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A New Generalized of Exponentiated Modified Weibull Distribution

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Abstract: In this paper, we propose a new generalization of exponentiated modified Weibull distribution, called the McDonald exponentiated modified Weibull distribution. The new distribution has a large number of well-known lifetime special sub-models such as the McDonald exponentiated Weibull, beta exponentiated Weibull, exponentiated Weibull, exponentiated expo- nential, linear exponential distribution, generalized Rayleigh, among others. Some structural properties of the new distribution are studied. Moreover, we discuss the method of maximum likelihood for estimating the model pa- rameters.

Key words: Hazard function, Moments, Maximum likelihood estimation, Exponentiated Modified Weibull Distributione.

1. Introduction

The exponential, Rayleigh, and linear exponential distribution are the most commonly used distributions in reliability and life testing, Lawless. These distri- butions have several desirable properties and nice physical interpretations. Un- fortunately the exponential distribution only has constant failure rate and the Rayleigh distribution has increasing failure rate. The generalized linear fail- ure rate distribution generalizes both these distributions which may have non- increasing hazard function also. Also, the Weibull distribution, having the ex- ponential, Rayleigh as special cases, is very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates, when modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, the

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Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub-shaped and the unimodal failure rates which are common in reliability and biological studies.

The models that present bathtub-shaped failure rate are very useful in sur- vival analysis. But, according to Nelson, the distributions presented in lifetime literature with this type of data, as the distributions proposed by Hjorth, are sufficiently complex and, therefore, difficult to be modeled. Later, other works had introduced new distributions for modeling bathtub-shaped failure rate. For example, Rajarshi and Rajarshi presented a review of these distributions and Haupt and Schabe considered a lifetime model with bathtub failure rates. How- ever, these models do not present much practicability to be used and in recent years new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub-shaped failure rate. Among these, the exponentiated Weibull (EW) distribution introduced by Mudholkar et al., the modified Weibull (MW) distribution proposed by Lai et al.

Elbatal introduced a generalized the modified Weibull distribution by pow- ering a positive real number to the cumulative distribution function (cdf). This new family of distribution called exponentiated modified Weibull distribution. The new distribution due to its flexibility in accommodating all the forms of the hazard rate function can be used in a variety of problems for modeling lifetime data. Another important characteristic of the distribution is that it contains, as special sub-models, the Weibull, exponentiated exponential (Gupta and Kundu), exponentiated Weibull distribution, generalized Rayleigh (Kundu and Rakab), generalized linear failure rate (Sarhan et al), modified Weibull distribution (Lai et al) and some other distributions. The exponentiated modified Weibull distri- bution is not only convenient for modeling comfortable bathtub-shaped failure rates data but is also suitable for testing goodness-of-fit of some special sub- models such as the exponentiated Weibull and modified Weibull distributions.

A random variable X is said to have the exponentiated modified Weibull distribution (EMW D) with four parameters (α, β, λ, θ) if its probability density function is given by

$$f(x) = \theta(\alpha + \lambda\beta x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}[1 - e^{-(\alpha x+\beta x^{\lambda})}]^{\theta-1} \tag{1}$$

while the cumulative distribution function is given by

$$F(x, \alpha, \beta, \theta) = [1 - e^{-(\alpha x+\beta x^{\lambda})}]^{\theta} \tag{2}$$

where α > 0, β > 0 are scale parameters of the distribution whereas the parame- ters λ > 0 and θ > 0 denote the shape parameters.

The aim of this paper is extend the (EMW D) distribution by introducing three extra shape parameters to define a new distribution refereed to as the Mc- Donald Exponentiated Modified Weibull (McEMW) distribution. The role of the three additional parameters is to introduce skewness and to vary tail weights and provide greater flexibility in the shape of the generalized distribution and consequently in modeling observed data. It may be mentioned that although several skewed distribution functions exist on the positive real axis, not many skewed distributions are available on the whole real line, which are easy to use for data analysis purpose. The main idea is to introduce three shape parameters, so that the (McEMW) distribution can be used to model skewed data, a feature which is very common in practice.

1.1 McDonald Generalized Distribution.

For an arbitrary parent cdf G(x), the probability density function (pdf) f (x) of the new class of distributions called the McDonald generalized distributions (denoted with the prefix ”Mc” for short) is defined by

$$f(x,a,b,c) = \frac{c}{B(a,b)} g(x)[G(x)]^{ac-1}[1-[G(x)]^c]^{b-1} \tag{3}$$

where a >0, b >0 and c >0 are additional shape parameters.(See Corderio et al. for additional details). Note that g(x) is the pdf of parent distribution ,$g(x) = \frac{dG(X)}{dx}$

Theclass of distributions (3) includes as special sub-models the beta generalized (Eugene et al;[6]) for c = 1 and Kumaraswamy (Kw) generalized distributions (Cordeiro and Castro, [4])for a = 1. For random variable X with density function (3), we write X~Mc−G(a, b, c).The probability density function (3) will be most tractable when G(x) and g(x) havesimple analytic expressions. The corresponding cumulative function for this generalization is given by

$$F(x,a,b,c) = I_{[G(x)]^c}(a,b) = \frac{1}{B(a,b)} \int_0^{[G(x)]^c} \omega^{(a-1)}(1-\omega)^{b-1}\, d\omega \tag{4}$$

Where$I_y(a,b) = \frac{1}{B(a,b)} \omega^{(a-1)}(1-\omega)^{b-1}\, d\omega$denotes the incomplete beta function ratio (Gradshteyn & Ryzhik, [8]). Equation (4) can also be rewritten as follows

$$F(x,a,b,c) = \frac{[G(x)]^{ac}}{aB(a,b)}\ {}_2F_1(a, 1-b; a+1; [G(x)]^c), \tag{5}$$

where

$$_2F_1(a,b;c;x) = B(b,c-b)^{-1}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}dt$$

is the well-known hypergeometric functions which are well established in the lit- erature (see,Gradshteyn and Ryzhik). Some mathematical properties of the cdf F (x) for any Mc-G distribution defined from a parent G(x) in equation (5), could, in principle, follow from the properties of the hypergeometric function, which are well established in the literature (Gradshteyn and Ryzhik, Sec. 9.1). One important benefit of this class is its ability to skewed data that cannot prop- erly benefitted by many other existing distributions. Mc-G family of densities allows for higher levels of flexibility of its tails and has a lot of applications in various fields including economics, finance, reliability, engineering, biology and medicine.

The hazard function (hf) and reverse hazard functions (rhf) of the Mc-G distribution are given by

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{cg(x)[G(x)]^{ac-1}[1-[G(x)]^c]^{b-1}}{B(a,b)\{1-I_{[G(x)]^c}(a,b)\}}, \tag{6}$$

and

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{cg(x)[G(x)]^{ac-1}[1-[G(x)]^c]^{b-1}}{B(a,b)\{I_{[G(x)]^c}(a,b)\}},$$

respectively. Recently Cordeiro et al. presented results on the McDonald normal distribution. Cordeiro et al. proposed McDonald Weibull distribution, and Fran- cisco et al. obtained the statistical properties of the McGamma and applied the model to reliability data. Oluyede and Rajasooriya introduced the McDagum distribution and its statistical properties with applications.Elbatal and Merovci introduced the McDonald exponentiated Pareto distribution. Elbatl & al. intro- duced the McDonald generalized linear failure rate distribution.

