μ_1'), dispersion (σ), skewness (γ_1) and kurtosis (γ_2) can be calculated from the moments.

For X~CRTMBIII$(\alpha, \beta, \gamma, \lambda_1, \lambda_2)$, the rth ordinary moment is

$$E\left(X^r\right) = \int_0^\infty x^r f\left(x\right) dx,$$

$$\mu_r' = E(X^r) = \int_0^\infty x^r \alpha\beta x^{-\beta-1}\left[\lambda_1\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2(\lambda_2 - \lambda_1)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1}\right.$$

$$\left. + 3(1 - \lambda_2)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1}\right] dx.$$

Letting $\gamma x^{-\beta} = y, x = \left(\dfrac{y}{\gamma}\right)^{-\frac{1}{\beta}}, \gamma\beta x^{-\beta-1}dx = -dy$, then

$$\mu'_r = E(X^r) = \left[\lambda_1 \frac{\alpha}{\gamma} \gamma^{\frac{r}{\beta}} B\left(1 - \frac{r}{\beta}, \frac{\alpha}{\gamma} + \frac{r}{\beta}\right) + 2(\lambda_2 - \lambda_1)\frac{\alpha}{\gamma}\gamma^{\frac{r}{\beta}} B\left(1 - \frac{r}{\beta}, \frac{2\alpha}{\gamma} + \frac{r}{\beta}\right)\right.$$

$$\left. + 3(1 - \lambda_2)\frac{\alpha}{\gamma}\gamma^{\frac{r}{\beta}} B\left(1 - \frac{r}{\beta}, \frac{3\alpha}{\gamma} + \frac{r}{\beta}\right)\right]$$

$$\mu'_r = E(X^r) = \gamma^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right)\left[\lambda_1 \frac{\Gamma\left(\frac{\alpha}{\gamma} + \frac{r}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} + (\lambda_2 - \lambda_1)\frac{\Gamma\left(\frac{2\alpha}{\gamma} + \frac{r}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}\right.$$

$$\left. + (1 - \lambda_2)\frac{\Gamma\left(\frac{3\alpha}{\gamma} + \frac{r}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right], r = 1,2,3,4,\ldots,$$

$$(19)$$

where $\Gamma(.,.)$ is a gamma function.

Mean and Variance of the CRTMBIII distribution are

$$E(X) = \gamma^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right)\left[\lambda_1 \frac{\Gamma\left(\frac{\alpha}{\gamma} + \frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} + (\lambda_2 - \lambda_1)\frac{\Gamma\left(\frac{2\alpha}{\gamma} + \frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + (1 - \lambda_2)\frac{\Gamma\left(\frac{3\alpha}{\gamma} + \frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right],$$

$Var(X)$

$$= \left\{ \begin{array}{l} \gamma^{\frac{2}{\beta}} \Gamma\left(1 - \frac{2}{\beta}\right)\left[\lambda_1 \frac{\Gamma\left(\frac{\alpha}{\gamma} + \frac{2}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} + (\lambda_2 - \lambda_1)\frac{\Gamma\left(\frac{2\alpha}{\gamma} + \frac{2}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + (1 - \lambda_2)\frac{\Gamma\left(\frac{3\alpha}{\gamma} + \frac{2}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right] - \\ \gamma^{\frac{2}{\beta}}\left(\Gamma\left(1 - \frac{1}{\beta}\right)\right)^2\left[\lambda_1 \frac{\Gamma\left(\frac{\alpha}{\gamma} + \frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} + (\lambda_2 - \lambda_1)\frac{\Gamma\left(\frac{2\alpha}{\gamma} + \frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + (1 - \lambda_2)\frac{\Gamma\left(\frac{3\alpha}{\gamma} + \frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right]^2 \end{array} \right\}$$

The Mellin transformation is applied to get the moments of a probability distribution. For X~CRTMBIII$(\alpha, \beta, \gamma, \lambda_1, \lambda_2)$, the Mellin transform is

$$M\{f(x); s\} = f^*(s) = E(X^{s-1}),$$

$$M\{f(x); s\} = \gamma^{\frac{s-1}{\beta}} \Gamma\left(1 - \frac{s-1}{\beta}\right)\left[\lambda_1 \frac{\Gamma\left(\frac{\alpha}{\gamma} + \frac{s-1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} + (\lambda_2 - \lambda_1)\frac{\Gamma\left(\frac{2\alpha}{\gamma} + \frac{s-1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + (1 - \lambda_2)\frac{\Gamma\left(\frac{3\alpha}{\gamma} + \frac{s-1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right].$$

$$(20)$$

The rth central moment (μ_r), coefficients of skewness (γ_1) and kurtosis (γ_2) for the CRTMBIII model are attained from $\mu_r = \sum_{\ell=1}^{r} (-1)^\ell \binom{r}{l} \mu'_\ell \mu'_{r-\ell}$, $\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$ and $\beta_2 = \frac{\mu_4}{(\mu_2)^2}$. The numerical values for the mean (μ'_1), median $(\tilde{\mu})$, standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) for the CRTMBIII distribution for selected values of $\alpha, \beta, \gamma, \lambda_1, \lambda_2$ are listed in Table 2. We also depict that the CRTMBIII model can be effective to model data sets in terms of the descriptive measures.

