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The Kummer Beta Normal: A Ne[1]w Useful-Skew Model

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Abstract: The normal distribution is the most popular model in applications to real data. We propose a new extension of this distribution, called the Kummer beta normal distribution, which presents greater flexibility to model scenarios involving skewed data. The new probability density function can be represented as a linear combination of exponentiated normal pdfs. We also propose analytical expressions for some mathematical quantities: Ordinary and incomplete moments, mean deviations and order statistics. The estimation of parameters is approached by the method of maximum likelihood and Bayesian analysis. Likelihood ratio statistics and formal goodnessof-fit tests are used to compare the proposed distribution with some of its sub-models and non-nested models. A real data set is used to illustrate the importance of the proposed model.

Key words: Bayesian analysis, Kummer beta generalized distribution, Maximum likelihood method, Moment, Normal distribution, Order statistic.

1. Introduction

The main motivation for statisticians to study new families of statistical distributions is to increase the flexibility to better model various data sets that cannot be properly fitted by the existing distributions. In many applied areas such as environmental and medical sciences, engineering, demography, biological studies, lifetime analysis, actuarial, economics, finance and insurance there is a clear need for extended forms of these distributions. Exponentiated generalized (EG), beta generalized (BG) (Eugene et al., 2002) and Kumaraswamy generalized (KwG) (Cordeiro and de Castro, 2011) families of distributions are very versatile to analyze different types of data. These families have been widely studied in statistics and some authors have developed several special EG, BG and KwG models. For EG models, Mudholkar and Srivastava (1993) and Mudholkar et al. (1995) defined the exponentiated Weibull (EW) distribution, Gupta et al. (1998) defined the exponentiated Pareto (EPa) distribution, Gupta and Kundu (2001) defined the exponentiated exponential (EE) distribution, Nadarajah and Gupta (2007) defined the exponentiated gamma (EGa) distribution and Cordeiro et al. (2011) defined the exponentiated generalized gamma (EGG) distribution. For BG models, Eugene et al. (2002) defined the beta normal (BN) distribution, Nadarajah and Kotz (2004) defined the beta Gumbel

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(BGu) distribution, Nadarajah and Gupta (2004) defined the beta Fr´echet (BF) distribution, Nadarajah and Kotz (2006) defined the beta exponential (BE) distribution and more recently, Pescim et al. (2010) and Parana´ıba et al. (2011) studied important mathematical properties of the beta generalized half-normal (BGHN) and beta Burr XII (BBXII) distributions, respectively. For KwG models, Cordeiro and de Castro (2011) defined the Kumaraswamy normal (KwN) distribution, Cordeiro et al. (2010) defined the Kumaraswamy Weibull (KwW) distribution, Pascoa et al. (2011) defined the Kumaraswamy generalized gamma distribution, Cordeiro et al. (2012b) defined the Kumaraswamy Gumbel (KwGu) distribution, Cordeiro et al. (2012c) defined the Kumaraswamy generalized half-normal (KwGHN) distribution and more recently, Parana´ıba et al. (2013) defined the Kumaraswamy Burr XII (KwBXII) distribution. However, the beta, Kumaraswamy and exponentiated generators do not provide flexibility to the extremes (right and left) of the probability density functions (pdfs). For this reason, they are not suitable for analyzing real data with high levels of asymmetry.

For an arbitrary baseline distribution $G(x; \gamma)$ with parameter vector γ and pdf $g(x; \gamma)$, Pescim et al. (2012) proposed the Kummer beta generalized (denoted by the prefix "KB-G" for short) family of distributions that provides greater flexibility to extremes. Its cumulative distribution function (cdf) is defined by

$$F_{\mathcal{KBG}}(x) = K \int_0^{G(x;\gamma)} t^{a-1}\ (1-t)^{b-1}\ \mathrm{e}^{-\mathrm{ct}}\ \mathrm{dt}, \tag{1}$$

where $a > 0$ and $b > 0$ are shape parameters which induce skewness, and thereby promote weight variation of the tails, whereas the parameter $-\infty < c < \infty$ "squeezes" the pdf to the left or right, i.e., it gives weights to the extremes of the pdfs. Here,

$$K^{-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ {}_1F_1(a; a+b; -c)$$

and,

$${}_1F_1(a; a+b; -c) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}\ (1-t)^{b-1}\ \mathrm{e}^{-\mathrm{ct}}\ \mathrm{dt} = \sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{a})_\mathrm{k}(-\mathrm{c})^\mathrm{k}}{(\mathrm{a}+\mathrm{b})_\mathrm{k}\ \mathrm{k}!}$$

is the confluent hypergeometric function (Abramowitz and Stegun, 1968), $\Gamma(\cdot)$ is the gamma function and $(d)k = d(d+1) \dots (d+k-1)$ denotes the ascending factorial.

The pdf corresponding to (1) can be expressed as

$$f_{\mathcal{KBG}}(x) = K\ g(x;\gamma)\ G(x;\gamma)^{a-1}\ [1-G(x;\gamma)]^{b-1}\ \exp\left[-c\ G(x;\gamma)\right]. \tag{2}$$

Equation (2) will be most tractable when both $G(x; \gamma)$ and $g(x; \gamma)$ have simple analytic expressions. Its major benefit is to offer more flexibility to extremes (right and/or left) of the pdfs and therefore it becomes suitable for analyzing data with high degree of asymmetry.

The KB-G class of distributions includes two important special cases: the beta-generalized (BG) distribution for c = 0 and the exponentiated generalized (EG) distribution for $c = 0$ and $b = 1$. Pescim et al. (2012), Cordeiro et al. (2014), Pescim et al. (2014) and Pescim and Nadarajah

(2015) defined the Kummer beta Weibull (KBW), Kummer beta generalized gamma (KBGG), Kummer beta Birnbaun-Saunders (KBBS) and Kummer beta gamma (KBGa) distributions by taking $G(x)$ and $g(x)$ to be the cdf and the pdf of the Weibull, generalized gamma, Birnbaum-Saunders and gamma distributions, respectively. They studied several mathematical properties of these distributions and showed clear evidence of the potential of the three skewness parameters when modeling real data.

The normal distribution is the most popular model in applications to real data. When the number of observations is large, it can serve as an approximation for other models. Over the past decades, several authors have proposed new generalizations based on the normal distribution for modeling real data sets; see, for example, Azzalini (1985), Eugene et al. (2002), Nadarajah (2005), Cordeiro and de Castro (2011), Cordeiro et al. (2012a), Nadarajah et al. (2014) and Alzaatreh et al. (2014) for skew-normal, beta normal, generalized normal, Kumaraswamy normal, McDonald normal, modified beta normal and gamma normal distributions, respectively. The emergence of such distributions in the statistics literature is only very recent.