The rest of the article is organized as follows. In Section 2, we define the cu- mulative, density and hazard functions of the (McEMW) distribution and some special cases. Section 3, includes rthmoment , moment generating function.The distribution of the order statistics is expressed in Section 4. Least Squares and Weighted Least Squares Estimators of McEMW distribution are presented in Section 5. Finally, Maximum likelihood estimation of the parameters is deter- mined in Section 6.

2. The McDonald Exponentiated Modified Weibull Distribution

In this section we studied the seven parameter McDonald Exponentiated Mod- ified Weibull (McEMW) distribution. Using G(x) and g(x) in (3) to be the cdf and pdf of (1) and (2). The pdf of the McEMW distribution is given by

$$f(x,\varphi) = \frac{c\theta}{B(a,b)}\left(\alpha + \lambda\beta x^{\lambda-1}\right) e^{-(\alpha x+\beta x^{\lambda})}\left[1-e^{-(\alpha x+\beta x^{\lambda})}\right]^{\theta a c-1} \times \left[1-(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}\right]^{b-1}, \quad x>0, \tag{7}$$

where α, β are scale parameters the other positive parameters λ, θ, a, b and c are shape parameters, and ϕ = (α, β, λ, θ, a, b, c). The corresponding cdf of the McEMW distribution is given by

$$F(x) = I_{[G(x)]^c}(a,b) = \frac{1}{B(a,b)}\int_0^{[G(x)]^c} \omega^{(a-1)}(1-\omega)^{b-1}\,d\omega = \frac{1}{B(a,b)}\int_0^{(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}} \omega^{(a-1)}(1-\omega)^{b-1}\,d\omega = I_{(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}}(a,b) \tag{8}$$

also, the cdf can be written as follows

$$F(x) = \frac{(1-e^{-(\alpha x+\beta x^{\lambda})})^{a\theta c}}{aB(a,b)}\,{}_2F_1\left(a,1-b;a+1;\left(1-e^{-(\alpha x+\beta x^{\lambda})}\right)^{\theta c}\right) \tag{9}$$

Where ${}_2F_1(a,b;c;x) = B(b,c-b)^{-1}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}\,dt$

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the McEMW distribution for selected values of the parameters.

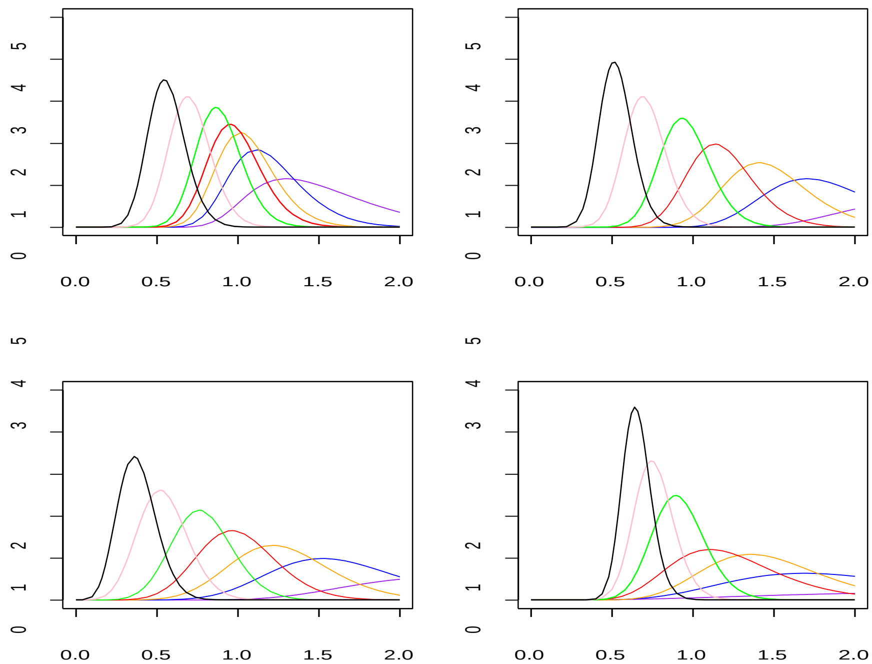

Figure 1: The pdf’s of various McEMW distributions for values of parameters

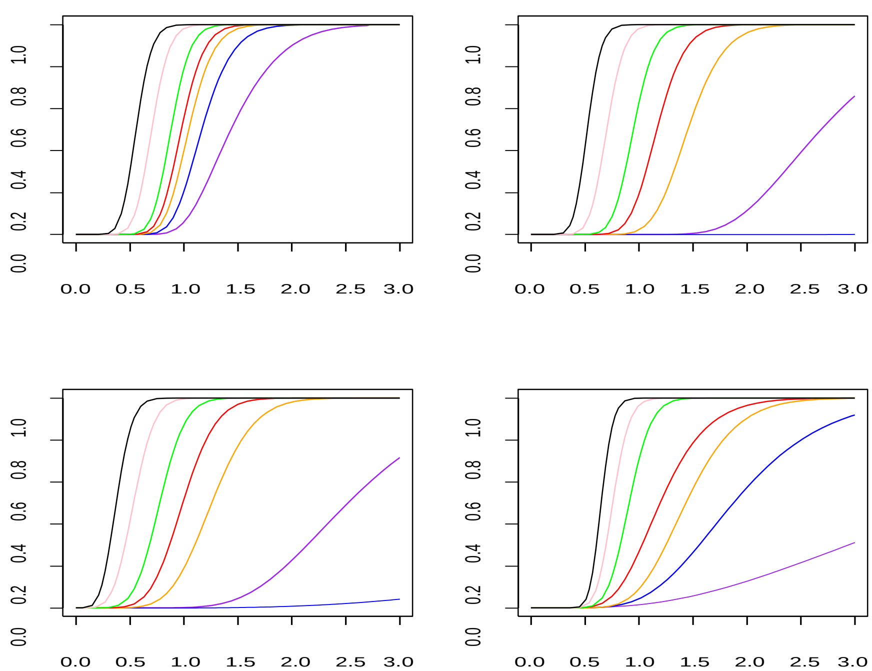

Figure 2: The cdf’s of various McEMW distributions for values of parameters.

The hazard rate function and reversed hazard rate function of the new distribution are given by

$$h(x)=\frac{f(x,\varphi)}{1-F(x,\varphi)}$$
$$=\frac{c\theta\mu\lambda^{\mu}[1-(\frac{\lambda}{x})^{\mu}]^{\theta ac-1}\{1-[1-(\frac{\lambda}{x})^{\mu}]^{\theta c}\}^{b-1}[1-(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}]^{b-1}}{B(a,b)\left\{1-\frac{(1-e^{-(\alpha x+\beta x^{\lambda})})^{a\theta c}}{aB(a,b)}\,{}_2F_1(a,1-b;a+1;(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}\right\}} \tag{10}$$

and

$$\tau(x)=\frac{f(x,\varphi)}{F(x,\varphi)}$$
$$=\frac{c\theta(\alpha+\lambda\beta x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}[1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta ac-1}[1-(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}]^{b-1}}{B(a,b)\left\{\frac{(1-e^{-(\alpha x+\beta x^{\lambda})})^{a\theta c}}{aB(a,b)}\,{}_2F_1(a,1-b;a+1;(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}\right\}} \tag{11}$$

respectively.