Table 2: μ_1', $\tilde{\mu}$, σ, γ_1 and γ_2 of the CRTMBIII Distribution

α	β	γ	λ_1	λ_2	μ_1'	$\tilde{\mu}$	σ	γ_1	γ_2
1.5	1.5	1	0.1	-0.1	7.0946	3.2782	80.9775	521.808	374033
0.5	3	1.5	0.1	-0.1	1.3711	1.1392	1.114	9.149	341.352
0.5	5	1.5	0.1	-0.1	1.1488	1.0813	0.4703	2.2561	22.8658
0.1	5	1.5	0.1	0.1	0.5778	0.5102	0.4486	1.937	35.0744
0.1	5	1.5	0.5	0.5	0.4518	0.3437	0.4419	1.9544	28.7892
0.1	5	1.5	0.5	-0.5	0.5822	0.525	0.4629	1.4152	11.7208
0.1	5	1.5	0.9	-0.5	0.508	0.4299	0.4702	1.4098	10.0304
0.1	5	1.5	0.9	0.5	0.3785	0.2292	0.4267	1.7812	11.2591
0.1	5	1.5	0.9	0.9	0.326	0.1683	0.397	2.3369	2.3369
0.5	10	1	0.5	0.5	0.9873	0.9859	0.2186	0.3888	5.8513
0.05	10	1	0.5	0.5	0.4815	0.4646	0.3378	0.3436	2.397
0.05	10	3	0.5	0.5	0.2712	0.1121	0.3353	1.3062	3.9946
0.05	10	3	0.1	0.1	0.3586	0.2449	0.3523	0.9644	3.1294
0.05	10	0.5	0.1	0.1	0.7113	0.7306	0.2515	-0.0875	3.7189
0.05	10	0.8	0.1	0.1	0.6368	0.6537	0.293	0.0204	2.933
0.05	10	0.5	0.1	0.1	0.7131	0.7306	0.2515	-0.0875	3.7189
0.05	8	0.5	0.5	0.5	0.5508	0.5562	0.3175	0.3224	3.4354
0.05	8	0.5	0.6	0.6	0.5334	0.5354	0.3209	0.3535	3.3438
3	5	1.5	0.5	0.5	1.5862	1.4800	0.5797	4.4762	167.305
5	5	5	0.1	0.6	1.8104	1.6882	0.6583	4.5965	183.073
5	5	5	0.5	0.5	1.7102	1.6066	0.6749	4.2527	177.0605
5	5	5	0.9	-0.9	1.8399	1.7654	0.7528	2.1844	26.3308
0.1	15	1	0.9	0.9	0.6335	0.6611	0.279	-0.198	2.3025
0.1	15	0.5	0.9	0.9	0.7519	0.7787	0.2092	521.808	374033

3.2 Order Statistics and their moments

Order statistics (OS) have wide applications in climatology, life testing and reliability. Moments of OS are also designed for replacement policy with the prediction of failure of future items determined from few early failures.

Let $X_1,...,X_n$ be a random sample from the CRTMBIII model and let $X_{1:n},...,X_{n:n}$ be the corresponding order statistics. The pdf of the ith order statistic, say $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} F^{r+i-1}(x),$$

where $B(.,.)$ is the beta function.

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{r=0}^{n-i} \sum_{s=0}^{r+i-1} \sum_{\ell=0}^{s} a_{r,s,\ell} \; \alpha\beta x^{-\beta-1} \left(1+\gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}(s+r+2\ell+i)-1}$$

$$\times \left[\lambda_1 + 2(\lambda_2 - \lambda_1)\left(1+\gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)\left(1+\gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}} \right],$$

(21)

where $a_{r,s,\ell} = \lambda_1^{r+i-s-1}\left(\lambda_2 - \lambda_1\right)^{s-\ell}\left(1-\lambda_2\right)^{\ell}\left(-1\right)^r \binom{n-i}{r}\binom{r+i-1}{s}\binom{s}{\ell}$.

The qth ordinary moment of $X_{i:n}$ say $\mu'_q = E\left(X_{i:n}^q\right)$ is determined from (21) as

3.3 Incomplete Moments

Mean inactivity life; mean waiting time and inequality measures can be obtained from incomplete moments. For X~CRTMBIII$\left(\alpha, \beta, \gamma, \lambda_1, \lambda_2\right)$, the incomplete lower moments are

$$M'_r(z) = E_{X \leq z}(X^r) = \int_0^z x^r\, \alpha\beta x^{-\beta-1}\left[\lambda_1(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} + 2(\lambda_2-\lambda_1)(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}-1} + 3(1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{3\alpha}{\gamma}-1}\right]dx.$$

(22)

Letting $\gamma x^{-\beta} = y, x = \left(\dfrac{y}{\gamma}\right)^{-\frac{1}{\beta}}, \gamma\beta x^{-\beta-1}dx = -dy$, then

$$E_{X \leq z}(X^r) = \frac{\alpha}{\gamma}\gamma^{\frac{r}{\beta}}\left\{\begin{array}{l}\lambda_1\left[B\left(1-\frac{r}{\beta},\frac{\alpha}{\gamma}+\frac{r}{\beta}\right) - B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{\alpha}{\gamma}+\frac{r}{\beta}\right)\right] + \\ 2(\lambda_2-\lambda_1)\left[B\left(1-\frac{r}{\beta},\frac{2\alpha}{\gamma}+\frac{r}{\beta}\right) - B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{2\alpha}{\gamma}+\frac{r}{\beta}\right)\right] + \\ 3(1-\lambda_2)\left[B\left(1-\frac{r}{\beta},\frac{3\alpha}{\gamma}+\frac{r}{\beta}\right) - B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{3\alpha}{\gamma}+\frac{r}{\beta}\right)\right]\end{array}\right\},$$