The cdf and the pdf of the normal distribution with location parameter $-\infty < \mu < \infty$ and scale parameter $\sigma > 0$ are given by

$$G(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

and

$$g(x;\mu,\sigma) = \frac{1}{\sqrt{2\ \pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] = \frac{1}{\sigma}\ \phi\left(\frac{x-\mu}{\sigma}\right), \tag{3}$$

respectively.

In this paper, we propose the Kummer beta normal (denoted with the prefix "KBN") distribution and provide a comprehensive description of some of its mathematical properties with the hope that it will attract wider applications in many areas of research. The main motivation for this extension is that the new distribution is a highly flexible distribution which admits different degrees of kurtosis and asymmetry. The normal distribution represents only a special case of the KBN distribution.

The cdf and the pdf of the KBN distribution are obtained from equations (1) and (2) as

$$F(x) = K\int_0^{\Phi\left(\frac{x-\mu}{\sigma}\right)} t^{a-1}\ (1-t)^{b-1}\ \exp(-c\ t)dt,$$

And

$$f(x) = \frac{K}{\sigma}\ \phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1}\exp\left[-c\ \Phi\left(\frac{x-\mu}{\sigma}\right)\right], \tag{5}$$

where $x \in R$, $\mu \in R$ is a location parameter, $\sigma > 0$ is a scale parameter, a and b are positive shape parameters, and $c \in R$ is a real-valued shape parameter. Hereafter, we denote by X a random variable following (5), and write $X \sim KBN(a, b, c, \mu, \sigma)$. For $\mu = 0$ and $\sigma = 1$, we have the standard KBN distribution. This pdf has three shape parameters a, b and c allowing for a high degree of flexibility. The parameter c controls tail weights to the extremes of the distribution.

The study of the new distribution is important since it extends some distributions previously considered in the literature. In fact, the normal distribution (with parameters μ and σ) is clearly a basic exemplar for $a = b = 1$ and $c = 0$, with a continuous crossover towards distributions with different shapes (e.g., a specified combination of skewness and kurtosis). The KBN distribution contains as sub-models the beta normal (BN) distribution for $c = 0$ and the exponentiated normal (EN) distribution for $c = 0$, $b = 1$. Plots of the KBN pdf for selected parameter values are displayed in Figure 1. It is evident that the shapes of the new pdf are much more flexible than its sub-models.

The article is outlined as follows. In Section 2, we provide useful expansions for the pdf of the KBN distribution. We obtain explicit expressions for the ordinary and incomplete moments (Section 3) and order statistics (Section 4). In Section 5, we discuss some statistical inference like maximum likelihood method and Bayesian approach. A real data application given in Section 6 reveals the usefulness of the new distribution for analyzing real data. Concluding remarks are addressed in Section 7.

2. Useful expansions

Expansions for equations (4) and (5) can be derived using the concept of exponentiated distributions. Consider the exponentiated normal (EN) distribution with power parameter a > 0 defined by Y ~ EN(a, μ, σ), with the cdf and the pdf given by $H(y; a) = \Phi\left(\frac{y-\mu}{\sigma}\right)^{a}$ and $h(y; a) = \frac{a}{b}\phi\left(\frac{y-\mu}{\sigma}\right)\Phi\left(\frac{y-\mu}{\sigma}\right)^{a-1}$, respectively.

By expanding the term $\exp\left[-c\Phi\left(\frac{x-\mu}{\sigma}\right)\right]$ and using the binomial in equation (5), we obtain the linear combination (for a >0 integer)

$$f(x) = \sum_{j,k=0}^{\infty} w_{j,k}\ h(x; a+j+k, \mu, \sigma), \tag{6}$$

Where h(x;a+j+k,μ ,σ) denotes the EN (a+j+k,μ , σ) pdf and the coefficient $W_{j,k}$ is given by

$$w_{j,k} = \frac{K(-1)^{j+k}c^{j}}{j!(a+j+k)}\binom{b-1}{k}.$$

By integrating (6), we obtain

$$F(x) = \sum_{j,k=0}^{\infty} w_{j,k} \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a+j+k} \tag{7}$$

If a is positive non-integer, we can expand $\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a+j+k}$ as

$$\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a+j+k} = \sum_{r=0}^{\infty} s_r(a+j+k) \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{r}, \tag{8}$$

Where

$$s_r(m) = \sum_{k=r}^{\infty} (-1)^{k+r} \binom{m}{k} \binom{k}{r}$$

Thus, from equation (3), (7) and (8), the KBN cdf can be express as

$$F(x) = \sum_{r=0}^{\infty} \sum_{j,k=0}^{\infty} w_{j,k}\ s_r(a+j+k) \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{r}. \tag{9}$$

By differentiating (9) and changing indices, we can obtain

$$f(x) = \sum_{r=0}^{\infty} b_r\ h(x; r, \mu, \sigma), \tag{10}$$

Where $b_r = \sum_{j,k=0}^{\infty} w_{j,k}\ s_r(a+j+k)$. Equation (10) reveals that the KBN pdf is a linear combination of EN pdfs. So, several properties of the KBN distribution can be obtained by knowing those properties of the EN distribution.

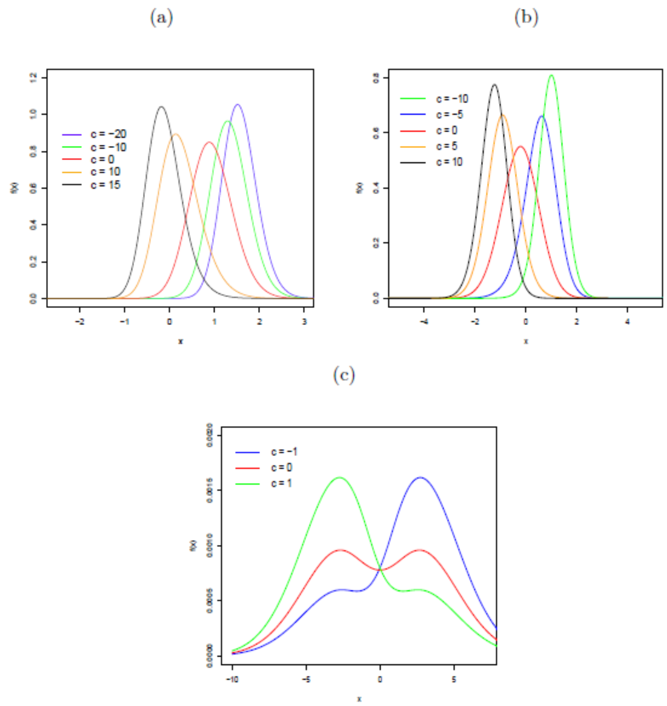

Figure 1: Plots of the KBN pdf for some parameter values: (a) KBN(8,2,*c*,0,1), (b) KBN(1.5, 2,*c*,0,1) and (c) KBN (0.1,0.1,*c*,0,1)pdfs (the red lines represent the BN pdfs).