Figure 3 illustrates some of the possible shapes of the hazard function of the McEMW distribution for selected values of the parameters.

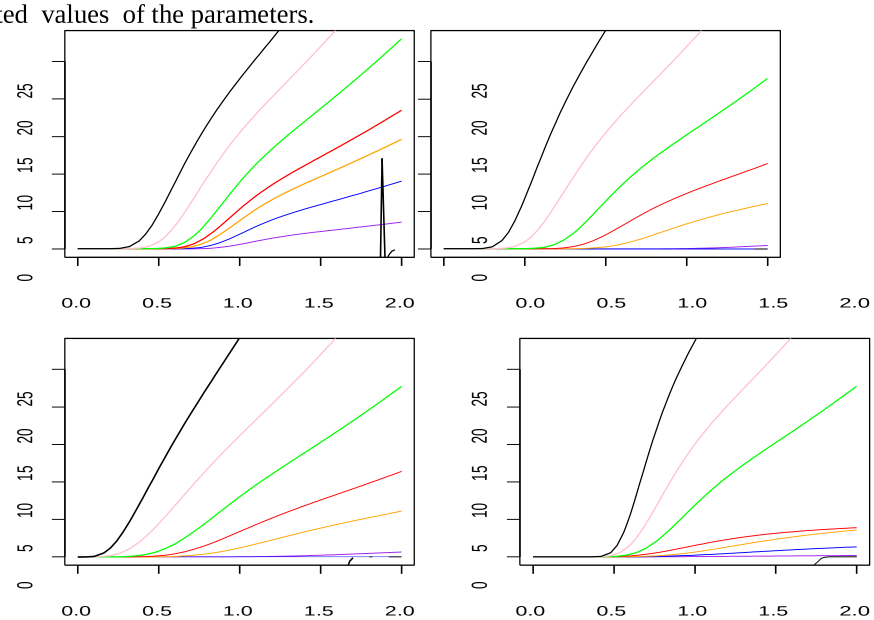

Figure 3: The hazard function's of various McEMW distributions for values of parameters.

2.1. Submodels

The McDonald Exponentiated Modified Weibull McEMW distribution is very flexible model that approaches to different distributions when its param- eters are changed. The McEMW distribution contains as special-models the following well known distributions. If X is a random variable with pdf (7), us- ing the notation $X \sim Mc{-}G(x, \alpha, \beta, \lambda, \theta, a, b, c)$ then we have the following cases.

•Exponentiated Modified Weibull distribution: For $a = b = 1$, the McEMW distribution reduces to EMW distribution which introduced by Elbatal.

•McDonald Modified Weibull distribution: For $\theta = 1$, the McEMW re- duces to McEW distribution.

•Beta Modified Weibull distribution: For $c = 1$, $\theta = 1$,the McEMW re- duces to BMW distribution.

•Kumaraswamy Modified Weibull distribution: For $a = 1, \theta = 1$, the McEMW distribution reduces to KMW distribution.

•Modified Weibull distribution: For $a = b = c = \theta = 1$, the McEMW distribution reduces to MW distribution.

•McDonald Exponentiated Weibull distribution: For $\alpha = 0$, the McEMW reduces to McEW distribution.

•Beta Exponentiated Weibull distribution: For $c = 1$, $\alpha = 0$,the McEMW reduces to BEW distribution.

•Kumaraswamy Exponentiated Weibull distribution:For $a = 1$, $\alpha = 0$,the McEMW distribution reduces to KEW distribution.

•Exponentiated Weibull distribution: For $a = b = c = 1, \alpha = 0$,the McEMW distribution reduces to EW distribution.

•McDonald Weibull distribution: For $\theta = 1$ and $\alpha = 0$, equation (7) be- comes McW distribution.

•Beta Weibull distribution: For $c = \theta = 1$, $\alpha = 0$,the McEMW reduces to BEW distribution.

•Weibull distribution: For $a = b = c = \theta = 1$, $\alpha = 0$,the McEMW distribu- tion reduces to KEW distribution.

•Exponential distribution: For $\alpha = 0$, $\lambda = 1$ and $\theta = 1$ the McEMW distri- bution reduces to McEW distribution.

•Generalized Rayleigh distribution: For $\alpha = 0$, and $\lambda = 2$ the McEMW distribution reduces to McGR distribution.

3. Statistical Properties

In this section we study the statistical properties of the (McEMW) dis- tribution, specifically moments and moment generating function. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distri- bution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 3.1

The rth moment of $(McEMW)$ distribution, $r = 1, 2, \ldots$ is given by

$$\mu_r' = \frac{c\theta}{B(a,b)} \sum_{i=j=k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{\theta c(a+i)-1}{j} \frac{(\beta(j+1))^k}{k!} \times \left[\frac{\alpha\Gamma(r+\lambda k+1)}{[\alpha(j+1)]^{r+\lambda k+1}} + \lambda\beta \frac{\Gamma(r+\lambda(k+1))}{[\alpha(k+1)]^{r+\lambda(k+1)}} \right] \tag{12}$$

Proof.

We start with the well known definition of the rth moment of the random variable X with probability density function f (x) given by

$$\mu_r' = \int_0^{\infty} x^r f(x,\varphi)\, dx$$

Substituting from (7) into the above relation, we get

$$\mu_r' = \frac{c\theta}{B(a,b)} \int_0^{\infty} x^r \left[\left(\alpha + \lambda\beta x^{\lambda-1}\right) e^{-(\alpha x+\beta x^{\lambda})} [1 - e^{-(\alpha x+\beta x^{\lambda})}]^{\theta a c-1} [1 - (1 - e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}]^{b-1}\, dx \right] \tag{13}$$

since $0 < e^{-(\alpha x+\beta x^{\lambda})} < 1$ for x > 0, the binomial series expansion of

$$[1 - (1 - e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}]^{b-1}\, dx$$

yields

$$[1 - (1 - e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}]^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} (1 - e^{-(\alpha x+\beta x^{\lambda})})^{\theta i c} \tag{14}$$

thus we get

$$\mu_r' = \frac{c\theta}{B(a,b)} \sum_{\mathrm{i=0}}^{\infty} (-1)^i \binom{b-1}{i} \times \int_0^{\infty} x^r\, \theta\left(\alpha + \lambda\beta x^{\lambda-1}\right) e^{-(\alpha x+\beta x^{\lambda})} [1 - e^{-\left(\alpha x+\beta x^{\lambda}\right)}]^{\theta c(a+i)-1}\, dx$$