$$\begin{aligned}&E_{X \leq z}(X^r)\\ &= \frac{\alpha}{\gamma}\gamma^{\frac{r}{\beta}}\left\{\begin{array}{l}\lambda_1 B\left(1-\frac{r}{\beta},\frac{\alpha}{\gamma}+\frac{r}{\beta}\right) + 2(\lambda_2-\lambda_1)B\left(1-\frac{r}{\beta},\frac{2\alpha}{\gamma}+\frac{r}{\beta}\right) + 3(1-\lambda_2)B\left(1-\frac{r}{\beta},\frac{3\alpha}{\gamma}+\frac{r}{\beta}\right) - \\ \lambda_1 B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{\alpha}{\gamma}+\frac{r}{\beta}\right) - 2(\lambda_2-\lambda_1)B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{2\alpha}{\gamma}+\frac{r}{\beta}\right) - 3(1-\lambda_2)B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{3\alpha}{\gamma}+\frac{r}{\beta}\right)\end{array}\right\},\end{aligned}$$

(23)

where $r < \beta$ and $B_{\gamma z^{-\beta}}(.,.)$ is the incomplete beta function.

For X~CRTMBIII$\left(\alpha, \beta, \gamma, \lambda_1, \lambda_2\right)$, the incomplete upper moments are

$$E_{X > z}(X^r) = \int_z^{\infty} x^r\, \alpha\beta x^{-\beta-1}\left[\lambda_1(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} + 2(\lambda_2-\lambda_1)(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}-1} + 3(1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{3\alpha}{\gamma}-1}\right]dx,$$

Letting $\gamma x^{-\beta} = y, x = \left(\dfrac{y}{\gamma}\right)^{-\frac{1}{\beta}}, \gamma\beta x^{-\beta-1}dx = -dy$, we arrive at

$$E_{X \geq z}(X^r) = \frac{\alpha}{\gamma}\gamma^{\frac{r}{\beta}}\left[\begin{array}{l}\lambda_1 B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{\alpha}{\gamma}+\frac{r}{\beta}\right) + 2(\lambda_2-\lambda_1)B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{2\alpha}{\gamma}+\frac{r}{\beta}\right) + \\ 3(1-\lambda_2)B_{\gamma z^{-\beta}}\left(1-\frac{r}{\beta},\frac{3\alpha}{\gamma}+\frac{r}{\beta}\right)\end{array}\right],$$

(24)

where $r < \beta$ and $B_{\gamma z^{-\beta}}(.,.)$ is the incomplete beta function.

The mean deviation about the mean $(\delta_1 = E|X - \mu|)$ and about the median $(\delta_2 = E|X - \tilde{\mu}|)$ can be written as $\delta_1 = 2\mu F(\mu) - 2\mu M_1'(\mu)$ and $\delta_2 = \mu - 2M_1'(\tilde{\mu})$ respectively, where $\mu = E(X)$ and $\tilde{\mu} = x_{0.5}$. The quantities $M_1'(\mu)$ and $M_1'(\tilde{\mu})$ can be obtained from (23). For specific probability p, Lorenz and Bonferroni curves are computed as $L(p) = \dfrac{M_1'(q)}{\mu'}$, $B(p) = L(p)|p$ and, where $q = Q(p)$.

3.4 Residual Life functions

For X~CRTMBIII $(\alpha, \beta, \gamma, \lambda_1, \lambda_2)$, the nth moment of the residual life, $m_n(z) = E[(X - z)^n | X > z]$, is

$$m_n(z) = \frac{1}{S(z)} \int_z^\infty (x - z)^n f(x) dx,$$

$$m_n(z) = \frac{1}{S(z)} \sum_{s=0}^n \binom{n}{s} (-z)^{n-s} E_{X>z}(X^s),$$

$$m_n(z) = \frac{1}{S(z)} \sum_{s=0}^n \binom{n}{s} (-z)^{n-s} \frac{\alpha}{\gamma} \gamma^{\frac{s}{\beta}} \left[\lambda_1 B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{\alpha}{\gamma} + \frac{s}{\beta} \right) + 2(\lambda_2 - \lambda_1) B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{2\alpha}{\gamma} + \frac{s}{\beta} \right) + 3(1 - \lambda_2) B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{3\alpha}{\gamma} + \frac{s}{\beta} \right) \right]. \quad (25)$$

The average remaining lifetime of a component at time z, say $m_1(z)$, or life expectancy is known as mean residual life (MRL) function is given by

$$m_1(z) = \frac{1}{S(z)} \sum_{s=0}^1 \binom{1}{s} (-z)^{1-s} \frac{\alpha}{\gamma} \gamma^{\frac{s}{\beta}} \left[\lambda_1 B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{\alpha}{\gamma} + \frac{s}{\beta} \right) \right.$$
$$\left. + 2(\lambda_2 - \lambda_1) B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{2\alpha}{\gamma} + \frac{s}{\beta} \right) + 3(1 - \lambda_2) B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{3\alpha}{\gamma} + \frac{s}{\beta} \right) \right].$$
$$(26)$$

