Journal of Data Science **9**(2011), 445-470

Bayesian Estimation and Prediction for the Power Law Process with Left-Truncated Data

Guo-Liang Tian[1], Man-Lai Tang[2] and Jun-Wu Yu[3]
[1] *The University of Hong Kong,* [2] *Hong Kong Baptist University*
and [3] *Hunan University of Science and Technology*

Abstract: The *power law process* (PLP) (i.e., the nonhomogeneous Poisson process with power intensity law) is perhaps the most widely used model for analyzing failure data from reliability growth studies. Statistical inferences and prediction analyses for the PLP with left-truncated data with classical methods were extensively studied by Yu *et al.* (2008) recently. However, the topics discussed in Yu *et al.* (2008) only included maximum likelihood estimates and confidence intervals for parameters of interest, hypothesis testing and goodness-of-fit test. In addition, the prediction limits of future failure times for failure-truncated case were also discussed. In this paper, with Bayesian method we consider seven totally different prediciton issues besides point estimates and prediction limits for x_{n+k}. Specifically, we develop estimation and prediction methods for the PLP in the presence of left-truncated data by using the Bayesian method. Bayesian point and credible interval estimates for the parameters of interest are derived. We show how five single-sample and three two-sample issues are addressed by the proposed Bayesian method. Two real examples from an engine development program and a repairable system are used to illustrate the proposed methodologies.

Key words: Bayesian method, nonhomogeneous Poisson process, noninformative prior, prediction intervals, reliability growth.

1. Introduction

When failure times from different systems during their development programs are collected and analyzed, an approximate straight line pattern in the corresponding log-log plot of the cumulative *mean time between failures* (MTBF) against the cumulative operating time is usually observed (see, Duane, 1964). Crow (1974) extended the Duane model to the *nonhomogeneous Poisson process* (NHPP) with a power intensity law, which is also known as AMSAA model due to its adoption by the U.S. Army Materiel Systems Analysis Activity. The

NHPP model is generally used to monitor the reliability growth of repairable systems, to assess the reliability growth of software and to predict failure behaviors. Statistically, a NHPP $\{N(t),\ t \geq 0\}$ with the power intensity law

$$\lambda(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}, \quad \alpha,\ \beta > 0, \tag{1.1}$$

is also known as *power law process* (PLP) or Weibull process. The corresponding mean function is defined by

$$m(t) = E\{N(t)\} = \int_0^t \lambda(s)\, ds = (t/\alpha)^\beta.$$

It was shown by Rigdon (2002) that a linear Duane plot does not imply a PLP, and a PLP does not imply a linear Duane plot.

Assume that we conduct a reliability growth test on some repairable system in the time interval $(0,\ t]$. For the *failure truncated* case, the number of failures, n, is predetermined. Let $0 < x_1 < x_2 < \cdots < x_n$ be the first n ordered failure times of the PLP. The time to the first failure (i.e., x_1) can be shown to follow Weibull distribution with scale parameter α and shape parameter β. That is, the *probability density function* (pdf) of x_1 is (Crowder *et al.*, 1991)

$$f_1(x_1) = \frac{\beta}{\alpha}\left(\frac{x_1}{\alpha}\right)^{\beta-1} e^{-(x_1/\alpha)^\beta}, \quad x_1 > 0$$

with the corresponding distribution function

$$F_1(x_1) = 1 - e^{-(x_1/\alpha)^\beta}, \quad x_1 > 0.$$

Let $f_i(x_i|x_1, \cdots, x_{i-1})$ denote the conditional density function of x_i given $x_1, \cdots,$ x_{i-1}, we have

$$f_i(x_i|x_1, \cdots, x_{i-1}) = \frac{\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x_i}{\alpha}\right)^\beta + \left(\frac{x_{i-1}}{\alpha}\right)^\beta\right], \quad x_i > x_{i-1}. \tag{1.2}$$

The joint pdf of $x_1, \cdots, x_n$ is thus given by

$$\begin{aligned} f(x_1, \cdots, x_n) &= f_1(x_1)\prod_{i=2}^n f_i(x_i|x_1, \cdots, x_{i-1}) \qquad (1.3)\\ &= \left(\frac{\beta}{\alpha}\right)^n e^{-(x_n/\alpha)^\beta} \prod_{i=1}^n \left(\frac{x_i}{\alpha}\right)^{\beta-1}, \qquad 0 < x_1 < \cdots < x_n. \end{aligned}$$

For the *time truncated* case, the test time t is predetermined. Let $0 < x_1 < \cdots < x_n < t$ be the observed failure times of the PLP. Similar to (1.3), the joint pdf of $(x_1, \cdots, x_n; N = n)$ is given by

$$\begin{aligned} f(x_1, \cdots, x_n; N = n) &= \left(\frac{\beta}{\alpha}\right)^n e^{-(t/\alpha)^\beta} \prod_{i=1}^n \left(\frac{x_i}{\alpha}\right)^{\beta-1}, \\ &\quad 0 < x_1 < \cdots < x_n < t, \quad n > 0, \\ \Pr\{N = 0\} &= e^{-(t/\alpha)^\beta}, \quad n = 0, \end{aligned} \tag{1.4}$$

where $N \mathrel{\hat=} N(t)$ and the symbol $\hat=$ means "equal by definition".