Again, the binomial series expansion of

$$1 - e^{-\left(\alpha x+\beta x^{\lambda}\right)}]^{\theta c(a+i)-1}$$

Yields

$$[1 - e^{-\left(\alpha x+\beta x^{\lambda}\right)}]^{\theta c(a+i)-1} = \sum_{i=0}^{\infty} (-1)^j \binom{\theta c(a+i)-1}{j} e^{-j(\alpha x+\beta x^{\lambda})} \tag{15}$$

we obtain

$$\mu_r' = \frac{c\theta}{B(a,b)} \sum_{i=j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \binom{\theta c(a+i)-1}{j} \times \int_0^{\infty} x^r \left(\alpha + \lambda\beta x^{\lambda-1}\right) e^{-(j+1)(\alpha x+\beta x^{\lambda})}\, dx \tag{16}$$

but the series expansion of $e^{-\beta(j+1)x^{\lambda}}$ is given by

$$e^{-\beta(j+1)x^{\lambda}} = \sum_{k=0}^{\infty} \frac{(-\beta(j+1))^k x^{\lambda k}}{k!} \tag{17}$$

substituting from (17) into (16), yields

$$\mu_r' = C_{i,j,k} \int_0^{\infty} x^{r+\lambda k} (\alpha + \lambda\beta^{\lambda-1}) e^{-\alpha(j+1)x}\, dx \tag{18}$$

where

$$C_{i,j,k} = \frac{c\theta}{B(a,b)} \sum_{i=0,j=0,k=0}^{\infty} (-1)^{i+j+k} \binom{b-1}{i} \binom{\theta c(a+i)-1}{j} \frac{(\beta(j+1))^k}{k!} \tag{19}$$

the integral in (18) can be computed as follows

$$\mu_r' = C_{i,j,k}\left[\alpha\int_0^\infty x^{r+\lambda k}e^{-\alpha(j+1)x}\,dx + \lambda\beta\int_0^\infty x^{r+\lambda k+\lambda-1}e^{-\alpha(j+1)x}\,dx\right]$$
$$= C_{i,j,k}\left[\frac{\alpha\Gamma(r+\lambda k+1)}{[\alpha(j+1)]^{r+\lambda k+1}} + \lambda\beta\frac{\Gamma(r+\lambda(k+1))}{[\alpha(j+1)]^{r+\lambda(k+1)}}\right] \tag{20}$$

wich completes the proof.

Theorem 3.2.

The moment generating function of (McEMW) distribution is given by

$$M(t) = C_{i,j,k}\left[\frac{\alpha\Gamma(\lambda k+1)}{[\alpha(j+1)-t]^{\lambda k+1}} + \lambda\beta\frac{\Gamma(\lambda(k+1))}{[\alpha(j+1)-t]^{\lambda(k+1)}}\right] \tag{21}$$

Proof

We start with the well known definition of the M (t) of the random variable

X with probability density function f (x) given by

$$M(t) = E(e^{tX}) = \int_0^\infty e^{tx}f(x,\varphi)\,dx$$

Substituting from (7) into the above relation, we get

$$M(t) = \frac{c\theta}{B(a,b)}\int_0^\infty \left[e^{tx}(\alpha+\lambda\beta x^{\lambda-1})e^{-(\alpha x+\beta x^\lambda)}\left[1-e^{-(\alpha x+\beta x^\lambda)}\right]^{\theta ac-1}\right.$$
$$\left.\times\left[1-(1-e^{-(\alpha x+\beta x^\lambda)})^{\theta c}\right]^{b-1}dx\right] \tag{22}$$

using the binomial series expansion given by (15) and (17) we get

$$M(t) = C_{i,j,k}\int_0^\infty x^{\lambda k}\left(\alpha+\lambda\beta^{\lambda-1}\right)e^{-x[\alpha(j+1)-t]}\,dx$$
$$= C_{i,j,k}\left[\alpha\int_0^\infty x^{\lambda k}e^{-x[\alpha(j+1)-t]}\,dx + \lambda\beta\int_0^\infty x^{\lambda(k+1)}e^{-x[\alpha(j+1)-t]}\,dx\right]$$
$$= C_{i,j,k}\left[\frac{\alpha\Gamma(\lambda k+1)}{[\alpha(j+1)-t]^{\lambda k+1}} + \lambda\beta\frac{\Gamma(\lambda(k+1))}{[\alpha(j+1)-t]^{\lambda(k+1)}}\right] \tag{23}$$

which completes the proof.

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4. Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the rth order statistic of the McEMW distribution, also, the measures of skewness and kurtosis of the distribution of the rth order statistic in a sample of size n for different choices of n; r are presented in this section. Let X1:n, X2:n, . . . , Xn:n be a simple random sample from (McEMW) distribution with pdf and cdf given by (7) and (8), respectively. Let X1, X2, . . . , Xn denote the order statistics obtained from this sample. We now give the probability density function of Xr:n, say fr:n(x, ϕ) and the moments of Xr:n, r = 1, 2, . . . , n. Therefore, the measures of skewness and kurtosis of the distribution of the Xr:n are presented. The probability density function of Xr:n is given by

$$f_{r;n}(x,\varphi)=\frac{1}{B(r,n-r+1)}[F(x,\varphi)^{r-1}[1-F(x,\varphi)]^{n-r}]f(x,\varphi) \tag{24}$$

where F (x, ϕ) and f (x, ϕ) are the cdf and pdf of the McEMW distribution given by (7), (8), respectively, and B(., .) is the beta function, since 0 < F (x, ϕ) < 1, for x > 0, by using the binomial series expansion of [1 − F (x, ϕ)]n−r , given by

$$[1-F(x,\varphi)]^{n-r}]=\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}[F(x,\varphi)]^j \tag{25}$$

we have

$$f_{r;n}(x,\varphi)=\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}[F(x,\varphi)]^{r+j-1}f(x,\varphi) \tag{26}$$

substituting from (7) and (8) into (26), we can express the k_{th} ordinary moment of the r_{th} order statistics $X_{r:n}$ say $E(X^k_{r:n})$ as a liner combination of the k_{th} moments of the $McEMW$ distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

5. Least Squares and Weighted Least Squares Estimators

In this section we provide the regression based method estimators of the unknown parameters of the McDonald exponentiated modified Weibull, which was originally suggested by Swain, Venkatraman and Wilson to estimate the parameters of beta distributions. It can be used some other cases also. Suppose X(1:n), . . . , X(n:n) is a random sample of size n from a distribution function G(.) and suppose X(i:n); i = 1, 2, . . . , n denotes the ordered sample. The proposed method uses the distribution of G(X(i:n)). For a sample of size n, we have

$$E\left(G(X_{(j:n)})\right) = \frac{j}{n+1}, V\left(G(X_{(j:n)})\right) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$$

$$\text{and } Cov\left(G(X_{(j:n)}), G(X_{(k:n)})\right) = \frac{j(n-k+1)}{(n+1)^2(n+2)}; \text{for } j < k,$$

see Johnson, Kotz and Balakrishnan. Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1 (Least Squares Estimators). Obtain the estimators by mini- mizing

$$\sum_{j=1}^{n}\left(G(X_{(j:n)} - \frac{j}{n+1}\right)^2 \tag{27}$$

with respect to the unknown parameters. Therefore in case of $(McEMW)$ distribution the least squares estimators of $\alpha, \beta, \lambda, \theta, a, b$ and c, say $\widehat{\alpha}_{LSE}, \widehat{\beta}_{LSE}, \widehat{\lambda}_{LSE}, \widehat{\theta}_{LSE}, \widehat{a}_{LSE}, \widehat{b}_{LSE}$ and $\widehat{c}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left(\frac{(1-e^{-(\alpha x+\beta x^{\lambda})})^{a\theta c}}{aB(a,b)}\, {}_2F_1\left(a, 1-b; a+1; (1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}\right) - \frac{j}{n+1}\right)^2,$$

with respect to $\alpha, \beta, \lambda, \theta, a, b$, and c.