3. Ordinary and incomplete moments

Hereafter, let X denote the KBN(a, b, c, μ, σ) random variable. The *s*th moment of *X* for μ = 0 and σ = 1 can be expressed from (10) as

$$\mu_s' = \mathrm{E}\left(X^s\right) = \sum_{r=0}^{\infty} b_r \int_{-\infty}^{\infty} x^s \; h(x; r, 0, 1) dx$$

and then

$$\mu_s' = \sum_{r=0}^{\infty} b_r^* \;\; \tau_{s,r},$$

where $b_r^* = r\, b_r$ and $\tau_{s,r} = \int_{-\infty}^{\infty} x^s \phi(x) \Phi(x)^{r-1} dx$ is the (*s*,*r*)th probability weighted moment (PWM) (for s and r positive integers) of the nrmal distribution.

Nadarahah(2008) demonstrated that the (*s.r*)th PWM of the normal distribution can be expressed in terms of the Laricella function of type A (Exton, 1978) as

$$\tau_{s,r} \;\; = \;\; \frac{2^{\frac{s}{2}+1-r}}{\sqrt{\pi}} \sum_{p=0, p+s \text{ even}}^{r-1} \binom{r-1}{p} \left(\frac{2}{\sqrt{\pi}}\right)^p \Gamma\left(\frac{p+s+1}{2}\right)$$

$$\times F_A^{(p)}\left(\frac{p+s+1}{2};\frac{1}{2},\ldots,\frac{1}{2};\frac{3}{2},\ldots,\frac{3}{2};-1,\ldots,-1\right),$$

where

$$F_A^{(n)}\left(a;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right) \;=\; \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}\,(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_{n!}}$$

is the Lauricella function of type A and the Pochhammer symbol $(a)k = a(a + 1) \ldots (a + k - 1)$ indicates the kth rising factorial power of a with the convention $(a)_0 = 1$.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis of the KBN distribution as a function of c for selected values of a and b for $\mu = 0$ and $\sigma = 1.0$ are displayed in Figures 2 and 3. Figures 2a and 2b immediately indicate that the additional parameter c promotes high levels of asymmetry.

(a) (b)

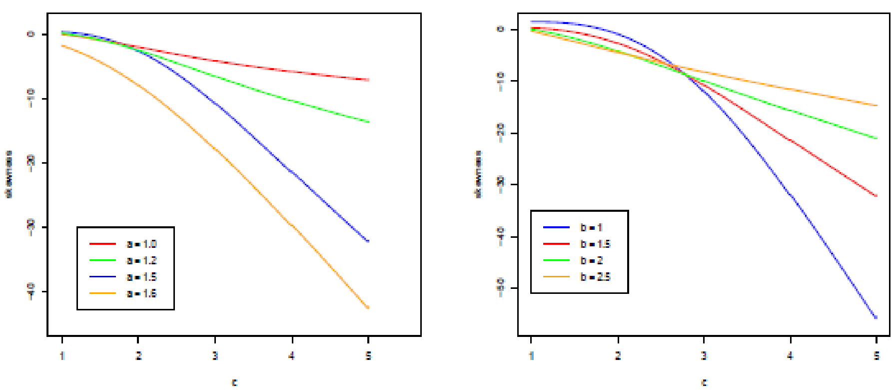

Figure 2: Skewness of the KBN distribution as a function of c for some values of a and b for μ =0 and σ =1.0 (a) b=1.5 and (b) a=1.5

(a) (b)

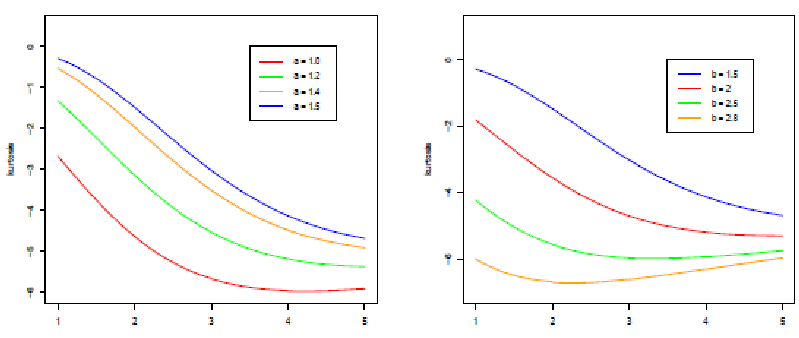

Figure 3: Kurtosis of the KBN distribution as a function of c for some values of a and b for μ =0 and σ =1.0. (a) b= 1.5 and (b) a =1.5.

The *s*th incomplete moment of *X* is defined by $m_s(y) = E(X^s I_{X<y}) = \int_{-\infty}^{y} x^s f(x)dx$. Consider the case $\mu = 0$ and $\sigma = 1$. Based on equation (10), $m_s(y)$ reduces to

$$m_s(y) = \sum_{r=0}^{\infty} b_r^* \int_{-\infty}^{y} x^s\ \phi(x)\ \Phi(x)^{r-1} dx, \tag{12}$$

where b_r^* is defined in (11).

We can write Φ(x) as a power series$\sum_{m=0}^{\infty} \Phi(x) = \eta_m x^m$, where $\eta_0 = \frac{\left(1+\sqrt{\frac{2}{\pi}}\right)^{-1}}{2}$,$\eta_{2m+1} = \frac{(-1)^m}{\sqrt{2\pi}2^m(2m+1)m!}$ for m = 0, 1, 2, … and $\eta_{2m} = 0$ for m = 1, 2, … . Using an identity given by Gradshteyn and Ryzhik (2007) for a power series raised to a positive integer *j*,

$$\left(\sum_{i=0}^{\infty} a_i\ x^i\right)^j = \sum_{i=0}^{\infty} c_{j,i}\ x^i, \tag{13}$$

where the coefficients $c_{j,i}$ (for i = 1,2,…) are easily obtained from the recurrence equation

$$c_{j,i} = (ia_0)^{-1} \sum_{m=1}^{i} [m(j+1) - i]\ \ a_m\ c_{j,i-m} \tag{14}$$

and $c_{j,0} = a_0^j$.