In practice, left-truncated data occur due to various reasons such as not being able to observe in the early developmental phase of a testing program. Recently, Yu *et al.* (2008) developed classical methods for statistical inferences and prediction analyses for the PLP with the first $r-1$ failure times (i.e., $\{x_i\}_{i=1}^{r-1}$) being missing. Using their notations, we denote the observed data as $Y_{\text{obs}}^{\text{ft}} = \{x_r, \cdots, x_n\}$ for the failure-truncated case and $Y_{\text{obs}}^{\text{tt}} = \{x_r, \cdots, x_n; t\}$ for the time-truncated case. First, the topics discussed in Yu *et al.* (2008) only included maximum likelihood estimates and confidence intervals for parameters of interest (e.g., α, β and MTBF), hypothesis testing on α and β, and goodness-of-fit test. In addition, the prediction limit of the $(n+k)$-th future failure time (i.e., x_{n+k}) for failure-truncated case was also discussed. In this paper, we will consider seven totally different issues (see, Sections 4.2–4.4 and 5) besides point estimates (see, Section 3) and prediction limits for x_{n+k} (see, Section 4.1), and solutions to the seven prediction issues are not yet available to date for both classical and Bayesian methods. Second, the prediction limit of x_{n+k} was available only for the failure-truncated case (Yu *et al.*, 2008). A similar result does not yet exist for the time-truncated case. The main reason is our inability to find an adequate prediction statistic in the framework of classical methods. Fortunately, the Bayesian method can be applied (see, Section 4.1). Finally, there is a computational challenge in obtaining exact solutions from the classical methods and the accuracy of approximate formulae are heavily dependent of large sample sizes, while the Bayesian method can facilitate the computation by employing the conditional sampling procedure (see, two conditional sampling procedures in Sections 3.1 and 3.3).

This article aims to develop Bayesian estimation and prediction methods for the PLP with the first $r-1$ failure times being missing. This left-truncated data pattern commonly occurs when (i) the importance of a reliability growth program is eventually recognized only when manufacturers reported the failures several times; and (ii) a new data-recording person may not be able to determine the exact failure times during the early stage of the process. This article is organized as follows. Section 2 presents the posterior and predictive distributions.

Section 3 derives Bayesian point estimates and credible interval estimates for the parameters of interest. Five practical single-sample issues are discussed and then addressed by the Bayesian approach in Section 4. In Section 5, we develop Bayesian methods for three two-sample issues. Two real examples from an engine development program and a repairable system are used to illustrate the proposed methodologies in Section 6. A brief discussion is given in Section 7.

2. Posterior and Predictive Distributions

In this article, we consider a common scenario in which missing data are produced only in the early stage of the test time (see, Yu *et al.*, 2008). That is, $\{x_i\}_{i=1}^{r-1}$ are missing. For the failure-truncated PLP, the joint pdf of the observed data $Y_{\text{obs}}^{\text{ft}} = \{x_i\}_{i=r}^{n}$ is

$$f(x_r, \cdots, x_n) = \frac{\beta^{n-r+1}\exp[-(x_n/\alpha)^\beta]}{(r-1)!\,\alpha^{n\beta}} \cdot x_r^{(r-1)\beta}\prod_{i=r}^{n} x_i^{\beta-1}, \quad 0 < x_1 < \cdots < x_n. \tag{2.1}$$

For the time-truncated PLP, we also assume that $\{x_i\}_{i=1}^{r-1}$ are missing. The joint pdf of the observed data $Y_{\text{obs}}^{\text{tt}} = \{x_r, \cdots, x_n; t\}$ is (Yu *et al.*, 2008)

$$f(x_r, \cdots, x_n; N = n) = \frac{\beta^{n-r+1}\exp[-(t/\alpha)^\beta]}{(r-1)!\,\alpha^{n\beta}} \cdot x_r^{(r-1)\beta}\prod_{i=r}^{n} x_i^{\beta-1}, \tag{2.2}$$

where $0 < x_1 < \cdots < x_n < t$. Let Y_{obs} represent $Y_{\text{obs}}^{\text{ft}}$ or $Y_{\text{obs}}^{\text{tt}}$. Combining (2.1) with (2.2), the likelihood function for α and β is then given by

$$L(\alpha, \beta|Y_{\text{obs}}) = [(r-1)!]^{-1}\alpha^{-n\beta}\beta^{n-r+1}\exp[-(\tau/\alpha)^\beta] \cdot x_r^{(r-1)\beta}u^{\beta-1}, \quad \alpha,\ \beta > 0, \tag{2.3}$$

where

$$u \mathrel{\hat{=}} \prod_{i=r}^{n} x_i, \quad \text{and} \quad \tau \mathrel{\hat{=}} \begin{cases} x_n, & \text{if } Y_{\text{obs}} = Y_{\text{obs}}^{\text{ft}}, \\ t, & \text{if } Y_{\text{obs}} = Y_{\text{obs}}^{\text{tt}}. \end{cases} \tag{2.4}$$

When prior information is not available, it is reasonable to use noninformative prior distributions.

Case 1: Shape parameter β is known. Following Guida *et al.* (1989), we choose the following noninformative prior density of α

$$g(\alpha) \propto 1/\alpha, \quad \alpha > 0. \tag{2.5}$$

The posterior distribution of α is thus given by

$$\begin{aligned} h(\alpha|Y_{\text{obs}}) &\propto L(\alpha, \beta|Y_{\text{obs}}) \times g(\alpha) \\ &= [\Gamma(n)]^{-1}\beta\tau^{n\beta}\alpha^{-n\beta-1}\exp[-(\tau/\alpha)^\beta], \quad \alpha > 0. \end{aligned} \tag{2.6}$$

Let y be any random variable that one wants to predict (e.g., x_{n+k} with k being any positive integer); the predictive density of y is

$$f(y|Y_{\text{obs}}) = \int_0^\infty f(y|Y_{\text{obs}}, \alpha)h(\alpha|Y_{\text{obs}})d\alpha. \tag{2.7}$$