Method 2 (Weighted Least Squares Estimators). The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=q}^{n}\omega j\left(G(X_{(j:n)} - \frac{j}{n+1}\right)^2 \tag{28}$$

with respect to the unknown parameters, where

$$\omega j = \frac{1}{V(G(X_{(j:n)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$$

Therefore, in case of $McEMW$ distribution the weighted least squares estimators of $\alpha, \beta, \lambda, \theta, a, b$ and c, say $\widehat{\alpha}_{WLSE}, \widehat{\beta}_{WLSE}, \widehat{\lambda}_{WLSE}, \widehat{\theta}_{WLSE}, \widehat{a}_{WLSE}, \widehat{b}_{WLSE}$ and $\widehat{c}_{WLSE}$, respectively, can be obtained by minimizing

$$\sum_{j=q}^{n}\omega j\left[I_{e^{-(\alpha x_i+\beta x_i^{\lambda})\theta c}}(a,b) - \frac{j}{n+1}\right]^2$$

with respect to the unknown parameters only.

6. Estimation and Inference

In this section we determine the maximum likelihood estimates MLE of the parameters of the McEMW (x, ϕ) distribution from complete samples only. Let X1, X2, . . . , Xn be a random sample from X ∼ McEMW (x, ϕ) with observed values x1, x2, . . . , xn and let ϕ = (α, β, λ, θ, a, b, c)T be the vector of the model parameters. The log likelihood function of (7) is defined as

$$\begin{aligned}\ell &= n\log c + n\log\theta + n\log\left[\Gamma(a+b)\right] - n\log\left[\Gamma(a)\right] - n\log\left[\Gamma(b)\right] \\ &+ \sum_{i=1}^{n}\log\left(\alpha+\lambda\beta x_i^{\lambda-1}\right) - \alpha\sum_{i=1}^{n}x_i - \beta\sum_{i=1}^{n}x_i^{\lambda} \\ &+ (\theta ac-1)\sum_{i=1}^{n}\log\left[1-e^{-(\alpha x+\beta x^{\lambda})}\right] \\ &+ (b-1)\sum_{i=1}^{n}\log\left[1-(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}\right], \qquad (29)\end{aligned}$$

Differentiating ℓ with respect to each parameter α, β, λ, θ, a, b andc and setting the result equals to zero, we obtain maximum likelihood estimates. The partial derivatives of L(ϕ) with respect to each parameter or the score function is given by:

$$U_n(\varphi) = \left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial c}\right),$$

where

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= \sum_{i=1}^{n}\frac{1}{\left(\alpha+\lambda\beta x_i^{\lambda-1}\right)} - \sum_{i=1}^{n}x_i + (\theta ac-1)\sum_{i=1}^{n}\frac{x_i}{1-e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)}} \\ &-\theta c(b-1)\sum_{i=1}^{n}\frac{(1-e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)})^{\theta c-1}e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)}}{\left[1-(1-e^{-(\alpha x+\beta x^{\lambda})})^{\theta c}\right]}, \qquad (30)\end{aligned}$$

$$\begin{aligned}\frac{\partial\,\ell}{\partial\beta} &= \sum_{i=1}^{n}\frac{\lambda x_i^{\lambda-1}}{\left(\alpha+\lambda\beta x_i^{\lambda-1}\right)} - \sum_{i=1}^{n}x_i^{\lambda} \\ &+\lambda(\theta ac-1)\sum_{i=1}^{n}\frac{x_i^{\lambda}e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)}}{\left[1-e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)}\right]} \\ &+\lambda\theta c(b-1)\sum_{i=1}^{n}\frac{x_i^{\lambda}(1-e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)})^{\theta c-1}e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)}}{\left[1-(1-e^{-\left(\alpha+\lambda\beta x_i^{\lambda}\right)})^{\theta c}\right]} \qquad (31)\end{aligned}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{\left[\lambda x_i^{\lambda-1}\ln(x_i) + x_i^{\lambda-1}\right]}{\left(\alpha + \lambda\beta x_i^{\lambda-1}\right)} - \sum_{i=1}^{n} x_i^{\lambda}\ln(x_i)$$
$$+\beta(\theta ac - 1)\sum_{i=1}^{n} \frac{e^{-(\alpha x+\beta x^{\lambda})} x_i^{\lambda}\ln(x_i)}{[1-e^{-(\alpha x+\beta x^{\lambda})}]}$$
$$+\beta c\theta(b-1)\sum_{i=1}^{n} \frac{e^{-(\alpha+\lambda\beta x_i^{\lambda})}(1-e^{-(\alpha+\lambda\beta x_i^{\lambda})})^{\theta c-1} x_i^{\lambda}\ln(x_i)}{\left[1-(1-e^{-(\alpha+\lambda\beta x_i^{\lambda})})^{\theta c}\right]}, \tag{32}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + ac\sum_{i=1}^{n} log[1-e^{-(\alpha+\lambda\beta x_1^{\lambda})}] + c(b-1)\sum_{i=1}^{n} \frac{(1-e^{-(\alpha+\lambda\beta x_1^{\lambda})})^{\theta c}\, log(1-e^{-(\alpha+\lambda\beta x_1^{\lambda})})}{[1-e^{-(\alpha+\lambda\beta x_i^{\lambda})}]} \tag{33}$$

$$\frac{\partial \ell}{\partial a} = n\psi(a+b) - n\psi(a) + \theta c\sum_{i=1}^{n} log[1-e^{-(\alpha+\lambda\beta x_1^{\lambda})}] \tag{34}$$

$$\frac{\partial \ell}{\partial b} = n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n} log[1-(1-e^{-(\alpha+\lambda\beta x_1^{\lambda})})^{\theta c}] \tag{35}$$

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \theta a\sum_{i=1}^{n} log[1-e^{-(\alpha+\lambda\beta x_1^{\lambda})}] + \theta(b-1)\sum_{i=1}^{n} \frac{(1-e^{-(\alpha+\lambda\beta x_1^{\lambda})})^{\theta c}\, log(1-e^{-(\alpha+\lambda\beta x_1^{\lambda})})}{[1-(1-e^{-(\alpha+\lambda\beta x_i^{\lambda})})^{\theta c}]} \tag{36}$$

and

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \theta a\sum_{i=1}^{n} \log\left[1-e^{-(\alpha+\lambda\beta x_i^{\lambda})}\right]$$
$$+\theta(b-1)\sum_{i=1}^{n} \frac{(1-e^{-(\alpha+\lambda\beta x_i^{\lambda})})^{\theta c}\log(1-e^{-(\alpha+\lambda\beta x_i^{\lambda})})}{\left[1-(1-e^{-(\alpha+\lambda\beta x_i^{\lambda})})^{\theta c}\right]}, \tag{36}$$

where $\psi(.)$ is digamma function $\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and the MLE of the parameters $\alpha, \beta, \lambda, \theta, a, b$ and c, say $\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\theta}, \widehat{a}, \widehat{b}$ and $\widehat{c}$ are obtained by solving the following equations, $\frac{\partial\ell}{\partial\alpha} = \frac{\partial\ell}{\partial\beta} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = \frac{\partial\ell}{\partial a} = \frac{\partial\ell}{\partial b} = \frac{\partial\ell}{\partial c} = 0$. There is no closed form solution to these equations, so numerical technique must be applied.