Further, using (13), we have

$$\Phi(x)^{r-1} = \sum_{m=0}^{\infty} \eta_{r-1,m}\ x^m, \tag{15}$$

where the coefficients $\eta_{r-1,m}$ can be determined from the recurrence equation (14). Thus, using (15) and changing variable in integral (12), the *s*th incomplete moment of *X* is given by

$$m_s(y) = \begin{cases} \sum_{r,m=0}^{\infty} b_r^* \eta_{r-1,m} \left[E\left(X^{s+m}\right) - \frac{1}{\sqrt{2\pi}} 2^{\frac{s+m-1}{2}} \Gamma\left(\frac{s+m+1}{2}, \frac{y^2}{2}\right)\right], & \text{if } y > 0, \\ \frac{1}{\sqrt{2\pi}} \sum_{r,m=0}^{\infty} b_r^* \eta_{r-1,m} (-1)^{m+s} 2^{\frac{s+m-1}{2}} \Gamma\left(\frac{s+m+1}{2}, \frac{y^2}{2}\right), & \text{if } y \le 0, \end{cases} \tag{16}$$

where $\Gamma(a,x) = \int_x^{\infty} t^{a-1}\, e^{-t} dt$ denotes the complementary incomplete gamma function.

Consider the case $\mu = 0$ and $\sigma = 1$. We can derive the mean deviations of *X* about the mean μ_1' and about the median M (δ_2) in terms of the first incomplete moments. The median can be obtained by inverting $F(M) = K\int_0^{\Phi(M)} t^{a-1}(1-t)^{b-1}e^{-ct}\, dt = \frac{1}{2}$ numerically. They can be expressed as

$$\delta_1 = 2\left[\mu_1'\ F\left(\mu_1'\right) - m_1\left(\mu_1'\right)\right], \qquad \delta_2 = \mu_1' - 2m_1\left(M\right),$$

Where $m_1(\cdot)$ is the first incomplete moment of X given by (16) with s = 1. We have

$$m_1(y) = \begin{cases} \sum_{r,m=0}^{\infty} b_r^* \eta_{r-1,m}\left[\mathrm{E}\left(X^{1+m}\right) - \frac{1}{\sqrt{2\pi}} 2^{\frac{m}{2}} \Gamma\left(\frac{m+2}{2}, \frac{y^2}{2}\right)\right], & \text{if } y > 0, \\ \frac{1}{\sqrt{2\pi}} \sum_{r,m=0}^{\infty} b_r^* \eta_{r-1,m}(-1)^{m+1} 2^{\frac{m}{2}} \Gamma\left(\frac{m+2}{2}, \frac{y^2}{2}\right), & \text{if } y \leq 0. \end{cases} \tag{17}$$

The measure δ_1 and δ_2 can be calculated from (17) by setting $y = \mu_1'$ and y = M, respectively. An application of the mean deviations is to the Lorenz and Bonferroni curves defined by $L(\pi) = m1(q)/\ \mu_1'$ and $B(\pi) = m1(q)/\pi\ \mu_1'$, respectively, where $q = F^{-1}(\pi)$ can be computed foe a given probability π by inverting (4) numerically. These curves have applications in several fields and can be calculated from equation (17).

4. Order Statisitcs

Order statistics have been used in a wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials.

Suppose Z_1 , ..., Z_n is a random sample from the standard KBN distribution and let $Z_{1:n} < \dots < Z_{i:n}$ denote the corresponding order statistics. The pdf $f_{i:n}(z)$ of the i^{th} order statistic can be written as

$$f_{i:n}(z) = \frac{f(z)\ n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(z)^{i+j-1}.$$

We now demonstrate that $f_{i:n}$ (z) can be written as a linear combination of standard EN pdfs. First, we provide an expansion for the cdf of the standard KBN distribution. Using (9) and (10), the pdf of the i^{th} order statistic, $Z_{i:n}$, can be expressed as

$$f_{i:n}(z) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\sum_{r=0}^{\infty} b_r\ r\ \phi(z)\ \Phi(z)^{r-1}\right] \left[\sum_{r=0}^{\infty} b_r\ \Phi(z)^r\right]^{i+j-1}.$$

From equation (13), we obtain

$$\left[\sum_{r=0}^{\infty} b_r\ \ \Phi(z)^r\right]^{i+j-1} = \sum_{r=0}^{\infty} b_{i+j-1,r}\ \Phi(z)^r,$$

where $b_{i+j-1,0} = b_0^{i+j-1}$ and

$$b_{i+j-1,r} = (rb_0)^{-1} \sum_{m=1}^{r} [m\ (i+j) - \ r]\ b_m\ b_{i+j-1,r-m}.$$

Hence, the pdf of the i^{th} order statistic for the standard KBN distribution can be expressed as

$$f_{i:n}(z) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{r}{r+s}\, b_{i+j-1,s}\, b_r\, h(z; r+s). \tag{18}$$

Equation (18) is the main result of this section. It gives the pdf of the standard KBN order statistics as a linear combination of standard EN pdfs. So, several mathematical quantities of standard KBN order statistics like ordinary moments, generating function, and mean deviations follow immediately from those quantities of the standard EN distribution.

5. Inference and estimation

5.1 Maximum likelihood method

In this section, the estimation of the model parameters of the KBN distribution will be investigated by maximum likelihood. Let $X = (X_1, \ldots, X_n)$ be a random sample from this distribution with unknown parameter vector $\theta = (a, b, c, \mu, \sigma)^T$. The total log-likelihood function for θ is

$$\begin{aligned} \ell(\boldsymbol{\theta}) \ &= \ n\log(K) - \frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 + (a-1)\sum_{i=1}^{n}\log\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right] \\ &+ (b-1)\sum_{i=1}^{n}\log\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right] - c\sum_{i=1}^{n}\Phi\left(\frac{x_i-\mu}{\sigma}\right). \end{aligned} \tag{19}$$

the elements of score vector are

$$\begin{aligned} U_\mu(\boldsymbol{\theta}) \ &= \ -\frac{n\mu}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{a-1}{\sigma\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]}{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right\} \\ &+ \frac{b-1}{\sigma\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]}{\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]}\right\} + \frac{c}{\sigma\sqrt{2\pi}}\sum_{i=1}^{n}\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right], \\ U_\sigma(\boldsymbol{\theta}) \ &= \ -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{a-1}{\sigma^2\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{(x_i-\mu)\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]}{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right\} \end{aligned}$$

$$+\frac{b-1}{\sigma^2\sqrt{2\pi}}\sum_{i=1}^{n}\left\{\frac{(x_i-\mu)\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]}{\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]}\right\}$$

$$+\frac{c}{\sigma^2\sqrt{2\pi}}\sum_{i=1}^{n}(x_i-\mu)\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right],$$