For X1~CRTMBIII $(\alpha, \beta, \gamma, \lambda_1, \lambda_2)$, the nth moment of the reverse residual life, $M_n(z) = E[(z - X)^n | X \le z]$ is

$$M_n(z) = \frac{1}{F(z)} \int_0^z (z - x)^n f(x) dx$$

$$M_n(z) = \frac{1}{F(z)} \sum_{s=0}^n (-1)^s \binom{n}{s} z^{n-s} E_{X \le z}(X^s),$$

$$M_n(z) = \frac{1}{F(z)} \sum_{s=0}^n (-1)^s \binom{n}{s} z^{n-s} m_n(z) = \frac{1}{S(z)} \sum_{s=0}^n \binom{n}{s} (-z)^{n-s} \frac{\alpha}{\gamma} \gamma^{\frac{s}{\beta}} \left\{ \begin{array}{l} \lambda_1 B \left(1 - \frac{s}{\beta}, \frac{\alpha}{\gamma} + \frac{s}{\beta} \right) + 2(\lambda_2 - \lambda_1) B \left(1 - \frac{s}{\beta}, \frac{2\alpha}{\gamma} + \frac{s}{\beta} \right) + 3(1 - \lambda_2) B \left(1 - \frac{r}{\beta}, \frac{3\alpha}{\gamma} + \frac{s}{\beta} \right) - \\ \lambda_1 B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{\alpha}{\gamma} + \frac{s}{\beta} \right) - 2(\lambda_2 - \lambda_1) B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{2\alpha}{\gamma} + \frac{s}{\beta} \right) - 3(1 - \lambda_2) B_{\gamma z^{-\beta}} \left(1 - \frac{s}{\beta}, \frac{3\alpha}{\gamma} + \frac{s}{\beta} \right) \end{array} \right\}.$$
$$(27)$$

The waiting time z for failure of a component has passed with condition that this failure had happened in the interval [0, z] is called mean waiting time (MWT) or mean inactivity

time. The waiting time z for failure of a component of X with CRTMBIII distribution is defined by

$$M_1(z) = \frac{1}{F(z)} \sum_{s=0}^{1} (-1)^s \binom{1}{s} z^{1-s} \frac{\alpha}{\gamma} \gamma^{\frac{s}{\beta}} \left\{ \begin{array}{l} \lambda_1 B\left(1-\frac{s}{\beta}, \frac{\alpha}{\gamma}+\frac{s}{\beta}\right) + 2(\lambda_2-\lambda_1)B\left(1-\frac{s}{\beta}, \frac{2\alpha}{\gamma}+\frac{s}{\beta}\right) + 3(1-\lambda_2)B\left(1-\frac{r}{\beta}, \frac{3\alpha}{\gamma}+\frac{s}{\beta}\right) - \\ \lambda_1 B_{\gamma z^{-\beta}}\left(1-\frac{s}{\beta}, \frac{\alpha}{\gamma}+\frac{s}{\beta}\right) - 2(\lambda_2-\lambda_1)B_{\gamma z^{-\beta}}\left(1-\frac{s}{\beta}, \frac{2\alpha}{\gamma}+\frac{s}{\beta}\right) - 3(1-\lambda_2)B_{\gamma z^{-\beta}}\left(1-\frac{s}{\beta}, \frac{3\alpha}{\gamma}+\frac{s}{\beta}\right) \end{array} \right\}. \quad (28)$$

3.5 Stress-strength Reliability for CRTMBIII Distribution

Let $X_1 \sim$CRTMBII$(\alpha_1, \beta, \gamma, \lambda_1, \lambda_2)$ and $X_2 \sim$CRTMBII$(\alpha_2, \beta, \gamma, \lambda_1, \lambda_2)$, such that X_1 represents strength and X_2 represents stress. Then reliability of the component is:

$$R = \Pr(X_2 < X_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f(x_1, x_2) dx_2 dx_1 = \int_0^{\infty} f_{x_1}(x) F_{x_2}(x) dx,$$

$$R = \int_0^{\infty} \alpha_1 \beta x^{-\beta-1} \left[\lambda_1 (1+\gamma x^{-\beta})^{-\frac{\alpha_1}{\gamma}-1} + 2(\lambda_2-\lambda_1)(1+\gamma x^{-\beta})^{-\frac{2\alpha_1}{\gamma}-1} + 3(1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{3\alpha_1}{\gamma}-1} \right] \times$$

$$\left[\lambda_1 (1+\gamma x^{-\beta})^{-\frac{\alpha_2}{\gamma}} + (\lambda_2-\lambda_1)(1+\gamma x^{-\beta})^{-\frac{2\alpha_2}{\gamma}} + (1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{3\alpha_2}{\gamma}} \right] dx,$$

$$R = \left\{ \begin{array}{l} \int_0^{\infty} \alpha_1 \beta x^{-\beta-1} \left[\lambda_1^2 (1+\gamma x^{-\beta})^{-\frac{\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}-1} + 2\lambda_1(\lambda_2-\lambda_1)(1+\gamma x^{-\beta})^{-\frac{2\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}-1} + 3(\lambda_1-\lambda_1\lambda_2)(1+\gamma x^{-\beta})^{-\frac{3\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}-1} \right] + \\ \int_0^{\infty} \alpha_1 \beta x^{-\beta-1} \left[(\lambda_1\lambda_2-\lambda_1^2)(1+\gamma x^{-\beta})^{-\frac{\alpha_1}{\gamma}-\frac{2\alpha_2}{\gamma}-1} + 2(\lambda_2-\lambda_1)^2(1+\gamma x^{-\beta})^{-\frac{2\alpha_1}{\gamma}-\frac{2\alpha_2}{\gamma}-1} + 3(1-\lambda_2)(\lambda_2-\lambda_1)(1+\gamma x^{-\beta})^{-\frac{3\alpha_1}{\gamma}-\frac{2\alpha_2}{\gamma}-1} \right] + \\ \int_0^{\infty} \alpha_1 \beta x^{-\beta-1} \left[(\lambda_1-\lambda_1\lambda_2)(1+\gamma x^{-\beta})^{-\frac{\alpha_1}{\gamma}-\frac{3\alpha_2}{\gamma}-1} + 2(\lambda_2-\lambda_1)(1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{2\alpha_1}{\gamma}-\frac{3\alpha_2}{\gamma}-1} + 3(1-\lambda_2)^2(1+\gamma x^{-\beta})^{-\frac{3\alpha_1}{\gamma}-\frac{3\alpha_2}{\gamma}-1} \right] \end{array} \right\},$$