Hence, the Bayesian *upper prediction limit* (UPL) of y with level γ, denoted as $y_U^{(\beta)}$, satisfies

$$\gamma = \int_{-\infty}^{y_U^{(\beta)}} f(y|Y_{\text{obs}})dy. \tag{2.8}$$

Case 2: Shape parameter β is unknown. Following Box & Tiao (1973), we consider the following noninformative joint prior density for (α, β)

$$g(\alpha, \beta) \propto 1/(\alpha\beta), \quad \alpha,\ \beta > 0. \tag{2.9}$$

Hence, the corresponding joint posterior density is

$$h(\alpha, \beta|Y_{\text{obs}}) = c^{-1}\beta^{n-r}(ux_r^{r-1})^\beta \cdot \alpha^{-n\beta-1}\exp[-(\tau/\alpha)^\beta], \quad \alpha,\ \beta > 0, \tag{2.10}$$

where

$$c \mathrel{\hat=} \Gamma(n)\Gamma(n-r)/z^{n-r},$$

and

$$z \mathrel{\hat=} \sum_{i=r+1}^{n} \ln(\tau/x_i) + r\ln(\tau/x_r). \tag{2.11}$$

Similar to (2.7) and (2.8), let y_U denote the Bayesian UPL of y with level γ. Hence, we have

$$f(y|Y_{\text{obs}}) = \int_0^\infty \int_0^\infty f(y|Y_{\text{obs}}, \alpha, \beta)h(\alpha, \beta|Y_{\text{obs}})d\alpha d\beta,$$

and

$$\gamma = \int_{-\infty}^{y_U} f(y|Y_{\text{obs}})dy. \tag{2.12}$$

3. Bayesian Point and Credible Interval Estimates

3.1 Bayesian estimates of α and β

We first consider the case in which β is known. Let

$$X \mathrel{\hat=} (\tau/\alpha)^\beta. \tag{3.1}$$

The inverse transformation is then $\alpha = \tau\, X^{-1/\beta}$. From (2.6), it is easy to verify that

$$X|Y_{\rm obs} \sim \Gamma(n, 1) \;\text{ or }\; 2X|Y_{\rm obs} \sim \chi^2(2n). \tag{3.2}$$

Therefore, the Bayes point estimator and two-sided $100\gamma\%$ Bayes *credible interval* (CI) for α are respectively given by

$$\tilde{\alpha} = E(\alpha|Y_{\rm obs}) = \tau\, E(X^{-1/\beta}|Y_{\rm obs}) = \tau\Gamma\left(n - \frac{1}{\beta}\right)\Big/\Gamma(n), \tag{3.3}$$

and

$$\left[\tau\left\{\frac{1}{2}\chi^2(2n;(1+\gamma)/2)\right\}^{-1/\beta},\; \tau\left\{\frac{1}{2}\chi^2(2n;(1-\gamma)/2)\right\}^{-1/\beta}\right], \tag{3.4}$$

where $\chi^2(n;\gamma)$ denotes the γ percentage point of the chi-square distribution with n degrees of freedom such that $\Pr\{\chi^2(n) \le \chi^2(n;\gamma)\} = \gamma$.

When β is unknown, we can integrate (2.10) with respect to α to obtain the following marginal posterior density of β

$$h(\beta|Y_{\rm obs}) = \frac{z^{n-r}}{\Gamma(n-r)}\beta^{n-r-1}e^{-\beta z}, \quad \beta > 0. \tag{3.5}$$

Obviously, (3.5) implies that

$$\beta|Y_{\rm obs} \sim \Gamma(n-r,\; z) \;\text{ or }\; 2z\beta|Y_{\rm obs} \sim \chi^2(2n-2r). \tag{3.6}$$

Based on (3.6), the Bayes point estimator and two-sided $100\gamma\%$ Bayes CI for β are respectively given by

$$\tilde{\beta} = (n-r)/z, \tag{3.7}$$

and

$$\left[\chi^2(2n-2r;(1-\gamma)/2)/(2z),\; \chi^2(2n-2r;(1+\gamma)/2)/(2z)\right]. \tag{3.8}$$

By comparing (3.8) with (3.11) in Yu, Tian and Tang (2008), we can immediately conclude that the Bayes CI for β is identical to the classical confidence interval for β when the joint prior is chosen to be (2.9).

To derive the Bayes estimates for α, we adopt the conditional sampling method. From (2.10), we have

$$h(\alpha|Y_{\rm obs},\beta) \propto \alpha^{-n\beta-1}\exp[-(\tau/\alpha)^\beta].$$

Similar to (3.2), we obtain

$$X|(Y_{\rm obs},\beta) \sim \Gamma(n,1), \tag{3.9}$$

where X is defined in (3.1). Hence, the conditional sampling algorithm can be summarized as follows.

The conditional sampling:

Step 1. Generate m i.i.d. posterior samples $\{\beta^{(\ell)}\}_{\ell=1}^{m}$ of β according to (3.6);

Step 2. Generate m i.i.d. posterior samples $\{X^{(\ell)}\}_{\ell=1}^{m}$ of X according to (3.9);

Step 3. For each given pair $(\beta^{(\ell)}, X^{(\ell)})$, calculate $\alpha^{(\ell)} = \tau\,(X^{(\ell)})^{-1/\beta^{(\ell)}}$. Here, $\{\alpha^{(\ell)}\}_{\ell=1}^{m}$ are i.i.d. posterior samples of α.

The Bayes estimates for α can then be obtained from the above i.i.d. posterior samples $\{\alpha^{(\ell)}\}_{\ell=1}^{m}$. For example, the Bayes point estimator of α is then given by $\tilde{\alpha} = \frac{1}{m}\sum_{\ell=1}^{m}\alpha^{(\ell)}$.