$$U_a(\boldsymbol{\theta})=\frac{n}{K}\frac{\partial K}{\partial a}+\sum_{i=1}^{n}\log\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right],$$

$$U_b(\boldsymbol{\theta})=\frac{n}{K}\frac{\partial K}{\partial b}+\sum_{i=1}^{n}\log\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right],$$

$$U_c(\boldsymbol{\theta})=\frac{n}{K}\frac{\partial K}{\partial c}-\sum_{i=1}^{n}\Phi\left(\frac{x_i-\mu}{\sigma}\right),$$

where the partial derivateives of *K* in relation to *a*, *b* and *c* are

$$\frac{\partial K}{\partial a}=-\frac{\left\{[\psi(a)-\psi(a+b)]\ {}_1F_1(a,a+b,-c)+\frac{\partial\ {}_1F_1(a,a+b,-c)}{\partial a}\right\}}{B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2},$$

$$\frac{\partial K}{\partial b}=-\frac{\left\{[\psi(b)-\psi(a+b)]\ {}_1F_1(a,a+b,-c)+\frac{\partial\ {}_1F_1(a,a+b,-c)}{\partial b}\right\}}{B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2},$$

$$\frac{\partial K}{\partial c}=\frac{a\ {}_1F_1(a+1,a+b+1,-c)}{(a+b)B(a,b){}_1F_1(a,a+b,-c)},$$

$$\frac{\partial\ {}_1F_1(a,a+b,-c)}{\partial a}=-[\psi(a)-\psi(a+b)]\ {}_1F_1(a,a+b,-c)-\sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{k!(a+b)_k}\ [\psi(a+b+k)-\psi(a+k)],$$

and

$$\frac{\partial\ {}_1F_1(a,a+b,-c)}{\partial b}=\psi(a+b)\ {}_1F_1(a,a+b,-c)+\sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{k!(a+b)_k}\ \psi(a+b+k),$$

where ψ (x) and d log Γ (x)/dx denotes the digamma function.

Maximization of (19) can be performed by using well established routines like the nlm routine or optimize in the R statistical package. Setting these equations to zero, **U**($\boldsymbol{\theta}$) = **0**, and solving them simultaneously yields the maximum likelihood estimate (MLE) $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques like the Newton-Raphson algorithm.

Most For interval estimation and hypothesis tests on the parameters in $\boldsymbol{\theta}$, we require the 5 X 5 total observed information matrix $\mathbf{J}(\boldsymbol{\theta}) = -\{ U_{rs} \}$, where the elements U_{rs}, for r, $s = \mu , \sigma$, a, b, c can be obtained numerically. The estimated asymptotic multivariate normal $N_5\,(0, J(\hat{\theta})^{-1})$ distribution of $\hat{\boldsymbol{\theta}}$ can be used to construct approximate confidence regions for parameters. An asymptotic confidence interval with significance level γ for each parameter θ_r is given by

$$\mathrm{ACI}\,(\theta_r, 100(1-\gamma)\%) = \left(\hat{\theta}_r - z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}, \hat{\theta}_r + z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}\right),$$

where $\hat{\kappa}^{\theta_r,\theta_r}$ is the rth diagonal element of $\mathbf{J}(\boldsymbol{\theta})^{-1}$ estimated at $\hat{\boldsymbol{\theta}}$ for r = 1,2,3,4,5 and $z_{\gamma/2}$ is the 1 $\gamma/2$ quantile of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for comparing the new distribution with some of its special models. For example, we may adopt the LR statistic to check if the fit using the KBN distribution is statistically "superior" to a fit using the normal distribution for a given data set. In any case, considering the partition $\theta = (\theta_1^{\mathrm{T}}, \theta_2^{\mathrm{T}})^{\mathrm{T}}$, tests of hypotheses of the type $H_0 : \theta_1 = \theta_1^{(0)}$ versus $H_A: \theta_1 \neq \theta_1^{(0)}$ can be performed using the LR statistic $w = 2\{\ell(\hat{\theta}) - \ell(\tilde{\theta})\}$, where $\hat{\theta}$ and $\tilde{\theta}$ are the estimates of θ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector θ_1 of interest. The LR test rejects H_0 if $w > \xi\gamma$, where $\xi\gamma$ denotes the upper 100γ % point of the χ_q^2 distribution.

5.2 Bayesian inference

As it is well-known, the Bayesian approach allows for the incorporation of previous knowledge of the parameters through informative prior pdfs. When this information is not available, we can consider a non-informative prior. In the Bayesian context, the information referring to the model parameters is obtained through a posterior marginal distribution. Two difficulties usually arise. The first one refers to obtaining the marginal posterior distribution, and the second to the calculation of the moments of interest. Both cases require numerical integration that, many times, do not present an analytical solution. To overcome these problems, we use the simulation methods based on the Markov Chain Monte Carlo (MCMC), like the Gibbs sampler and Metropolis-Hastings algorithms.

Since we have no prior information from historical data or from previous experiments, we assign conjugate but weakly informative prior distributions to the parameters. Since we assume an informative (but weakly) prior distribution, the posterior distribution is a well-defined proper distribution. We assume that the parameters (a, b, c, μ and σ) have independence priors and consider that the joint prior distribution of all unknown parameters has a pdf given by

$$\pi(a|x,b,c,\mu,\sigma) \propto K^n \prod_{i=1}^{n} \left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{a-1} \cdot \pi(a),$$

$$\pi(b|x,a,c,\mu,\sigma) \propto K^n \prod_{i=1}^{n} \left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{b-1} \cdot \pi(b),$$

$$\pi(c|x,a,b,\mu,\sigma) \propto K^n \exp\left[-c\sum_{i=1}^{n}\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right] \cdot \pi(c),$$

$$\pi(\mu|x,a,b,c,\sigma) \propto \exp\left[-c\sum_{i=1}^{n}\Phi\left(\frac{x_i-\mu}{\sigma}\right) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$
$$\cdot\prod_{i=1}^{n}\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{b-1}\cdot\pi(\mu),$$

and

$$\pi(\sigma|x,a,b,c,\mu) \propto \frac{1}{\sigma^n} \exp\left[-c\sum_{i=1}^{n}\Phi\left(\frac{x_i-\mu}{\sigma}\right) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$
$$\cdot\prod_{i=1}^{n}\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{b-1}\cdot\pi(\sigma).$$

Since the full conditional distributions do not have explicit expressions, we require the use of the Metropolis-Hastings algorithm to generate the variables a, b, c, μ and σ for the KBN distribution.