The elements of Hessian matrix are:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\sum_{i=1}^{n} \left(\alpha + \lambda \beta {x_i}^{\lambda-1}\right)^{-2} - (\theta ac - 1) \sum_{i=1}^{n} \frac{{x_i}^2 e^{-\alpha x_i - \beta {x_i}^{\lambda}}}{\left(-1 + e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^2}$$

$$- (b-1) \sum_{i=1}^{n} \frac{\left(1 - e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^{\theta c} \theta c {x_i}^2 e^{-\alpha x_i - \beta {x_i}^{\lambda}} \times \alpha_1}{\left(-1 + e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^2 \left(-1 + \left(1 - e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^{\theta c}\right)^2}$$

$$\alpha_1 = \theta c e^{-\alpha x_i - \beta {x_i}^{\lambda}} - 1 + \left(1 - e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^{\theta c}$$

$$\frac{\partial^2 \ell}{\partial \alpha \beta} = -\sum_{i=1}^{n} \frac{\lambda {x_i}^{\lambda-1}}{\left(\alpha + \lambda \beta {x_i}^{\lambda-1}\right)^2} - (\theta ac - 1) \sum_{i=1}^{n} \frac{x_i {x_i}^{\lambda} e^{-\alpha x_i - \beta {x_i}^{\lambda}}}{\left(-1 + e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^2}$$

$$- (b-1) \sum_{i=1}^{n} \frac{\left(1 - e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^{\theta c} \theta c x_i {x_i}^{\lambda} e^{-\alpha x_i - \beta {x_i}^{\lambda}} \times \beta_1}{\left(-1 + e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^2 \left(-1 + \left(1 - e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^{\theta c}\right)^2}$$

$$\beta_1 = \theta c e^{-\alpha x_i - \beta {x_i}^{\lambda}} - 1 + \left(1 - e^{-\alpha x_i - \beta {x_i}^{\lambda}}\right)^{\theta c}$$

$$\frac{\partial^2 \ell}{\partial\alpha\lambda} = -\sum_{i=1}^{n} \frac{\beta\, {x_i}^{\lambda-1} + \lambda\,\beta\, {x_i}^{\lambda-1} \ln(x_i)}{(\alpha + \lambda\,\beta\, {x_i}^{\lambda-1})^2}$$

$$-(\theta\, ac - 1)\sum_{i=1}^{n} \frac{\beta\, x_i {x_i}^{\lambda} \ln(x_i)\, \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^2}$$

$$-(b-1)\sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, c x_i \beta\, {x_i}^{\lambda} \ln(x_i)\, \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} \times \lambda_1}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^2 \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$\lambda_1 = \left(\theta\, c\mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} - 1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)$$

$$\frac{\partial^2 \ell}{\partial\alpha\theta} = ac\sum_{i=1}^{n} \frac{x_i \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}$$

$$-(b-1)\sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} c x_i \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} \theta_1}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$\theta_1 = -\theta\, c \ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) - 1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}$$

$$\frac{\partial^2 \ell}{\partial\alpha a} = \theta\, c\sum_{i=1}^{n} \frac{x_i \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}$$

$$\frac{\partial^2 \ell}{\partial\alpha b} = -\sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, c x_i \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) \left(1 - \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)}$$

$$\frac{\partial^2 \ell}{\partial\alpha b} = \theta\, a\sum_{i=1}^{n} \frac{x_i \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}$$

$$-(b-1)\sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, x_i \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} \times b_1}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$b_1 = \left(-\theta\, c \ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) - 1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \sum_{i=1}^{n} -\frac{\lambda^2 \left(x_i^{\lambda-1}\right)^2}{\left(\alpha + \lambda\, \beta\, x_i^{\lambda-1}\right)^2} - (\theta ac - 1) \sum_{i=1}^{n} \frac{\left(x_i^{\lambda}\right)^2 \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^2}$$

$$- (b-1) \sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c} \theta\, c \left(x_i^{\lambda}\right)^2 \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}} \beta_{11}}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^2 \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$\beta_{11} = \left(\theta\, c\mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}} - 1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c}\right)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \lambda} = \sum_{i=1}^{n} \frac{x_i^{\lambda-1} \alpha \, \left(1 + \lambda\, \ln\left(x_i\right)\right)}{\left(\alpha + \lambda\, \beta\, x_i^{\lambda-1}\right)^2} - \sum_{i=1}^{n} x_i^{\lambda} \ln\left(x_i\right)$$

$$- (\theta\, ac - 1) \sum_{i=1}^{n} \frac{x_i^{\lambda} \ln\left(x_i\right) \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}} \left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}} + \beta\, x_i^{\lambda}\right)}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^2}$$

$$+ (b-1) \sum_{i=1}^{n} (A_i + B_i)$$

$$A_i = \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c} \theta\, c x_i^{\lambda} \ln\left(x_i\right) \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}} \cdot K_i}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^2 \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c}\right)}$$

$$B_i = -\frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c} \theta\, c \left(x_i^{\lambda}\right)^2 \left(\mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^2 \beta\, \ln\left(x_i\right) \cdot L_i}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^2 \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, x_i^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$K_i = \left(\theta\, c\beta\, {x_i}^{\lambda} \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} + 1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} - \beta\, {x_i}^{\lambda} + {x_i}^{\lambda} \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} \beta\right)$$

$$L_i = \left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, c\right)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial a} = \theta\, c \sum_{i=1}^{n} \frac{{x_i}^{\lambda} \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial b} = -\sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, c{x_i}^{\lambda} \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)\left(1 - \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial c} = \theta\, a \sum_{i=1}^{n} \frac{{x_i}^{\lambda} \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}$$
$$- (b-1) \sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, {x_i}^{\lambda} \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} \times \beta_{12}}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)\left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$\beta_{12} = \left(-\theta\, c \ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) - 1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)$$

$$\frac{\partial^2 \lambda}{\partial \lambda^2} = -\sum_{i=1}^{n} \frac{\left(-2\, \ln(x_i)\, \alpha - \alpha\, \lambda\, (\ln(x_i))^2 + \beta\, {x_i}^{\lambda-1}\right) {x_i}^{\lambda-1} \beta}{(\alpha + \lambda\, \beta\, {x_i}^{\lambda-1})^2} - \beta \sum_{i=1}^{n} {x_i}^{\lambda} (\ln(x_i))^2$$
$$- (\theta\, ac - 1) \sum_{i=1}^{n} \frac{\beta\, {x_i}^{\lambda} (\ln(x_i))^2 \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} \left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} + \beta\, {x_i}^{\lambda}\right)}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^2}$$
$$+ (b-1) \sum_{i=1}^{n} (C_i + D_i)$$