3.2 The Pseudo Bayesian Estimate of $M(x_n)$

In this subsection, we restrict our discussion to the failure-truncated case. If no improvements are incorporated into the repairable system after the time of the n-th failure (i.e., x_n) and the intensity $\lambda(x_n) = (\beta/\alpha)(x_n/\alpha)^{\beta-1}$ remains constant thereafter, then the subsequent times between failures of the system independently follow exponential distribution with the common failure rate $\lambda(x_n)$ and the MTBF $M(x_n) \hat{=} 1/\lambda(x_n)$. Since $M(x_n)$ involves the random variable x_n, its posterior density cannot be obtained.[1] Following the idea of Higgins and Tsokos (1981), we consider the pseudo Bayes point and CI estimates for $\lambda(x_n)$ (or $M(x_n)$ equivalently) instead. For this purpose, we noted that (see, Theorem 1 in Yu *et al.*, 2008)

$$\frac{\lambda(x_n)}{\hat{\lambda}_M(x_n)} = \frac{Z \cdot S}{4n(n-r+1)},$$

where $Z \sim \chi^2(2n-2r)$ is independent of $S \sim \chi^2(2n)$ and $\hat{\lambda}_M(x_n) = n(n-r+1)/(zx_n)$ is the maximum likelihood estimate of $\lambda(x_n)$. For the sake of convenience, we define

$$\lambda_n \hat{=} \lambda(x_n), \quad \hat{\lambda}_M \hat{=} \hat{\lambda}_M(x_n), \quad a \hat{=} \frac{2n(n-r+1)}{2n-r+1}, \quad \text{and} \quad Q \hat{=} \frac{a\lambda_n}{\hat{\lambda}_M}.$$

As a result, we have

$$E(Q) = n^* \quad \text{and} \quad \mathrm{Var}(Q) = 2n^*, \tag{3.10}$$

where

$$n^* = \frac{2n(n-r)}{2n-r+1}. \tag{3.11}$$

[1]Some researchers think that although the failure time x_n is a random variable, it has been observed before making posterior inference. Therefore, the posterior Bayesian estimates of $\lambda(x_n)$ and $M(x_n)$ in a failure-truncated PLP can be easily obtained through the same procedure used for the time-truncated case (see Section 3.3).

By treating λ_n as a parameter, we consider the transformation $\hat{\lambda}_M = a\lambda_n/Q$. Since (3.10) implies that Q may be approximated by the chi-square random variable $\chi^2(n^*)$, a pseudo likelihood function for $\hat{\lambda}_M$ takes the following form

$$L(\hat{\lambda}_M|\lambda_n) \propto \frac{1}{\hat{\lambda}_M}\left(\frac{\lambda_n}{\hat{\lambda}_M}\right)^{n^*/2} \cdot \exp\left[-\frac{a\lambda_n}{2\hat{\lambda}_M}\right], \quad \hat{\lambda}_M > 0.$$

If we adopt a noninformative prior distribution of λ_n (i.e., $g(\lambda_n) \propto 1/\lambda_n$), then the pseudo-posterior density of λ_n is

$$f(\lambda_n|\hat{\lambda}_M) \propto \lambda_n^{n^*/2-1} \cdot \exp\left[-\frac{a\lambda_n}{2\hat{\lambda}_M}\right].$$

Therefore, the pseudo Bayes point estimator and the two-sided $100\gamma\%$ Bayes CI for λ_n are respectively given by

$$\tilde{\lambda}_n = n^*\hat{\lambda}_M/a, \tag{3.12}$$

and

$$\Big[a^{-1}\hat{\lambda}_M\,\chi^2(n^*;(1-\gamma)/2),\ a^{-1}\hat{\lambda}_M\,\chi^2(n^*;(1+\gamma)/2)\Big]. \tag{3.13}$$

Usually, n^* in (3.11) is not a positive integer. In this case, we can approximate $\chi^2(n^*;\gamma)$ by

$$\chi^2(n^*;\gamma) \doteq n^*\left(1-\frac{2}{9n^*}+u_\gamma\sqrt{\frac{2}{9n^*}}\right)^3,$$

where u_γ denotes the γ percentage point of the standard normal distribution $\mathcal{N}(0,1)$. On the other hand, there exist algorithms to calculate $\chi^2(n^*;\gamma)$ for fractional degrees of freedom. For example, the built-in S-plus function `qchisq` can handle this case.

3.3 The Bayesian estimate of $M(t)$

In this subsection, we consider the time-truncated case. Equivalently, we consider the posterior estimate for $\lambda_t \hat{=} \lambda(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}$.

When β is known, from (2.6), the posterior density of λ_t is

$$h(\lambda_t|Y_{\text{obs}}) \propto \lambda_t^{n-1}\exp(-\lambda_t t\beta^{-1}), \quad \lambda_t > 0, \tag{3.14}$$

that is,

$$\lambda_t \sim \Gamma(n, t\beta^{-1}) \quad \text{or} \quad 2t\beta^{-1}\lambda_t \sim \chi^2(2n).$$

Therefore, the Bayes point estimator and the two-sided $100\gamma\%$ Bayes CI for λ_t are respectively given by

$$\tilde{\lambda}_t(\beta) = n\beta/t, \tag{3.15}$$

and

$$\left[\frac{\beta}{2t}\,\chi^2(2n;(1-\gamma)/2),\ \frac{\beta}{2t}\,\chi^2(2n;(1+\gamma)/2)\right]. \tag{3.16}$$