6. Application – INPC data

This section contains an application of the Kummer beta normal (KBN) distribution to a real data referred to as INPC data. We shall compare the fits of the KBN distribution with those of two sub-models (the beta normal (BN) and normal distributions) and also to the following non-nested models: the McDonald Normal (McN) (Cordeiro et al., 2012a), the gamma-normal (GN) (Alzaatreh et al., 2014) and the modified beta normal (MBN) (Nadarajah et al., 2014) distributions.

The INPC is a national index of consumer prices of Brazil, released by IBGE (Brazilian Institute of Geography and Statistics). The period of collection goes from day 01 to 30 of the reference month and the target population includes families dwelling in the urban areas, whose head of the household is considered the main employee. The survey was conducted in the metropolitan regions of Bel´em, Belo Horizonte, Bras´ılia, Curitiba, Fortaleza, Goiˆania, Porto Alegre, Recife, Rio de Janeiro, Sao Paulo and Salvador. The data set was extracted from the IBGE database available at http : //www.ibge.gov.br and reported by De Morais (2009). Table 1 presents a descriptive summary for the INPC data set and suggests a skewed distribution with high degrees of skewness and kurtosis.

Table 2: MLEs and the corresponding SEs (given in parentheses) of the model parameters for the INPC data and the measure AIC and BIC.

Model	μ	σ	a	b	c	AIC	BIC
KBN	0.4514	0.5806	4.3419	0.2658	9.4879	238.5	253.8
	(0.4378)	(0.1646)	(1.8986)	(0.0035)	(0.0025)		
BN	-0.4391	0.4686	5.3041	0.2905	0	256.0	268.3
	(0.1590)	(0.0028)	(2.4669)	(0.0431)	(-)		
Normal	0.6442	0.5988	1	1	0	288.5	294.6
	(0.0477)	(0.0337)	(-)	(-)	(-)		
Gamma-normal	μ	σ	α	β			
	0.3098	0.3771	0.7881	3.3616	(-)	278.2	290.4
	(0.2000)	(0.0028)	(0.2176)	(0.4486)	(-)		
McN	μ	σ	a_1	b_1	c_1		
	-1.2530	0.5993	13.9336	0.2858	3.8102	251.1	266.4
	(0.0205)	(0.0178)	(0.0631)	(0.0307)	(0.0412)		
MBN	μ	σ	a_2	b_2	β_1		
	-0.3192	0.4578	5.9186	0.2954	1.70004	260.1	275.3
	(0.0037)	(0.0037)	(4.0527)	(0.0272)	(1.3441)		

Table 3: LR statistics for the INPC data.

Model	Hypotheses	Statistic w	p-value
KBN vs BN	$H_0 : c = 0$ vs $H_1 : H_0$ is false	19.53	< 0.0001
KBN vs Normal	$H_0 : a = b = 1$ and $c = 0$ vs $H_1 : H_0$ is false	55.99	< 0.0001

(i) Maximum likelihood estimation

Table 2 gives the MLEs and the corresponding SEs (given in parentheses) of the model parameters and the values of the following statistics for some models: Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The computations were performed using the statistical software R. The AIC and BIC values for the KBN model are the smallest values among those fitted submodels and non-nested models.

A formal test of the need for the third skewness parameter in KB-G distributions can be based on the LR statistics. The results of this test are shown in Table 3 for the INPC data set. We reject the null hypotheses of the LR test in favor of the KBN distribution. The rejection is extremely highly significant and it gives clear evidence of the potential need for the three skewness parameters when modeling real data.

Figure 4 displays the histogram of the data and fitted pdfs of the KBN distribution, its sub-models and non-nested models. Further, Figure 5 plots the empirical cdf and estimated cdfs of the KBN distribution, its sub-models and non-nested models. We note that the KBN distribution produces better fit than the other models.

We also apply formal goodness-of-fit tests in order to verify which distribution fits the data better. We consider the Cram´er-Von Mises (W^*) and AndersonDarling (A^*) statistics. In general, the smaller the values of the statistics, W^* and A^*, the better the fit to the data. Let $F(x; \widehat{\boldsymbol{\theta}})$ be the cdf, where the form of F is known but $\widehat{\boldsymbol{\theta}}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics, W^* and A^*, we proceed as follows:

(i) compute $v_i = F(x_i, \hat{\theta})$, where the x_i's are in ascending order, $y_i = \Phi^{-1}(v_i)$ is the standard normal quantile function and $u_i = \Phi\{(yi - \bar{y} / s_y)\}$, where

$$\overline{y} = n^{-1} \sum_{i=1}^{n} y_i \text{ and } s_y^2 = (n-1)^{-1} \sum_{i=1}^{n} (y_i - \overline{y})^2;$$

(ii) compute

$$W^2 = \sum_{i=1}^{n} \{u_i - (2i-1)/(2n)\}^2 + 1/(12n)$$

and

$$A^2 = -n - n^{-1} \sum_{i=1}^{n} \{(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)\};$$

(iii) modify W^2 into $W^* = W^2(1 + 0.5/n)$ and A^* into $A^* = A^2(1 + 075/n + 2.25/n^2)$.

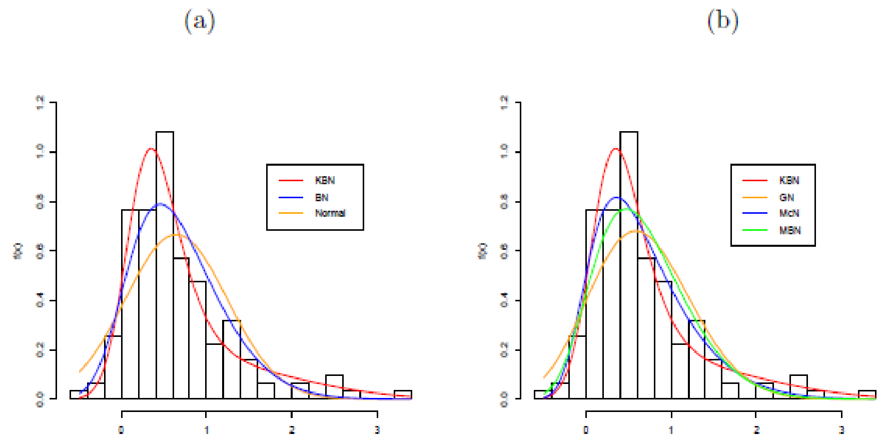

Figure 4: (a) Estimated pdfs of the KBN distribution and its sub-models.
(b)Estimated pdfs of the KBN, GN, McN and MBN models.