$$Ci = \frac{\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\theta\,c\beta\,x_i{}^{\lambda}\left(\ln x_i\right)^2\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}C_{1i}}{\left(-1+\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^2\left[-1+\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\right]}$$

$$C_{1i} = \left(\theta\,c\beta\,x_i{}^{\lambda}\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}+1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}-\beta\,x_i{}^{\lambda}+x_i{}^{\lambda}\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\beta\right)$$

$$D_i = -\frac{\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\theta\,c\beta^2\left(x_i{}^{\lambda}\right)^2\left(\ln\left(x_i\right)\right)^2\left(\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^2 D_{1i}}{\left(-1+\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^2\left(-1+\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\right)^2}$$

$$D_1 i = \left(-1+\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}+\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\theta\,c\right)$$

$$\frac{\partial^2\ell}{\partial\lambda\partial a} = \theta\,c\sum_{i=1}^{n}\frac{\beta\,x_i{}^{\lambda}\ln\left(x_i\right)\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}}{1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}}$$

$$\frac{\partial^2\ell}{\partial\lambda\partial b} = -\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\theta\,c\beta\,x_i{}^{\lambda}\ln\left(x_i\right)\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}}{\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)\left(1-\left(1-\mathrm{e}^{-\alpha\,x_i-\beta\,x_i{}^{\lambda}}\right)^{\theta\,c}\right)}$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial c} = \theta\, a \sum_{i=1}^{n} \frac{\beta\, {x_i}^{\lambda} \ln(x_i)\, \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}{1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}}$$

$$- (b-1) \sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, \beta\, {x_i}^{\lambda} \ln(x_i)\, \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}} S_i}{\left(-1 + \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)\left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)^2}$$

$$S_i = \left(-\theta\, c \ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right) - 1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)$$

$$\frac{\partial^2 \ell}{\partial a^2} = n\psi'(a+b) - n\psi'(a)$$

$$\frac{\partial^2 \ell}{\partial a\partial b} = n\psi'(a+b)$$

$$\frac{\partial^2 \ell}{\partial a\partial c} = \theta \sum_{i=1}^{n} \ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)$$

$$\frac{\partial^2 \ell}{\partial b^2} = n\psi'(a+b) - n\psi'(b)$$

$$\frac{\partial^2 \ell}{\partial b\partial c} = \sum_{i=1}^{n} -\frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta\, \ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)}{1 - \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}}$$

$$\frac{\partial^2 \ell}{\partial c^2} = -\frac{n}{c^2} - (b-1) \sum_{i=1}^{n} \frac{\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c} \theta^2 \left(\ln\left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)\right)^2}{\left(-1 + \left(1 - \mathrm{e}^{-\alpha\, x_i - \beta\, {x_i}^{\lambda}}\right)^{\theta\, c}\right)^2}$$

7. Application

In this section, we use a real data set to show that the McEMW distribution can be a better model than one based on the EMW distribution, MW and Weibull distribution. The data set given in Table 1 taken from Murthy et al. page 180 represents the failure times of 50 components(per 1000h):

Table 1: Failure Times of 50 Components(per 1000 hours)

0.036	0.058	0.061	0.074	0.078	0.086	0.102	0.103	0.114	0.116
0.148	0.183	0.192	0.254	0.262	0.379	0.381	0.538	0.570	0.574
0.590	0.618	0.645	0.961	1.228	1.600	2.006	2.054	2.804	3.058
3.076	3.147	3.625	3.704	3.931	4.073	4.393	4.534	4.893	6.274
6.816	7.896	7.904	8.022	9.337	10.940	11.020	13.880	14.730	15.080

Table 2: Estimated parameters and criteria for comparison of the Weibull, Modified Weibull, Exponentiated Weibull and McEMW distribution for the failure times of 50 components(per 1000h).

Model	Parameter Estimate	Standard Error	$-\ell(\cdot;x)$	$-2\ell(\cdot;x)$	AIC
Weibull	$\hat{\alpha}=0.429$	0.104	103.0572	206.114	210.114
	$\hat{\lambda}=0.638$	0.073			
Modified	$\hat{\alpha}=0.043$	0.131	102.320	204.64	210.640
Weibull	$\hat{\beta}=0.492$	0.181			
	$\hat{\gamma}=0.619$	0.154			
EMW	$\hat{\lambda}=0.644$	0.200	100.767	201.534	209.534
	$\hat{\alpha}=0.101$	0.119			
	$\hat{\beta}=0.263$	0.207			
	$\hat{\theta}=0.702$	0.149			
McEMW	$\hat{\alpha}=1.090$	0.002	97.657	195.314	209.314
	$\hat{\beta}=10.792$	0.054			
	$\hat{\lambda}=0.189$	0.005			
	$\hat{\theta}=12.887$	3.238			
	$\hat{a}=6.902$	2.934			
	$\hat{b}=0.113$	0.017			
	$\hat{c}=4.118$	1.034			

The variance covariance matrix $I(\hat{\varphi})^{-1}$ of the MLEs under the McEMW dis- tribution is computed as

$$\begin{pmatrix} 4.1\cdot10^{-6} & -1.2\cdot10^{-5} & -3.1\cdot10^{-5} & 1.6\cdot10^{-4} & 0.3\cdot10^{-2} & 8.5\cdot10^{-6} & 5.2\cdot10^{-5} \\ -1.2\cdot10^{-5} & 2.9\cdot10^{-3} & 1.3\cdot10^{-5} & -1.8\cdot10^{-3} & 0.9\cdot10^{-2} & -2.1\cdot10^{-5} & -5.9\cdot10^{-4} \\ -3.1\cdot10^{-5} & 1.3\cdot10^{-5} & 2.8\cdot10^{-5} & 2.2\cdot10^{-5} & -0.3\cdot10^{-2} & -6.8\cdot10^{-6} & 7.1\cdot10^{-6} \\ 1.6\cdot10^{-4} & -1.8\cdot10^{-3} & 2.2\cdot10^{-5} & 10.4 & 0.5\cdot10^{-1} & -8.3\cdot10^{-6} & -3.3 \\ 3.2\cdot10^{-3} & 9.5\cdot10^{-3} & -3.0\cdot10^{-3} & 5.1\cdot10^{-2} & 8.6 & 1.6\cdot10^{-2} & 1.6\cdot10^{-2} \\ 8.5\cdot10^{-6} & -2.1\cdot10^{-5} & -6.8\cdot10^{-6} & -8.3\cdot10^{-6} & 0.1\cdot10^{-1} & 2.8\cdot10^{-4} & -2.6\cdot10^{-6} \\ 5.2\cdot10^{-5} & -5.9\cdot10^{-4} & 7.1\cdot10^{-6} & -3.3 & 0.1\cdot10^{-1} & -2.6\cdot10^{-6} & 1.1 \end{pmatrix}$$

Thus, the variances of the MLE of α, β, λ, θ, a, b and c is var(αˆ) = 4.195 · 10−6, var(βˆ) = 2.939 · 10−3, var(λˆ) = 2.805 · 10−5, var(θˆ) = 10.491, var(aˆ) = 8.611, var(ˆb) = 2.893 · 10−4, var(cˆ) = 1.071.
Therefore, 95% confidence intervals for α, β, λ, γ, a, b and c are [1.001, 1.178], [10.685, 10.898], [0.179, 0.199], [6.539, 19.235]
[1.150, 12.653], [0.080, 0.147] and [2.090, 6.147] respectively.