When β is unknown, we consider the following transformation

$$\begin{cases} \lambda_t &= (\beta/\alpha)(t/\alpha)^{\beta-1}, \quad \lambda_t > 0, \\ \beta &= \beta, \qquad\qquad\qquad\quad \beta > 0. \end{cases}$$

According to (2.10), the joint posterior distribution of (λ_t, β) can be shown to be

$$\begin{aligned} h(\lambda_t,\beta|Y_{\text{obs}}) &= h(\alpha,\beta|Y_{\text{obs}})\cdot J(\alpha,\beta\to\lambda_t,\beta) & (3.17)\\ &= c^{-1}\beta^{n-r}(ux_r^{r-1})^{\beta}\cdot\alpha^{-n\beta-1}\exp[-(t/\alpha)^{\beta}]\cdot(\lambda_t\beta/\alpha)^{-1}\\ &= c^{-1}t^n\beta^{-r-1}\lambda_t^{n-1}\exp(-\beta z - t\lambda_t/\beta), \quad \lambda_t>0,\ \beta>0. \end{aligned}$$

Integrating (3.17) with respect to β yield the following posterior distribution of λ_t

$$h(\lambda_t|Y_{\text{obs}}) = c^{-1}t^n\lambda_t^{n-1}\cdot\int_0^{\infty}\beta^{-r-1}\exp(-\beta z - t\lambda_t/\beta)\,d\beta, \quad \lambda_t > 0. \tag{3.18}$$

Therefore, the Bayes point estimator of λ_t is

$$\begin{aligned} \tilde{\lambda}_t &= E(\lambda_t|Y_{\text{obs}})\\ &= c^{-1}t^n\cdot\int_0^{\infty}\left\{\beta^{-r-1}e^{-\beta z}\int_0^{\infty}\lambda_t^n\exp(-t\lambda_t/\beta)\,d\lambda_t\right\}d\beta\\ &= n(n-r)/(tz). & (3.19) \end{aligned}$$

From (3.17), we obtain $h(\lambda_t|Y_{\text{obs}},\beta)\propto\lambda_t^{n-1}\exp(-t\lambda_t/\beta)$, i.e.,

$$\lambda_t|(Y_{\text{obs}},\beta)\sim\Gamma(n,\ t/\beta). \tag{3.20}$$

To construct a Bayes CI of λ_t, we consider the following conditional sampling method to obtain i.i.d. posterior samples of λ_t.

The conditional sampling:

Step 1. Generate m i.i.d. posterior samples $\{\beta^{(\ell)}\}_{\ell=1}^m$ of β according to (3.6);

Step 2. For each given $\beta^{(\ell)}$, generate $\lambda_t^{(\ell)}$ according to (3.20), $\ell = 1, \cdots, m$. $\{\lambda_t^{(\ell)}\}_{\ell=1}^m$ are then m i.i.d. posterior samples of λ_t.

4. Bayesian Predictions and Estimations for Single-Sample Problems

In this section, we consider five practical single-sample issues.

4.1 Prediction limits for x_{n+k}

4.1.1 Shape parameter β is known

Let $f(x_{n+k}|Y_{\rm obs})$ denote the predictive density function of the $(n+k)$-th future failure time (i.e., x_{n+k}) given the observed data $Y_{\rm obs}$. According to (2.8), the Bayesian UPL $x^{(\beta)}_{U,B}(n,k,\gamma)$ of x_{n+k} with confidence level γ should satisfy

$$\gamma = \Pr\left\{x_{n+k} \le x^{(\beta)}_{U,B}(n,k,\gamma)\Big|Y_{\rm obs}\right\} = \int_{\tau}^{x^{(\beta)}_{U,B}(n,k,\gamma)} f(x_{n+k}|Y_{\rm obs})\,dx_{n+k}, \qquad (4.1)$$

where

$$f(x_{n+k}|Y_{\rm obs}) = \int_0^\infty h(\alpha|Y_{\rm obs})\cdot f(x_{n+k}|Y_{\rm obs},\alpha)\,d\alpha. \qquad (4.2)$$

It should be noted that $h(\alpha|Y_{\rm obs})$ is given by (2.6) and

$$\begin{aligned} f(x_{n+k}|Y_{\rm obs},\alpha) &= \frac{f(Y_{\rm obs},\ x_{n+k}|\alpha)}{f(Y_{\rm obs}|\alpha)} \qquad (4.3)\\ &= \frac{\int_{\tau<x_{n+1}<\cdots<x_{n+k-1}<x_{n+k}} f(x_r,x_{r+1},\cdots,x_{n+k}|\alpha)\ \prod_{i=n+1}^{n+k-1} dx_i}{f(Y_{\rm obs}|\alpha)}, \end{aligned}$$

where $f(Y_{\rm obs}|\alpha)$ is given by (2.3) and $f(x_r,x_{r+1},\cdots,x_{n+k}|\alpha)$ is also given by (2.3) with (n,τ) being replaced by $(n+k,\ x_{n+k})$, that is,

$$f(x_r,x_{r+1},\cdots,x_{n+k}|\alpha) = \frac{\beta^{n+k-r+1}\exp[-(x_{n+k}/\alpha)^\beta]}{(r-1)!\alpha^{(n+k)\beta}}\cdot x_r^{(r-1)\beta}\prod_{i=r}^{n+k} x_i^{\beta-1}.$$

By using the identity (2.2) in Yu *et al.* (2008), we immediately have the numerator in the right-hand side of (4.3) being equal to

$$\frac{\beta^{n-r+2}\exp[-(x_{n+k}/\alpha)^\beta]}{(r-1)!(k-1)!\alpha^{(n+k)\beta}}\cdot x_r^{(r-1)\beta}u^{\beta-1}\cdot x_{n+k}^{\beta-1}\left[x_{n+k}^\beta-\tau^\beta\right]^{k-1}.$$

