ON THE ESTIMATION OF THE INVERSE WEIBULL DISTRIBUTION

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*Abstract:* We consider estimation of the probability density function and the cumulative density function of the inverse Weibull distribution. The following estimators are considered: uniformly minimum variance unbiased estimator, maximum likelihood estimator. Analytical expressions are derived for the bias and mean squared error. Simulation studies and a real data application show that the maximum likelihood estimator performs better.

*Key words*: Inverse Weibull distribution, Maximum likelihood estimator, Model selection criteria.

1. Introduction

A random variable X is said to have the inverse Weibull (IW) distribution if its cumulative distribution function (CDF) and probability density function (PDF) are given by

and

respectively, for x > α, where α is a location parameter, β > 0 is a shape parameter, and λ > 0 is a scale parameter.

The IW distribution is very flexible. By an appropriate choice of the shape parameter β, the PDF can assume a wide variety of shapes. The particular case for λ = 1 is the Fr´echet distribution. The particular case for β = 1 is the inverse exponential distribution. The particular case for β = 2 is the inverse Raleigh distribution.

Various methods have been developed for estimating the parameters of the IW distribution: estimation of confidence limits for reliability and tolerance limits (Calabria and Pulcini, 1989); maximum likelihood (ML) estimation of a two-parameter IW distribution with unique solutions (Calabria and Pulcini, 1990); Bayes two-sample prediction (Calabria and Pulcini, 1994); condi-tional confidence interval estimation based on censored generalized order statistics (Maswadah, 2003); estimation based on lower record values (Sultan, 2008); Bayesian and non-Bayesian esti-mation procedures based on generalized order statistics (Abd Ellah, 2012); estimation based on progressively censored data (Mohie El-Din et al., 2013); Bayes estimation for complete, type I and type II censored samples under general entropy and squared error loss functions (Kumar Singh et al., 2013); Bayesian estimation of the parameters of the proportional IW distribution under diﬀerent loss functions (Sindhu and Aslam, 2013); Bayes estimation of the scale parameter by con-sidering

quasi, gamma, and uniform priors under the squared error, entropy, and precautionary loss functions (Yahgmaei et al., 2013); ML estimation of the scale parameter (Yahgmaei et al., 2013); statistical inference of a two-parameter IW distribution based on a progressively type II censored sample (Sultan et al., 2014). Estimation methods for related Weibull distributions can be found in Teimouri and Gupta (2013) and Erisoglu and Erisoglu (2014).

The recent applications of the IW distribution have been widespread. We mention: life testing of Brazilian friction-resistant low alloy-high strength steel rails (De Souza et al., 2009); risk assessment and managing technical systems in case of mining industry (Radosavljevic et al., 2009); kinetic modeling of native cassava starch thermo-oxidative degradation (Jankovic, 2014).

Because of the numerous applications of the IW distribution, we feel the importance to investi-gate eﬃcient estimation of its PDF and CDF. We consider uniformly minimum variance unbiased (UMVU) and ML estimation methods. We have chosen these estimation methods because they are some of the most popular ones. In particular, ML estimation is the most widely used method.

Estimation of PDFs and CDFs is important for several reasons: i) assessment of the closeness between empirical and estimated CDFs in PP plots, see Section 6; ii) assessment of the close-ness between empirical and estimated inverse CDFs in QQ plots, see Section 6; iii) assessment of the closeness between empirical and estimated PDFs in density plots, see Section 6; iv) estima-tion of functionals of PDFs like the diﬀerential entropy, the negentropy, the R´enyi entropy, the Kullback-Leibler divergence and the Fisher information; v) estimation of functionals of CDFs like the cumulative residual entropy, the Bonferroni curve and the Lorenz curve; vi) estimation of func-tionals of both PDFs and CDFs like probability weighted moments, the hazard rate function, the reverse hazard rate function and the mean deviation about the mean.

Studies estimating PDFs and CDFs have appeared in the recent literature for other distribu-tions. Bagheri et al. (2014) derived estimators of the PDF and the CDF of a three-parameter generalized exponential-Poisson distribution when all but its shape parameter are assumed known.Bagheri et al. (2016a) derived estimators of the PDF and the CDF of a three-parameter expo-nentiated Gumbel distribution when all but its shape parameter are assumed known. Alizadeh et al. (2013) derived estimators of the PDF and the CDF of a two-parameter generalized Rayleigh distribution when all but its shape parameter are assumed known. Alizadeh et al. (2015) derived estimators of the PDF and the CDF of a three-parameter exponentiated Weibull distribution when all but its shape parameter are assumed known. Bagheri et al. (2016b) derived estimators of the PDF and the CDF of a three-parameter Weibull extension model when all but its shape parameter are assumed known.