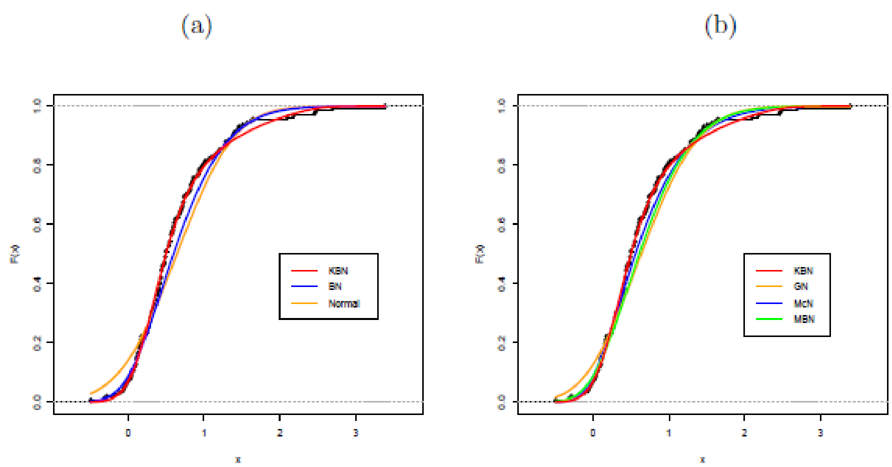

Figure 5: (a) Empirical and estimated cdfs of the KBN distribution and its sub-models.
(b) Empirical and estimated cdfs of the KBN, GN, McB and MBN models.

For further details, the reader is referred to Chen and Balakrishnan (1995). Thevalues of the statistics, W^* and A^*, for the distributions are given in Table 4. Overall, by comparing the measures of these formal goodness-of-fit tests in Table 4, we conclude that the KBN distribution outperforms all the distributions considered in this study. So, the proposed distribution can yield better fits than the normal, BN, GN, McN and MBN distributions and therefore may be an interesting alternative to these distributions for modeling skewed data sets. These results illustrate the potentiality of the new distribution and the necessity for additional shape parameters.

Table 4: Formal goodness-of-fit tests for the INPC data.

Model	Statistic	
	W^*	A^*
KBN	0.0715	0.5513
BN	0.2465	1.5050
Normal	0.7635	4.4915
GN	0.5742	3.4008
McN	0.1401	0.8813
MBN	0.2699	1.6412

The QQ plots of the normalized quantile residuals was introduced by Dunn and Smyth (1996) and more recently used by Cordeiro et al. (2013). Figures 6 and 7 indicate the improved fit achieved using the KBN distribution over other distributions. We also emphasize the gain yielded by the KBN distribution in relation to the normal, BN, GN, McN and MBN distributions.

(ii) Bayesian analysis

For the INPC data set, the following independent priors were considered to perform the Metropolis-Hastings algorithm: a ~ Ga(0.001, 0.001), b ~ Ga(0.001, 0.001), c ~ N(0, 1000), μ ~ N(0, 1000) and σ ~ Ga(0.001, 0.001), so that we have a vague prior distribution. Considering these prior pdfs, we generated two parallel independent runs of the Metropolis-Hastings with size 300,000 for each parameter. Disregarding the first 30,000 iterations to eliminate the effect of the initial values and, to avoid correlation problems, we considered a spacing of size 10, obtaining a sample of size 27,000 from each chain. To monitor the convergence of the Metropolis-Hastings, we performed the methods suggested by Cowles and Carlin (1996). To monitor the convergence of the samples, we used the between and within sequence information, following the approach developed in Gelman and Rubin (1992) to obtain the potential scale reduction, Rb. In all cases, these values were close to one, indicating the convergence of the chain.

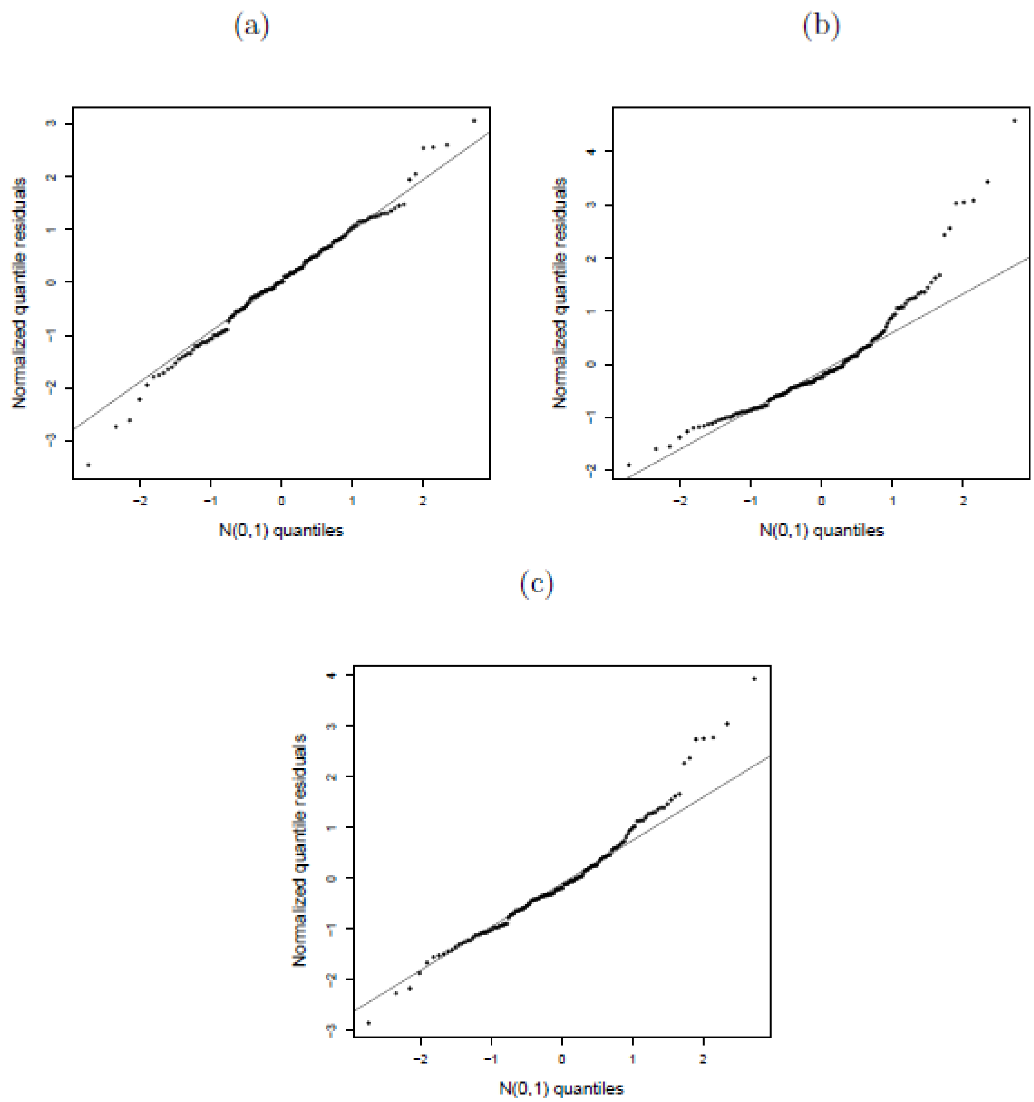

Figure 6: QQ plot of the normalized quantile residuals with an identity line for the distributions: (a) KBN, (b) Normal and (c) BN