In order to compare the two distribution models, we consider criteria like , -2ℓ and AIC (Akaike information criterion) for the data set. The better distribution corresponds to smaller -2ℓ and AIC values:

$$AIC = 2k - 2\ell$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model. Table 3 shows the MLEs under both distributions and the values of -2ℓ and AIC values. The values in table 3 indicate that the McEMW distribution

leads to a better fit than the EMW, MW distribution and Weibull distribution. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 4). The fitted density for the McEMW

model is closer to the empirical histogram than the fits of the EMW,MW and Weibull sub-models.

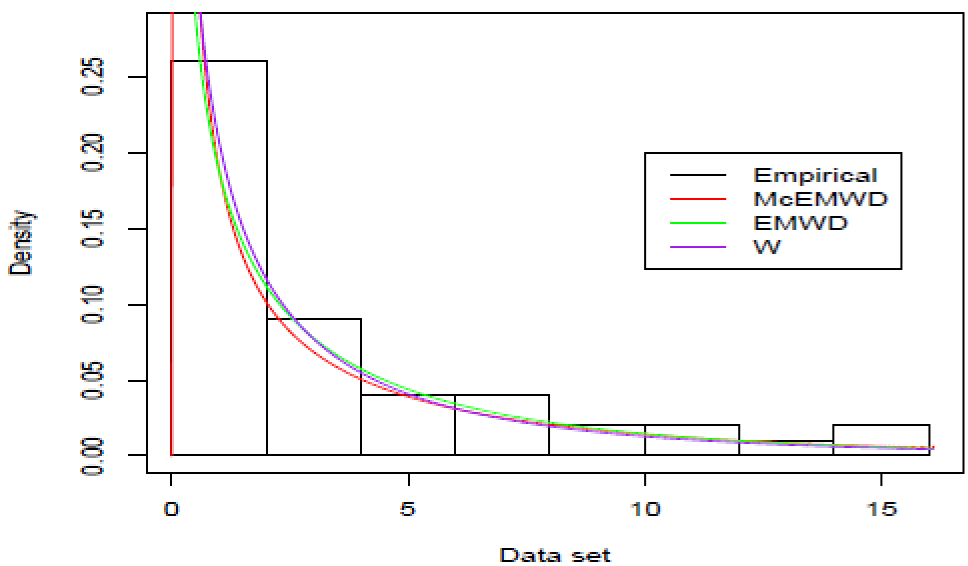

Figure 4: Estimated densities of the models for the failure times of 50 components.

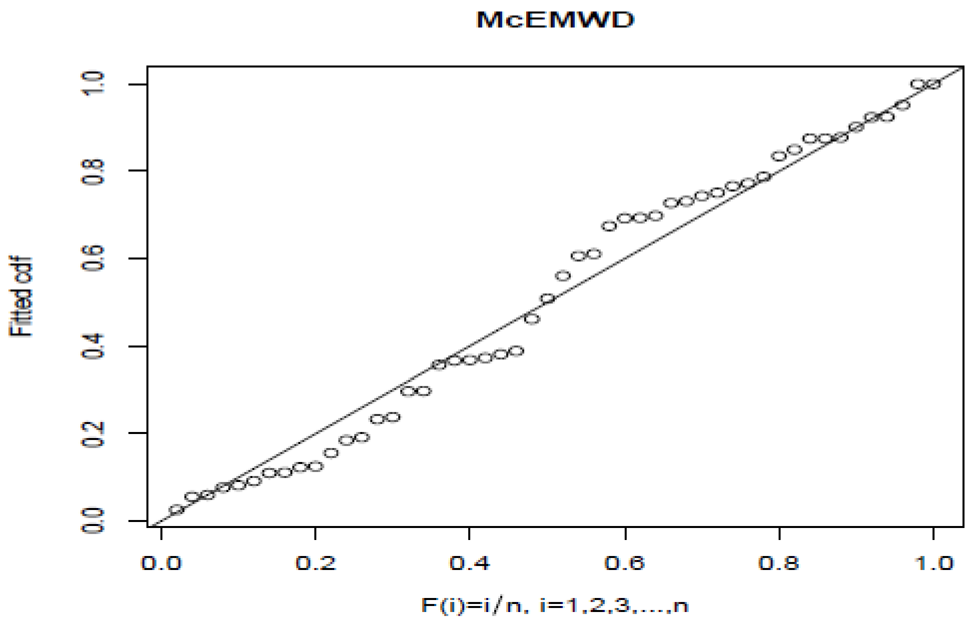

Figure 5: PP plots for fitted McEMD distribution.

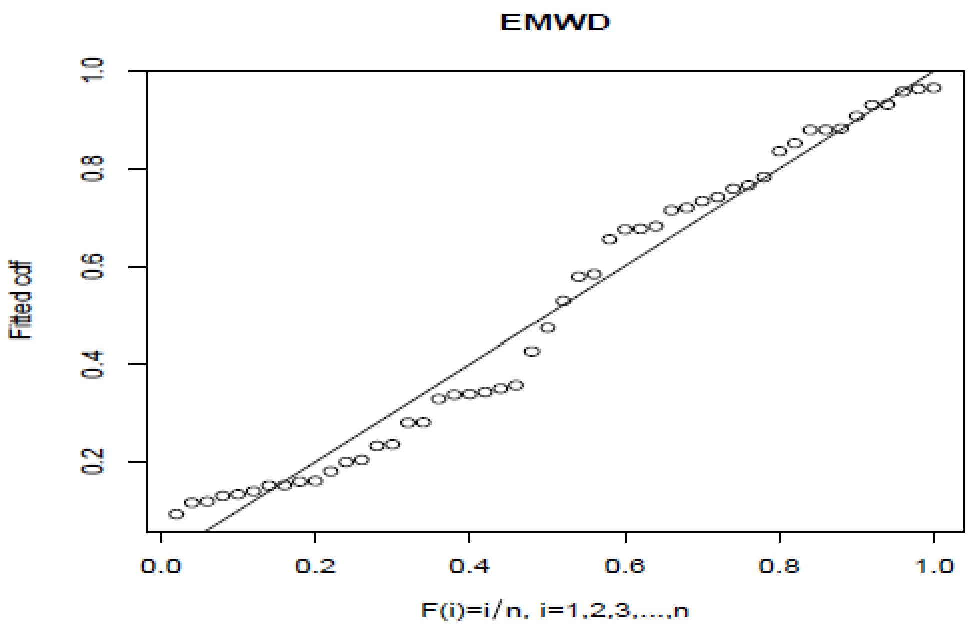

Figure 6: PP plots for fitted EMW distribution.

8. Conclusion

Here, we propose a new model, the so-called the McEMW dis- tribution which extends the EMW distribution in the analysis of data with rea lsupport. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modelling real data. We derive expansions for the moments and for the moment generating function. The esti- mation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. An application of the McEMW distribution to real data show that the new distribution can be used quite effectively to provide better fits than the EMW distribution.

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