The contents of this paper are organized as follows. The MLE and the UMVUE of the PDF and the CDF and their mean squared errors (MSEs) are derived in Sections 2 to 4. The estimators are compared by simulation and a real data application in Sections 5 and 6. Some conclusions are noted in Section 7. Throughout the paper (except for Sections 3 and 4), we assume all three parameters are unknown.

1. ML estimators of the PDF and the CDF

Here, we consider ML estimation of the PDF and the CDF when all three parameters of the IW distribution are unknown. Suppose X1, X2, . . . , Xn is a random sample from the IW distribution. Then, the log-likelihood function is

. (3)

The MLEs of α, β and λ say αb, βb and λb, respectively, can be obtained as the simultaneous solutions of

The second order partial derivatives of log L are

Using Lemmas A.1 to A.3 in the appendix, we obtain

where

and

1. **ML estimators of the PDF and the CDF when α and β are known**

Here, we consider the special case of the results in Section 2 when α and β are known. In this case, the MLE of λ is . The ML estimators of the PDF and the CDF reduce to

and

respectively. We know that the PDF of

is

for t > 0. By some elementary algebra, we can ﬁnd the PDF ofb = W say as

(4)

for w > 0. Theorem 1 calculatesand.

Theorem 1. We have

where Kν(·) denotes the modiﬁed Bessel function of the second kind of order ν.

Proof. By settingand using equation (4), we can write

where the last step follows by equation (3.471.9) in Gradshteyn and Ryzhik (2000). The proof is similar for.

1. **UMVU estimators of the PDF and the CDF when α and β are known**

In this section, we derive the UMVU estimators of the PDF and the CDF and their MSEs. We suppose X1, . . . , Xn is a random sample from the IW distribution given by (1)-(2) with both α and β known. Then,

is a complete suﬃcient statistic for the unknown parameter λ (when α and β are known) and the PDF of T is

(5)

for t > 0. According to Lehmann Scheﬀe theorem if fX1|T (x1|t) is the conditional PDF of X1 given T, we have

Where denotes the joint PDF of and T. Therefore is the UMVUE of f(x).

Lemma 1. The joint PDF of X1 and T can be expressed as

(6)

|  |  |  |  |
| --- | --- | --- | --- |
| Proof. The joint PDF of (X1, X2, . . . , Xn) is  .  In order to find the joint PDF of (X1, T ), we apply the transformation:  . |  |  |  |

We obtain the result by integrating the joint PDF of (Y1, Y2, . . . , Yn−1, T ) with respect to y2, y3, . . . , yn−1.

Theorem 2. Let T = t be given. Then,

is a UMVUE for f(x), and

Proof The proof that fe(x) is a UMVUE follows from (5) and Lemma . The proof that Fe(x) is a UMVUE follows by noting that (7) is the derivative of (8) with respect to x, i.e.,

，

which is the same as (7).

|  |  |  |
| --- | --- | --- |
| Theorem. The MSEs of and are  and  ,  respectively, where  ，  and  denotes the complementary incomplete gamma function.  Proof. By setting D2 = (n − 1)β(x − α)−(β+1) and k1 = (x−α)−β , we can write |  |  |

where the last step follows by the definition of the complementary incomplete gamma function. One can easily ﬁndand by setting r = 1, 2 into (9), and hence the MSE forThe proof for the expression for the MSE of is similar.

Note that the UMVU estimator of λ is and MSE

1. **Simulation study**

Here, we perform a simulation study to compare the finite sample performances of the following estimators: MLEs of the PDF and the CDF when all three parameters are unknown; MLEs of the PDF and the CDF assuming α and β are known; UMVUEs of the PDF and the CDF assuming α and β are known. The comparison is based on integrated mean squared errors (IMSEs), i.e., the integral of the MSE of the estimator of f(x) over 0 < x < ∞ or the integral of the MSE of the estimator of F (x) over 0 < x < ∞. The IMSEs were computed by generating one thousand replications of samples of size n = 5, 6, . . . , 100 from the IW distribution with (α, λ, β) = (1, 1, 0.1), (1, 1, 0.5), (1, 1, 5), (1, 0.1, 2), (1, 0.5, 2), (1, 0.5, 9). These parameter values were chosen to correspond to diﬀerent shapes of the IW distribution, see Figure 1. Figures 2 and 3 plot the deviations of the

IMSEs of the MLE from the IMSEs of the UMVUE versus n.