$$R = \left\{ \begin{array}{l} \dfrac{\lambda_1^2 \alpha_1}{(\alpha_1+\alpha_2)} + \dfrac{2\alpha_1(\lambda_1\lambda_2-\lambda_1^2)}{(2\alpha_1+\alpha_2)} + \dfrac{3\alpha_1(\lambda_1-\lambda_1\lambda_2)}{(3\alpha_1+\alpha_2)} + \\ \dfrac{\alpha_1(\lambda_2-\lambda_1)^2}{(\alpha_1+2\alpha_2)} + \dfrac{2\alpha_1(\lambda_1\lambda_2-\lambda_1^2)}{(2\alpha_1+2\alpha_2)} + \dfrac{3\alpha_1(1-\lambda_2)(\lambda_2-\lambda_1)}{(3\alpha_1+2\alpha_2)} + \\ \dfrac{\alpha_1(\lambda_1-\lambda_1\lambda_2)}{(\alpha_1+3\alpha_2)} + \dfrac{2\alpha_1(\lambda_2-\lambda_1)(1-\lambda_2)}{(2\alpha_1+3\alpha_2)} + \dfrac{3\alpha_1(1-\lambda_2)^2}{(3\alpha_1+3\alpha_2)} \end{array} \right\}. \quad (29)$$

Therefore the stress-strength reliability parameter R is independent of parameters β, γ, λ_1 and λ_2.

4. CHARACTERIZATIONS

In order to develop a stochastic function in a certain problem, it is necessary to know whether the selected function fulfills the requirements of the specific underlying probability distribution. To this end, it is required to study characterizations of the specific probability distribution. Here, we present two characterizations of the CRTMBIII distribution (i) ratio of the truncated moments and (ii) double truncated moments.

4.1 Ratio of Truncated Moments

We characterize the CRTMBIII distribution on the basis of a simple relationship between two truncated moments of functions of X [Theorem G (Glänzel; 1987)].

Proportion 4.1.1: Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let

$$h_1(x) = \frac{1}{\alpha}\left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2(\lambda_2 - \lambda_1)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1} + 3(1 - \lambda_2)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1}\right]^{-1}$$

and

$$h_2(x) = \frac{2}{\alpha}x^{-\beta}\left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2(\lambda_2 - \lambda_1)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1} + 3(1 - \lambda_2)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1}\right]^{-1}, x > 0.$$

The pdf of X is (10), if and only if $q(x)$ (in Theorem G) has the form $q(x) = x^\beta$, $x > 0$.

Proof. If X has pdf (10), then

$$\left(1 - F(x)\right)E\left(h_1(x)\big| X \geq x\right) = x^{-\beta}, x > 0,$$

$$\left(1 - F(x)\right)E\left(h_2(x)\big| X \geq x\right) = x^{-2\beta}, \ x > 0,$$

and

$$\frac{E[h_1(X)|X \geq x]}{E[h_2(X)|X \geq x]} = q(x) = x^\beta.$$

The differential equation $s'(x) = \frac{q'(x)h_2(x)}{q(x)h_2(x) - h_1(x)} = \frac{2\beta}{x}$ has solution $(x) = \ln x^{2\beta}$.

Therefore according to theorem G, X has pdf (11).

Corollary 4.1.1: Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let

$$h_2(x) = \frac{2}{\alpha}x^{-\beta}\left[\lambda_1 \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1} + 2(\lambda_2 - \lambda_1)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}-1} + 3(1 - \lambda_2)\left(1 + \gamma x^{-\beta}\right)^{-\frac{3\alpha}{\gamma}-1}\right]^{-1}, x > 0.$$

The pdf of X is (10) if and only if functions $q(x)$ and $h_1(x)$ satisfy the equation

$$\frac{q'(x)h_2(x)}{q(x)h_2(x) - h_1(x)} = \frac{2\beta}{x}.$$

Remark 4.1.1: The general solution of the above differential equation is

$$q(x) = x^{2\beta}\left[-\int 2\beta x^{-(2\beta+1)}\left(h_2(x)\right)^{-1} h_1(x)dx + D\right],$$

where D is a constant.

4.2 Doubly Truncated Moment

Here, we characterize the CRTMBIII distribution via doubly truncated moment.