Hence, (4.3) (cf. Calabria, Guida and Pulcini, 1990) and (4.2) become

$$f(x_{n+k}|Y_{\rm obs},\alpha) = \frac{\beta\exp[-(x_{n+k}^\beta-\tau^\beta)/\alpha^\beta]}{(k-1)!\alpha^{k\beta}}\cdot x_{n+k}^{\beta-1}\left[x_{n+k}^\beta-\tau^\beta\right]^{k-1}, \qquad (4.4)$$

and

$$f(x_{n+k}|Y_{\rm obs}) = \frac{1}{B(n,k)}\cdot\frac{\beta}{x_{n+k}}\left(\frac{\tau}{x_{n+k}}\right)^{n\beta}\left[1-\left(\frac{\tau}{x_{n+k}}\right)^\beta\right]^{k-1}, \qquad (4.5)$$

respectively. Substituting (4.5) into (4.1) yields

$$\gamma = \frac{1}{B(n,k)} \int_{[\tau / x_{U,B}^{(\beta)}(n,k,\gamma)]^\beta}^{1} y^{n-1}(1-y)^{k-1}\, dy.$$

Thus, $[\tau / x_{U,B}^{(\beta)}(n,k,\gamma)]^\beta$ equals to the $1-\gamma$ percentage point of the $\mathcal{B}eta(n,k)$ distribution. From the relationship between the quantiles of beta distribution and F-distribution, we have

$$x_{U,B}^{(\beta)}(n,k,\gamma) = \tau \left[\frac{k}{n} F(2k, 2n; \gamma) + 1 \right]^{1/\beta}, \tag{4.6}$$

where $F(m,n;\gamma)$ represents the γ percentage point of the F-distribution with m and n degrees of freedom.

4.1.2 Shape parameter β is unknown

The Bayes UPL for x_{n+k} with confidence level γ satisfies

$$\begin{aligned} \gamma &= \int_{\tau}^{x_{U,B}(n,k,r,\gamma)} f(x_{n+k}|Y_{\text{obs}})\, dx_{n+k} \qquad (4.7) \\ &= \int_{\tau}^{x_{U,B}(n,k,r,\gamma)} \left\{ \int_0^\infty \int_0^\infty h(\alpha,\beta|Y_{\text{obs}}) \cdot f(x_{n+k}|Y_{\text{obs}},\alpha,\beta)\, d\alpha d\beta \right\} dx_{n+k}, \end{aligned}$$

where $h(\alpha,\beta|Y_{\text{obs}})$ is given in (2.10) while $f(x_{n+k}|Y_{\text{obs}},\alpha,\beta)$ is given in (4.4). Thus,

$$\begin{aligned} & \int_0^\infty \int_0^\infty h(\alpha,\beta|Y_{\text{obs}}) \cdot f(x_{n+k}|Y_{\text{obs}},\alpha,\beta)\, d\alpha d\beta \\ = \; & \frac{\Gamma(n+k)}{c(k-1)!} \cdot \int_0^\infty \beta^{n-r} e^{-\beta z} \frac{1}{x_{n+k}} \left(\frac{\tau}{x_{n+k}} \right)^{n\beta} \left[1 - \left(\frac{\tau}{x_{n+k}} \right)^\beta \right]^{k-1} d\beta. \quad (4.8) \end{aligned}$$

It can be shown that

$$x_{U,B}(n,k,r,\gamma) = \tau \exp\left\{ y_\gamma \cdot z \Big/ [(n-r)(n-r+1)] \right\}, \tag{4.9}$$

where y_γ is the solution to the following equation

$$\gamma = \sum_{j=1}^{k} \left(\prod_{i=1, i\neq j}^{k} \frac{n+k-i}{j-i} \right) \cdot \left[1 - \left(1 + \frac{(n+k-j) y_\gamma}{(n-r)(n-r+1)} \right)^{-n+r} \right]. \tag{4.10}$$

To prove (4.9) and (4.10), we substitute both (4.9) and (4.8) into (4.7) and then make the transformation $x = (\tau/x_{n+k})^\beta$, which yields

$$\gamma = \frac{\Gamma(n+k)}{c(k-1)!} \cdot \int_0^\infty \beta^{n-r-1} e^{-\beta z} \left\{ \int_R^1 x^{n-1}(1-x)^{k-1}\, dx \right\} d\beta,$$

where $R = e^{-\beta y_\gamma z/[(n-r)(n-r+1)]}$. Using the following identity

$$\frac{1}{B(n,k)} \int_R^1 x^{n-1}(1-x)^{k-1}\, dx = 1 - \sum_{j=1}^{k} \binom{n+k-1}{j-1} R^{n+k-j}(1-R)^{j-1}$$

and expanding the term $(1-R)^{j-1}$, we can readily obtain (4.10).

4.2 Estimating the probability of $N(\tau, T) \le k$

Based on $Y_{\rm obs}^{\rm ft}$ or $Y_{\rm obs}^{\rm tt}$, we are interested in the following question:

Issue A1: *What is the probability that at most k failures will occur in the future time period* $(\tau,\, T]$ *with* $T > \tau$?

Equivalently, we wish to estimate the following probability:

$$\gamma_k = \Pr\{N(\tau, T) \le k | Y_{\rm obs}\} = \Pr\{N(T) \le n+k | Y_{\rm obs}\}, \tag{4.11}$$

where $N(\tau, T) \hat{=} N(T) - N(\tau) = N(T) - n$.