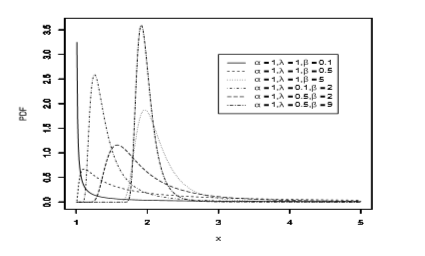
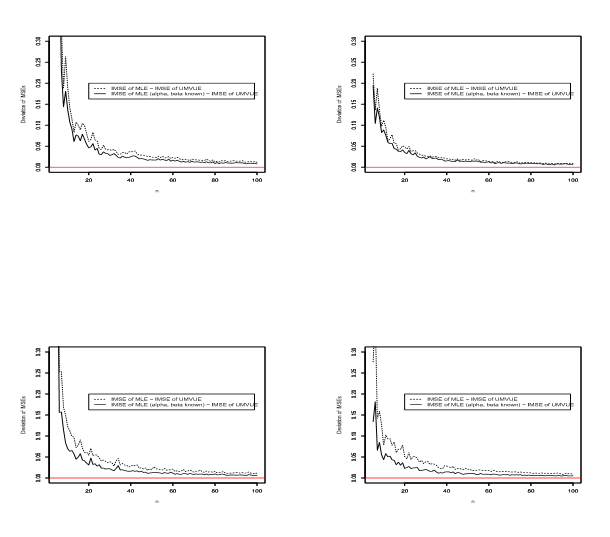


Figure 1: PDFs of the IW distribution for selected parameter values.



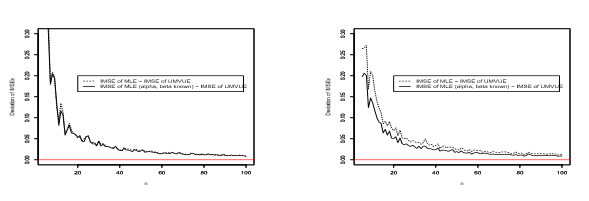


Figure 2: Deviations of the IMSEs of the MLE from the IMSEs of the UMVUE for f and (α,β,λ) = (1,1,0.1) (top left), F and (α,β,λ) = (1,1,0.1) (top right), f and (α,β,λ) = (1,1,0.5) (middle left), F and (α,β,λ) = (1,1,0.5) (middle right), f and (α,β,λ) = (1,1,5) (bottom left) and F and (α,β,λ) = (1,1,5) (bottom right).