Proposition 4.2.1: Let X:$\Omega \to (0, +\infty)$ be a continuous random variable. Then X has pdf (10) if and only if

$$E\left[\left[\lambda_1 + 2(\lambda_2 - \lambda_1)\left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}} + 3(1 - \lambda_2)\left(1 + \gamma x^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]^{-1}\bigg| x < X < y\right]$$

$$= \frac{\left(1 + \gamma y^{-\beta}\right)^{-\frac{\alpha}{\gamma}} - \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha}{\gamma}}}{[\bar{F}(x) - \bar{F}(y)]}, x > 0, y > 0. \quad (30)$$

Proof:
For random variable X with pdf (10), we have

$$E\left[\left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}}\right]^{-1} \Big| x < X < y\right]$$

$$= \frac{\int_y^x \left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma u^{-\beta})^{-\frac{2\alpha}{\gamma}}\right]^{-1} f(u)\, du}{\left[\bar{F}(x) - \bar{F}(y)\right]},$$

$$= \frac{\int_y^x \left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma u^{-\beta})^{-\frac{2\alpha}{\gamma}}\right]^{-1} \alpha\beta u^{-\beta-1}(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}-1}\left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma u^{-\beta})^{-\frac{2\alpha}{\gamma}}\right] du}{\left[\bar{F}(x) - \bar{F}(y)\right]},$$

$$= \frac{\int_y^x \alpha\beta u^{-\beta-1}(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}-1}\, du}{\left[F(y) - F(x)\right]} = \frac{(1+\gamma y^{-\beta})^{-\frac{\alpha}{\gamma}} - (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}}{\left[\bar{F}(x) - \bar{F}(y)\right]}.$$

Conversely, if (30) holds, then

$$\frac{\int_y^x \left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma u^{-\beta})^{-\frac{2\alpha}{\gamma}}\right]^{-1} f(u)\, du}{\left[\bar{F}(x) - \bar{F}(y)\right]} = \frac{(1+\gamma y^{-\beta})^{-\frac{\alpha}{\gamma}} - (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}}{\left[\bar{F}(x) - \bar{F}(y)\right]}$$

$$\int_y^x \left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma u^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma u^{-\beta})^{-\frac{2\alpha}{\gamma}}\right]^{-1} f(u)\, du = (1+\gamma y^{-\beta})^{-\frac{\alpha}{\gamma}} - (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}.$$

Differentiating with respect to y, we have

$$\left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma y^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma y^{-\beta})^{-\frac{2\alpha}{\gamma}}\right]^{-1} f(y) = \alpha\beta y^{-\beta-1}(1+\gamma y^{-\beta})^{-\frac{\alpha}{\gamma}-1},$$

or

$$f(x) = \alpha\beta x^{-\beta-1}(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1}\left[\lambda_1 + 2(\lambda_2 - \lambda_1)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} + 3(1-\lambda_2)(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}}\right], \quad x > 0,$$

which is pdf of the CRTMBIII distribution.

5. MAXIMUM LIKELIHOOD ESTIMATION

Here, we adopt maximum likelihood estimation technique for the CRTMBIII parameters. Let $\Phi = (\alpha, \beta, \gamma, \lambda_1, \lambda_2)$ be unknown parameter vector. The log likelihood function $\ell(\Phi)$ for the CRTMBIII distribution is

We can compute the maximum likelihood estimators (MLEs) of $\alpha, \beta, \gamma, \lambda_1, \lambda_2$ by solving equations 32-36 either directly or using quasi-Newton procedure, computer packages/ softwares such as R, SAS, Ox, MATLAB, MAPLE and MATHEMATICA,.

$$\frac{\partial}{\partial \alpha}\left(\ln L\left(x_i;\Phi\right)\right)=\frac{n}{\alpha}-\frac{1}{\gamma}\sum \ln\left(1+\gamma x_i^{-\beta}\right)-\frac{2}{\gamma}\sum \frac{\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}\ln\left(1+\gamma x_i^{-\beta}\right)\left[\left(\lambda_2-\lambda_1\right)+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}\right]}{\left[\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]}=0, \quad (32)$$

$$\frac{\partial}{\partial \beta}\left(\ln L\left(x_i;\Phi\right)\right)=\frac{n}{\beta}-\sum \ln x_i +\left(\frac{\alpha}{\gamma}+1\right)\sum \frac{\gamma x_i^{-\beta}\ln x_i}{\left(1+\gamma x_i^{-\beta}\right)}+2\sum x_i^{-\beta}\ln x_i\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1}\left\{\frac{\alpha\left(\lambda_2-\lambda_1\right)+3\alpha\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}}{\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}\right\}=0 \quad (33)$$

$$\frac{\partial}{\partial \gamma}\left(\ln L\left(x_i;\Phi\right)\right)=\frac{\alpha}{\gamma^2}\sum \ln\left(1+\gamma x_i^{-\beta}\right)-\left(1+\frac{\alpha}{\gamma}\right)\sum x_i^{-\beta}\left(1+\gamma x_i^{-\beta}\right)^{-1}+\frac{2\alpha}{\gamma}\sum \frac{\left\{\frac{\left(\lambda_2-\lambda_1\right)}{\left(1+\gamma x_i^{-\beta}\right)^{\frac{\alpha}{\gamma}}}\left[\frac{1}{\gamma}\ln\left(1+\gamma x_i^{-\beta}\right)-\frac{x_i^{-\beta}}{\left(1+\gamma x_i^{-\beta}\right)}\right]+\frac{3\left(1-\lambda_2\right)}{\left(1+\gamma x_i^{-\beta}\right)^{\frac{2\alpha}{\gamma}}}\left[\frac{\alpha}{\gamma^2}\ln\left(1+\gamma x_i^{-\beta}\right)-\frac{x_i^{-\beta}}{\left(1+\gamma x_i^{-\beta}\right)}\right]\right\}}{\left[\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]}=0, (34)$$

$$\frac{\partial}{\partial \lambda_1}\left(\ln L\left(x_i;\Phi\right)\right)=\sum\left[1-2\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}\right]\left[\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]^{-1}=0, \quad (35)$$

$$\frac{\partial}{\partial \lambda_2}\left(\ln L\left(x_i;\Phi\right)\right)=\sum\left[2\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}-3\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]\left[\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]^{-1}=0. \quad (36)$$

6. SIMULATION STUDY

In this section, we survey the performance of the MLEs of the parameters of the CRTMBIII distribution with respect to sample size n. This performance is done based on the following simulation study:

Step 1: Generate 1000 samples of size n from the CRTMBIII distribution based on the inverse cdf method.