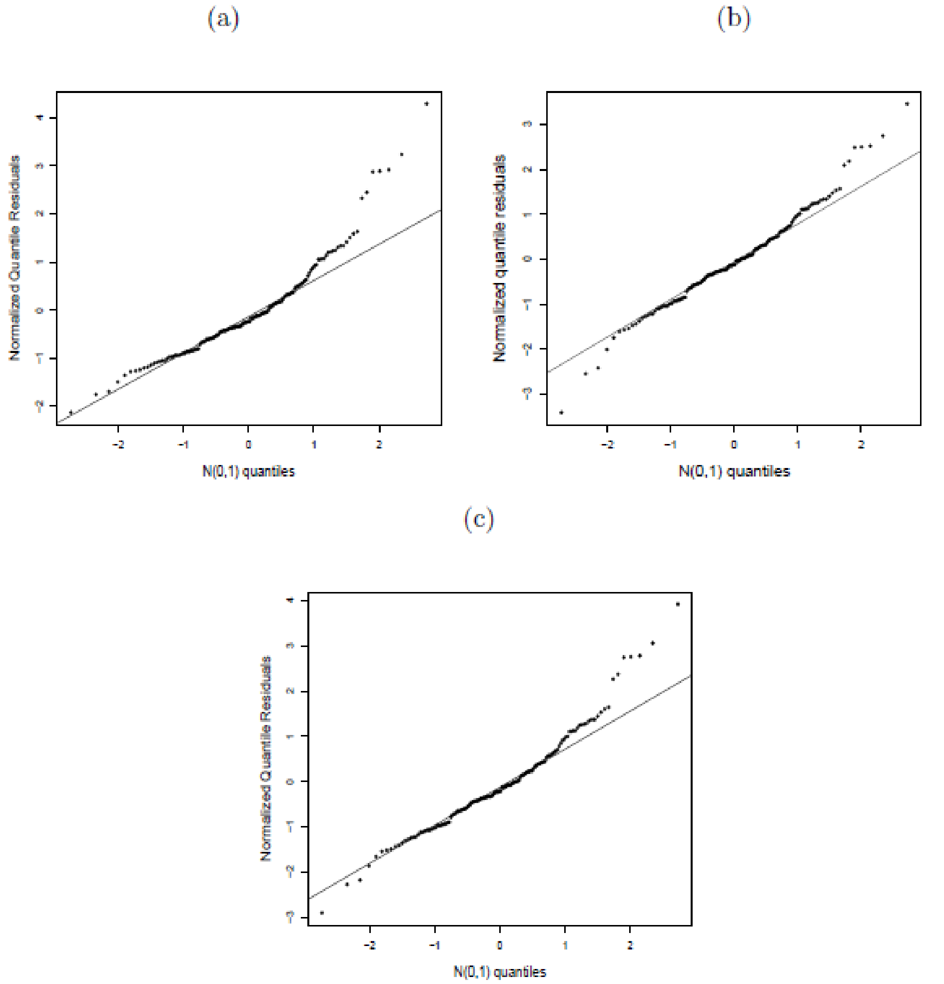

Figure 7: QQ plot of the normalized quantile residuals with an identity line for the distribution: (a) GN, (b) McN, (c) MBN.

The approximate posterior marginal pdfs of the parameters are illustrated in Figure 8. Table 5 reports the posterior summaries (posterior means, standard deviation (SD) and the 95% highest posterior density (HPD) intervals) for all parameters of the KBN distribution. We note that the values for posterior means (Table 5) are in good agreement with the MLEs.

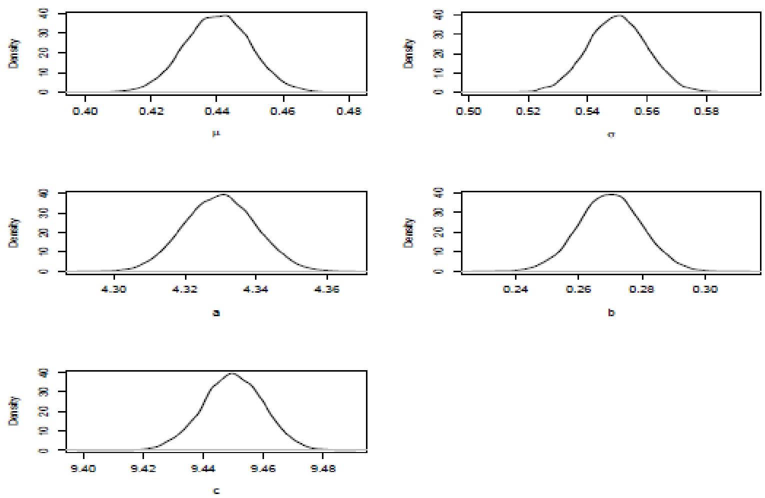

Figure 8: Approximate posterior marginal pdfs for the parameters of the KBN model for the INPC data.

7. Concluding remarks

We have introduced the Kummer beta normal (KBN) distribution with three shape parameters. The new distribution has proved to be versatile and analytically tractable. The KBN pdf can be expressed as a linear combination of exponentiated normal pdfs which allows us to derive some of its mathematical properties like its ordinary and incomplete moments, mean deviations and order statistics. The estimation of parameters has been approached by the method of maximum likelihood and Bayesian analysis. The usefulness of the KBN distribution has been illustrated by an application to a real data set. The new distribution provides a rather flexible mechanism for fitting a real world data and it may attract wider applications in many areas of research.

Table 5: Posterior summaries for the parameters of the KBN model for the INPC data.

Parameter	Mean	SD	HPD (95%)	$\widehat{R}$
a	4.3298	0.0098	(4.3107; 4.3493)	0.9999
b	0.2699	0.0102	(0.2505; 0.2898)	1.0007
c	9.4499	0.0102	(9.4301; 9.4696)	0.9998
μ	0.4401	0.0099	(0.4199; 0.4589)	1.0004
σ	0.5515	0.0101	(0.5307; 0.5699)	1.0010

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