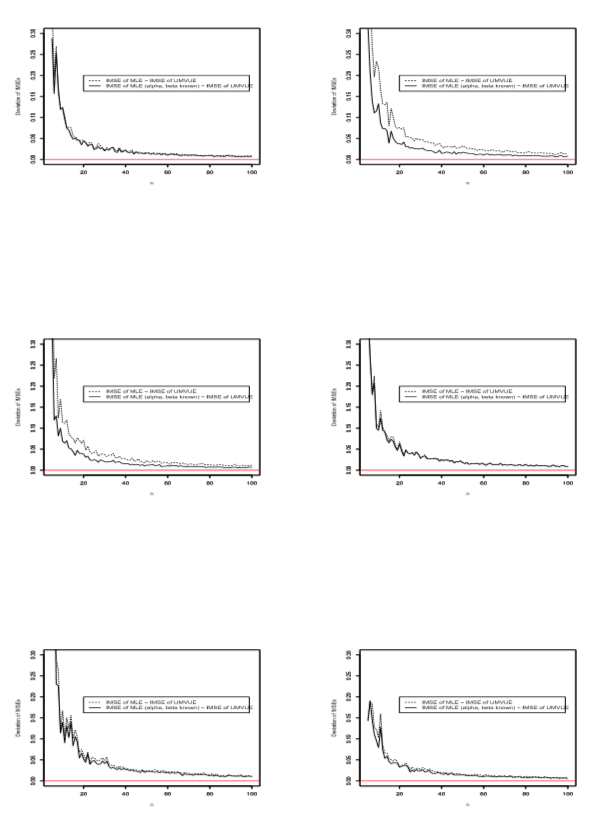


Figure 3: Deviations of the IMSEs of the MLE from the IMSEs of the UMVUE for f and (α,β,λ) = (1,0.1,2) (top left), F and (α,β,λ) = (1,0.1,2) (top right), f and (α,β,λ) = (1,0.5,2) (middle left), F and (α,β,λ) = (1,0.5,2) (middle right), f and (α,β,λ) = (1,0.5,9) (bottom left) and F and (α,β,λ) = (1,0.5,9) (bottom right).

We can see from the ﬁgures that the UMVU estimators of the PDF and the CDF assuming α and β are known are the most eﬃcient for all n. The ML estimators when α and β are known are the second most eﬃcient for all n. Note that the IMSEs of all the three estimators are nearly identical for all large n as expected by theory.

1. **Data analysis**

In this section, we consider a real life data set and illustrate the methods proposed in the previous sections. The data gives the breaking strengths of single carbon fibers of diﬀerent lengths:

2.247, 2.640, 2.842, 2.908, 3.099, 3.126, 3.245, 3.328, 3.355, 3.383, 3.572, 3.581, 3.681, 3.726, 3.727, 3.728, 3.783, 3.785, 3.786, 3.898, 3.912, 3.964, 4.050, 4.063, 4.082, 4.111, 4.118, 4.141, 4.216, 4.251, 4.262, 4.326, 4.402, 4.457, 4.466, 4.519, 4.542, 4.555, 4.614, 4.632, 4.634, 4.636, 4.678, 4.698, 4.738, 4.832, 4.924, 5.043, 5.099, 5.134, 5.359, 5.473, 5.571, 5.684, 5.721, 5.998, 6.060.

The data was taken from Crowder (2001).

In practical applications, all of the parameters of a model are usually unknown. Let x1, x2, . . . , xn denote the observations of the real data set, assumed to be a random sample from the IW distri-bution with α, β and λ unknown. We use the following procedures to estimate them:

The MLEs of α, β and λ say αb, βb and λb, respectively, as described in Section 2.

The percentile estimators (PCEs) of α, β and λ say αbpc, βbpc and λbpc, respectively, can be obtained by minimizing

,

where x(1) < x(2) < · · · < x(n) are the order statistics in the ascending order and .

The least squares estimators (LSEs) of α, β and λ say, ls and ,ls, respectively, can be obtained by minimizing

Where

The IW distribution was ﬁtted to the ﬁbre data by the MLE, the PCE, the LSE and the WLSE. The MLEs were computed by minimizing the minus of (3) numerically. For the minimization, we used the optim function in the R software (R Development Core Team, 2016). Table 1 gives the estimates of α, β, λ, the corresponding log-likelihoods and the corresponding p-values based on the Kolmogorov-Smirnov statistics. The p-values were computed as:

Kolmogorov-Smirnov statistics. The p-values were computed as:

1. compute the two-sample Kolmogorov-Smirnov statistic for H0 :

(α, β, λ) = the given estimates in Table 1 versus H1 : (α, β, λ) = not equal to the given estimates in Table 1;

1. compute the corresponding p-value;
2. simulate a random sample of size n = 57 from the empirical distribution of the data;
3. refit the inverse Weibull distribution to the simulated data by each of the four methods;
4. compute the two-sample Kolmogorov-Smirnov statistic for H0 : (α, β, λ) = the estimated parameters versus H1 : (α, β, λ) = not equal to the estimated parameters;
5. compute the corresponding p-value;
6. repeat steps 3 to 6 B times;
7. determine the empirical distribution of the B p-values obtained in step 7;
8. compare the observed p-value in step 2 to the empirical distribution in step 8 to yield the p-value reported in Table 1.

If the p-value is greater than 0.05 than the fitted model can be said to provide an adequate fit to the data. The larger the p-value the better the fit. The estimation method giving the largest p-value can be regarded as giving the best fit to the data. We took B = 1000.

We see from Table 1 that the log-likelihood value is the largest for the MLE, second largest for the WLSE, third largest for the PCE and the smallest for the LSE. The p-value is the largest for the MLE, second largest for the WLSE, third largest for the PCE and the smallest for the LSE.