Step 2: Compute the MLEs for 1000 samples, say $(\hat{\alpha},\hat{\beta},\gamma,\lambda_1,\hat{\lambda}_2)$ for i=1,2,...,1000 based on non-linear optimization algorithm with constraint matching to range of parameters.

Step 3: Compute the biases, mean squared errors and coverage probability of MLEs.

For this purpose, we have selected different arbitrarily parameter values and n=50,100,150,200 sample sizes. All codes are written in R and the results are summarized in Table 3. The result clearly shows that when the sample size increases, the mean square errors (MSEs) decrease. This shows the consistency of MLE estimators.

Table 3: Bias, MSEs and coverage probability (CP) of the CRTMBIII (1,8,1,0.5,0.5) , (2,3,1,0.5,0.5) (0.5,5,1.2,0.9,0.9) distribution.

Sample	Statistics	$\alpha = 1$	$\beta = 8$	$\gamma = 1$	$\lambda_1 = 0.5$	$\lambda_2 = 0.5$
n=50	Bias	0.472	1.638	0.761	0.104	0.180
	MSE	0.992	3.507	1.329	0.867	0.772
	CP	94.90	95.12	94.11	95.23	94.97
n=100	Bias	0.289	1.419	0.434	0.091	0.094
	MSE	0.718	1.127	1.071	0.530	0.423
	CP	95.30	94.89	95.04	95.01	94.99
n=200	Bias	0.117	0.912	0.210	0.070	0.087
	MSE	0.482	1.067	0.975	0.313	0.189
	CP	95.00	94.98	94.99	94.95	95.04
n=300	Bias	0.073	0.150	0.097	0.003	0.050
	MSE	0.210	0.829	0.151	0.113	0.093
	CP	95.07	95.01	94.95	94.99	95.02

| Sample	Statistics	$\alpha = 3$	$\beta = 2$	$\gamma = 1$	$\lambda_1 = 0.5$	$\lambda_2 = 0.5$
n=50	Bias	0.846	0.721	0.779	0.112	0.177
	MSE	1.232	1.221	1.211	0.921	0.646
	CP	95.10	94.97	94.64	95.09	94.98
n=100	Bias	0.789	0.544	0.452	0.095	0.820
	MSE	0.921	0.988	1.010	0.621	0.524
	CP	95.03	95.09	94.93	94.96	95.04
n=200	Bias	0.327	0.477	0.188	0.087	0.066
	MSE	0.768	0.661	0.923	0.320	0.168
	CP	95.09	95.04	95.03	95.05	95.02
n=300	Bias	0.112	0.187	0.078	0.005	0.032
	MSE	0.395	0.522	0.130	0.121	0.104
	CP	94.99	95.03	95.01	95.03	94.99
Sample	Statistics	$\alpha = 0.5$	$\beta = 5$	$\gamma = 1.2$	$\lambda_1 = 0.9$	$\lambda_2 = 0.9$
n=50	Bias	0.112	1.822	0.323	0.298	0.310
	MSE	0.315	2.301	0.748	0.664	0.734
	CP	95.08	95.61	96.06	94.78	95.22
n=100	Bias	0.099	0.921	0.214	0.166	0.201
	MSE	0.277	1.321	0.432	0.422	0.512
	CP	95.10	94.93	95.08	94.97	95.09
n=200	Bias	0.046	0.733	0.181	0.084	0.092
	MSE	0.121	0.887	0.312	0.128	0.196
	CP	95.04	95.31	95.09	95.05	94.89
n=300	Bias	0.010	0.171	0.084	0.055	0.069
	MSE	0.111	0.634	0.139	0.105	0.113
	CP	95.03	94.98	95.12	95.00	95.06

7. APPLICATIONS

We consider an application to data set such as strengths of 1.5 cm glass fibers for authentication of the flexibility, utility and potentiality of the CRTMBIII distribution. We compare the CRTMBIII distribution with TMBIII, MBIII, BIII, LL distributions. For selection of the optimum distribution, we compute the estimate of likelihood ratio statistics $(-2\hat{\ell})$, Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises (W*), Anderson Darling (A*), and Kolmogorov- Smirnov [K-S] statistics with p-values for all competing and sub distributions. We compute the MLEs and their standard errors (in parentheses). We also compute goodness of fit statistics (**GOFs**) values for the CRTMBIII, TMBIII, MBIII, BIII, LL models.