4.2.1 Shape parameter β is known

In this case, we denote γ_k in (4.11) by $\gamma_k^{(\beta)}$ and we have

$$\begin{aligned} \gamma_k^{(\beta)} &= \int_0^\infty h(\alpha|Y_{\rm obs}) \cdot \Pr\{N(T) \le n+k|Y_{\rm obs}, \alpha\}\, d\alpha \\ &= \int_0^\infty h(\alpha|Y_{\rm obs}) \cdot \sum_{j=n}^{n+k} \Pr\{N(T) = j|Y_{\rm obs}, \alpha\}\, d\alpha \\ &= \int_0^\infty h(\alpha|Y_{\rm obs}) \cdot \sum_{j=n}^{n+k} \frac{f(Y_{\rm obs},\ N(T)=j|\alpha)}{f(Y_{\rm obs}|\alpha)}\, d\alpha \end{aligned} \tag{4.12}$$

where $h(\alpha|Y_{\text{obs}})$ and $f(Y_{\text{obs}}|\alpha)$ are respectively given by (2.6) and (2.3), and

$$\begin{aligned}
& f(Y_{\text{obs}},\ N(T)=j|\alpha) \\
= & \int_{\tau<x_{n+1}<\cdots<x_j<T} f(Y_{\text{obs}}, x_{n+1}, \cdots, x_j, N(T)=j|\alpha) \prod_{\ell=n+1}^{j} dx_\ell \\
\overset{(2.3)}{=} & \int_{\tau<x_{n+1}<\cdots<x_j<T} \frac{\beta^{j-r+1} e^{-(T/\alpha)^\beta} \cdot x_r^{(r-1)\beta} \prod_{i=r}^{j} x_i^{\beta-1}}{(r-1)!\alpha^{j\beta}} \prod_{\ell=n+1}^{j} dx_\ell \\
= & \frac{\beta^{n-r+1} e^{-(T/\alpha)^\beta} \cdot x_r^{(r-1)\beta} u^{\beta-1} [T^\beta - \tau^\beta]^{j-n}}{(r-1)!(j-n)!\alpha^{j\beta}}.
\end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.12) yields[2]

$$\begin{aligned}
\gamma_k^{(\beta)} &= \sum_{j=n}^{n+k} \int_0^\infty \frac{\beta\tau^{n\beta}[T^\beta - \tau^\beta]^{j-n}}{\Gamma(n)(j-n)!} \cdot \alpha^{-j\beta-1} e^{-(T/\alpha)^\beta}\, d\alpha \\
&= \sum_{j=n}^{n+k} \binom{j-1}{n-1} \delta^n (1-\delta)^{j-n} \\
&= \left(\frac{\delta}{1-\delta}\right)^n \sum_{j=n}^{n+k} \binom{j-1}{n-1} (1-\delta)^j,
\end{aligned} \tag{4.14}$$

where

$$\delta \mathrel{\hat{=}} (\tau/T)^\beta. \tag{4.15}$$

In particular, from (4.14), we have the recursive formula

$$\gamma_0^{(\beta)} = \delta^n, \qquad \gamma_1^{(\beta)} = \delta^n(n+1-n\delta), \tag{4.16}$$

and

$$\gamma_k^{(\beta)} = \gamma_{k-1}^{(\beta)} + \binom{n+k-1}{n-1}(1-\delta)^k \delta^n, \quad k \geq 2. \tag{4.17}$$

4.2.2 Shape parameter β is unknown

[2]Alternatively, formula (4.14) can also be obtained by exploiting the following result:

$$\Pr\{N(\tau,T) \leq k|Y_{\text{obs}}, \alpha, \beta\} = \sum_{i=0}^{k} \frac{[(T/\alpha)^\beta - (\tau/\alpha)^\beta]^i}{i!} \cdot \exp[-(T/\alpha)^\beta + (\tau/\alpha)^\beta].$$

In this case, (4.11) becomes

$$\begin{aligned}\gamma_k &= \int_0^\infty \int_0^\infty h(\alpha,\beta|Y_{\rm obs}) \cdot \Pr\{N(T) \le n+k|Y_{\rm obs},\alpha,\beta\}\, d\alpha d\beta \\ &= \int_0^\infty \int_0^\infty h(\alpha,\beta|Y_{\rm obs}) \cdot \sum_{j=n}^{n+k} \frac{f(Y_{\rm obs},\ N(T)=j|\alpha,\beta)}{f(Y_{\rm obs}|\alpha,\beta)}\, d\alpha d\beta,\end{aligned}$$

where $h(\alpha,\beta|Y_{\rm obs})$, $f(Y_{\rm obs}|\alpha,\beta)$ and $f(Y_{\rm obs},\ N(T)=j|\alpha,\beta)$ are respectively given in (2.10), (2.3) and (4.13). Thus, we have

$$\begin{aligned}\gamma_k &= \sum_{j=n}^{n+k} \frac{z^{n-r}\Gamma(j)}{\Gamma(n)\Gamma(n-r)(j-n)!} \cdot \int_0^\infty \beta^{n-r-1}(ux_r^{r-1}/T^n)^\beta [1-(\tau/T)^\beta]^{j-n}\, d\beta \\ &= \sum_{j=n}^{n+k}\sum_{i=0}^{j-n} \frac{z^{n-r}\Gamma(j)(-1)^i}{(n-1)!(j-n-i)!i!}\Big[z+(n+i)\ln(T/\tau)\Big]^{-n+r}. \qquad (4.18)\end{aligned}$$