Each p-value suggests that the IW distribution is a reasonable model for the fibre data.

Table 1: Estimates of the parameters, the corresponding log-likelihoods and the corresponding p-values.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Estimate of α | Estimate of β | Estimate of λ | Log-likelihood | p-value |
|  |  |  |  |  |  |
| MLE | -1.332681 | 5.881455 | 14462.558235 | -78.37166 | 0.061 |
| LSE | -0.1010659 | 6.9754366 | 14462.5582408 | -110.9826 | 0.051 |
| WLSE | -0.4817415 | 6.4801069 | 14462.5582412 | -90.49647 | 0.060 |
| PCE | -0.3809359 | 6.6041882 | 14462.5582413 | -93.97134 | 0.056 |
|  |  |  |  |  |  |

Table 2: Mean absolute and mean squared deviations based on CDFs.

MAD MSD

MLE 0.204 0.060

PCE 0.265 0.094

LSE 0.245 0.081

WLSE 0.250 0.085

Table 3: Mean absolute and mean squared deviations based on inverse CDFs.

MAD MSD

MLE 0.095 0.013

PCE 0.18 0.045

LSE 0.143 0.029

WLSE 0.153 0.032

Table 4: Mean absolute and mean squared deviations based on PDFs.

MAD MSD

MLE 0.313 0.150

PCE 0.380 0.186

LSE 0.391 0.194

WLSE 0.378 0.150

As mentioned in Section 1, one use of the estimators of PDFs and CDFs is in assessing the goodness of fit with respect to PP plots, QQ plots and density plots. Table 2 shows the mean absolute deviations (MADs) and mean squared deviations (MSDs) between the estimated and empirical CDFs. Table 3 shows the MADs and MSDs between the estimated and empirical inverse CDFs. Table 4 shows the MADs and MSDs between the estimated and empirical PDFs. Each of these tables shows that the values of MAD and MSD are smallest for the ML estimator.

We also compared the estimation methods by means of model selection criteria. The ones we considered are:

‘pure’ maximum likelihood (ML)

Akaike information criterion (AIC)

n corrected AIC (AICc) =

Bayes information criterion (BIC, also known as Schwarz criterion)

Hannan-Quinn criterion (HQC)

where log L(θ) denotes the log-likelihood, n denotes the number of observations and k denotes the number of parameters of the distribution. The smaller the values of these criteria the better the fit. For more discussion on these criteria, see Burnham and Anderson (2004) and Fang (2011).

Table 5: The values of model selection criteria.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | ML | AIC | BIC | AICc | HQC |
| MLE | 156.7433 | 162.7433 | 168.8725 | 163.1962 | 165.1253 |
| LSE | 221.9652 | 227.9652 | 234.0943 | 228.4180 | 230.3472 |
| WLSE | 180.9929 | 186.9929 | 193.1221 | 187.4458 | 189.3749 |
| PCE | 187.9427 | 193.9427 | 200.0718 | 194.3955 | 196.3247 |
|  |  |  |  |  |  |

Table 5 gives values of the model selection criteria for the four diﬀerent estimation methods.

We can see that the ML estimators give the smallest values for all five model selection criteria.

The WLS estimators give the second smallest values for all five criteria. The PC estimators give the third smallest values for all five criteria. The LS estimators give the largest values for all five criteria.

1. **Conclusions**

We have compared the UMVU and ML estimators for the PDF and the CDF of the IW distri-bution. Explicit expressions are given for the biases and MSEs of the UMVU and ML estimators.

We have compared the finite sample performances of the estimators by simulation and a real data application. The results show that the UMVU estimator performs the best in terms of the

IMSEs in the simulation study. The ML estimator performs the best in terms of the log-likelihood values, the p-values based on the Kolmogorov-Smirnov statistics, the mean absolute and mean squared deviations based on empirical and estimated CDFs, the mean absolute and mean squared deviations based on empirical and estimated inverse CDFs, the mean absolute and mean squared deviations based on empirical and estimated PDFs, AIC, AICc, BIC and HQC.

Some future work are to: i) consider other estimation methods like method of moments, general-ized method of moments, Bayesian estimation, bootstrapping, jackknifing, and empirical likelihood method; ii) extend this study to reparameterizations and extensions of the IW distribution.

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