7.1 Strengths of Glass Fibers: The values of data about strengths of 1.5 cm glass fibers (Smith and Naylor; 1987 and Arifa et al.;2017) are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

A descriptive summary for the strengths of 1.5 cm glass fibers data set provides the following values: 63 (sample size), 0.55 (minimum), 2.24 (maximum), 1.59 (median), 1.506825 (mean), 0.3241257 (standard deviation), 21.5105 (coefficient of variation), -0.89993 (coefficient of skewness) and 3.92376 (coefficient of kurtosis). The boxplot (**Fig.**

3(a)) for strengths of glass fibers data is negatively skewed. The TTT (total time on test) plot (**Fig. 3(b)**) for strengths of glass fibers data is concave, which infers increasing failure rate. So, the BIII-ME distribution is suitable to model these data.
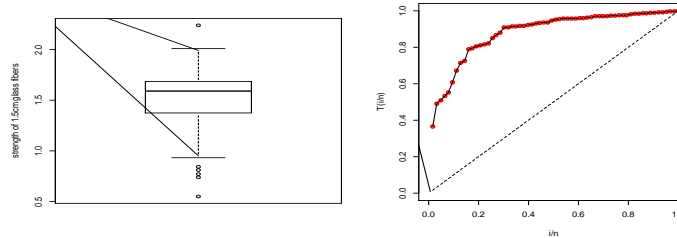
(a)   (b)

Figure 3 Boxplot (a) and TTT plot (b) for glass fiber data

Table 4 reports the MLEs (standard errors) and measures W*, A*, KS (p-values). Table 5 displays the values $-2\widehat{\ell}$, AIC, CAIC, BIC and HQIC.

Table 4: MLEs, (standard errors) and W*, A*, KS (p-values) for Strengths of Glass Fibers

Model	α	β	γ	λ_1	λ_2	W	A	K-S (p-value)
CRT MBIII	3343.164 (4457.274)	18.4436 (2.302738)	16264.06 (23802.86)	0.6463 (0.3772)	1.000000e-10 (0.8656)	**0.0390**	**0.2318**	**0.0782 (0.8359)**
TMBI II	15062.42 (3881.1386)	20.0016 (0.7276)	82081.20 (8543.3265)	0.4698 (0.3181)	1	0.0807	0.4523	0.1077 (0.4574)
MBIII	38353.3724 (4604.0351)	20.70762 (0.7271)	172677.4910 (6315.7470)	1	1	0.1135	0.6271	0.1292 (0.2434)
BIII	3.4417 (0.4505)	4.0886 (0.3357)	1	1	1	0.9554	5.1997	0.2462 (0.001)
LL	1	3.4506 (0.3414)	1	1	1	0.8045	4.4035	0.5346 (4.441e-16)

Table 5: $-2\widehat{\ell}$, AIC, CAIC, BIC and HQIC for Strengths of Glass Fibers

Model	$-2\widehat{\ell}$	AIC	CAIC	BIC	HQIC
CRTMBIII	19.17072	29.17072	30.22335	39.88639	33.38525
TMBIII	22.18592	30.18593	30.87558	38.75846	33.55755
MBIII	23.82574	29.82573	30.23251	36.25514	32.35445
BIII	73.76724	77.76725	77.96725	82.05352	79.45306
LL	136.9448	138.9448	139.0104	141.088	139.7877

From the tables 4 and 5, it is clear that our proposed model is best fitted, with smallest values for all GOFs and maximum p-value.

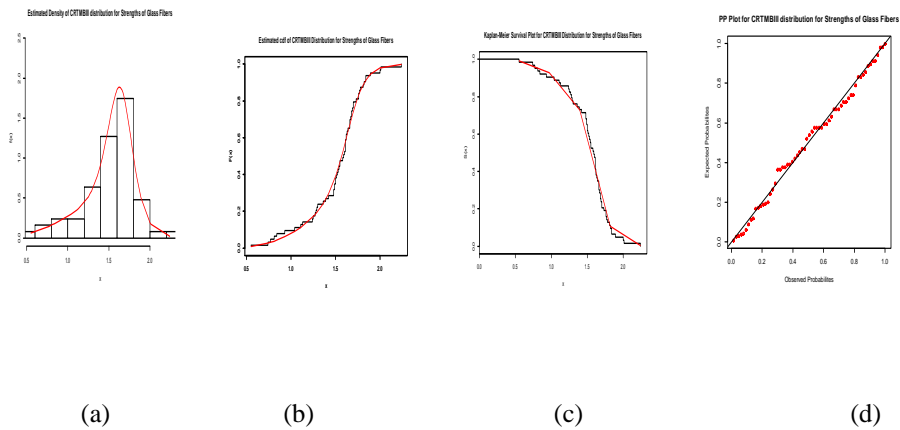   (a)   (b)   (c)   (d)

Figure 4: Fitted (a) pdf, (b) cdf, (c) survival and (d) PP plots for the CRTMBIII distribution to Strengths.

Figure 4 infers that the proposed model is closely fitted to strengths of glass fibers.

8. CONCLUDING REMARKS

We propose a very flexible distribution on the basis of the cubic transmuted mapping that is suitable for applications in survival analysis, reliability and actuarial science. The density function of CRTMBIII is symmetrical, right-skewed, left-skewed, exponential, arc, J and bimodal shaped. The flexible hazard rate of the proposed model can accommodate almost all types of shapes such as unimodal, bimodal, arc, increasing, decreasing, decreasing-increasing-decreasing, inverted bathtub and modified bathtub. We derive the important mathematical properties of the proposed distribution such as survival function, hazard function, reverse hazard function, cumulative hazard function, mills ratio, elasticity, quantile function, moments about the origin, and moments of order statistics, incomplete moments, inequality measures and stress-strength reliability measures. We characterize the proposed distribution via ratio of truncated moments and doubly truncated moment. We address the maximum likelihood estimation for the model parameters. We evaluate the precision of the maximum likelihood estimators via simulation study. We consider an application to real data set to illustrate the flexibility, utility and potentiality of the proposed model. We compute goodness of fit tests for examining the adequacy and competency of the proposed model. We ascertain empirically that the proposed model is suitable for strengths of glass fibers analysis.

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