In particular,

$$\gamma_0 = \left[\frac{z}{z+n\ln(T/\tau)}\right]^{n-r}, \qquad (4.19)$$

and

$$\gamma_1 = (n+1)\gamma_0 - n\left[\frac{z}{z+(n+1)\ln(T/\tau)}\right]^{n-r}. \qquad (4.20)$$

4.3 Estimating the probability of $\lambda(T) \le \lambda_0$

Based on $Y_{\rm obs}^{\rm ft}$ or $Y_{\rm obs}^{\rm tt}$, we are interested in the following question

Issue B1: *Given that a pre-determined target value λ_0 for the failure rate of the system undergoing development testing is not achieved at time τ, what is the probability that the target λ_0 will be achieved at time T with $T > \tau$?*

Equivalently, we wish to estimate the following probability:

$$\gamma = \Pr\{\lambda(T) \le \lambda_0 | Y_{\rm obs}\}, \qquad (4.21)$$

where $\lambda(T) = (\beta/\alpha)(T/\alpha)^{\beta-1} \hat{=} \lambda_T$.

4.3.1 Shape parameter β is known

Similar to (3.14), the posterior density of λ_T can be easily shown to be

$$h(\lambda_T|Y_{\rm obs}) \propto \lambda_T^{n-1}\exp[-\lambda_T \cdot \tau^\beta(\beta T^{\beta-1})^{-1}], \quad \lambda_T > 0. \qquad (4.22)$$

Substituting (4.22) into (4.21) and using the following relationship between the gamma and Poisson distribution functions

$$\frac{b^a}{\Gamma(a)}\int_0^{\lambda} x^{a-1}e^{-xb}\,dx = 1-\sum_{h=0}^{a-1}\frac{(b\lambda)^h}{h!}e^{-b\lambda}, \tag{4.23}$$

we have

$$\begin{aligned}\gamma^{(\beta)} &= \int_0^{\lambda_0} h(\lambda_T|Y_{\text{obs}})\,d\lambda_T \\ &= 1-\sum_{h=0}^{n-1}\frac{[\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_0]^h}{h!}\exp\Big[-\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_0\Big] \\ &= 1-\sum_{h=0}^{n-1}\text{Poisson}(h|\theta(\beta)),\end{aligned} \tag{4.24}$$

where $\theta(\beta)\,\hat{=}\,\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_0$ and $\text{Poisson}(h|\theta)\,\hat{=}\,\theta^h e^{-\theta}/h!$.

4.3.2 Shape parameter β is unknown

Consider the following transformation

$$\begin{cases}\lambda_T = (\beta/\alpha)(T/\alpha)^{\beta-1}, & \lambda_T>0,\\ \beta = \beta, & \beta>0.\end{cases}$$

Similar to (3.17) and (3.18), the posterior distribution of λ_T is

$$h(\lambda_T|Y_{\text{obs}}) = \frac{z^{n-r}}{\Gamma(n)\Gamma(n-r)}\int_0^\infty \frac{\beta^{n-r-1}(ux_r^{r-1})^\beta}{(\beta T^{\beta-1})^n}\cdot\lambda_T^{n-1}\exp[-\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_T]\,d\beta.$$

Substituting this expression into (4.21), we have

$$\begin{aligned}\gamma &= \int_0^{\lambda_0} h(\lambda_T|Y_{\text{obs}})\,d\lambda_T \\ &= 1-\sum_{h=0}^{n-1}\int_0^\infty \frac{[\theta(\beta)]^h}{h!}e^{-\theta(\beta)}\cdot\frac{z^{n-r}}{\Gamma(n-r)}\beta^{n-r-1}e^{-\beta z}\,d\beta \\ &= 1-\sum_{h=0}^{n-1}\int_0^\infty \text{Poisson}(h|\theta(\beta))\cdot\frac{z^{n-r}}{\Gamma(n-r)}\beta^{n-r-1}e^{-\beta z}\,d\beta.\end{aligned} \tag{4.25}$$

4.4 Predicting the time T required to achieve the target failure-rate

In this subsection, we are interested in the following issue:

ISSUE C1: *Given that the target value λ_0 for the system failure-rate is not achieved at τ, how long will it take in order that the system failure rate will be attained at λ_0?*

Statistically, for a given confidence level γ and the target failure-rate λ_0, we wish to estimate the time T ($T > \tau$) that satisfies (4.21). When β is known, it is easy to see from (4.22) that

$$2\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_T | Y_{\rm obs} \sim \chi^2(2n).$$

From (4.21), we have

$$\gamma = \Pr\left\{2\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_T \le 2\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_0 | Y_{\rm obs}\right\},$$

which implies $2\tau^\beta(\beta T^{\beta-1})^{-1}\lambda_0 = \chi^2(2n;\gamma)$ or

$$T = \left[\frac{2\lambda_0\tau^\beta}{\beta\chi^2(2n;\gamma)}\right]^{1/(\beta-1)}. \tag{4.26}$$

When β is unknown, the desired T that satisfies (4.25) can be solved by using an iterative algorithm.

Equivalently, we notice that Issue C1 can be re-formulated as follows:

ISSUE D1: *Based on Y_{obs}^{ft} or Y_{obs}^{tt}, what is the Bayes UPL of $\lambda(T)$ with T being a pre-specified value larger than τ?*

Statistically, it is desired to find $\lambda_{U,B}(T)$ satisfying $\gamma = \Pr\{\lambda(T) \le \lambda_{U,B}(T) | Y_{\rm obs}\}$ for a given level γ. When β is known, solving λ_0 from (4.26), we obtain

$$\lambda_{U,B}(T) = \frac{\beta T^{\beta-1}\chi^2(2n;\gamma)}{2\tau^\beta}. \tag{4.27}$$

When β is unknown, the corresponding $\lambda_{U,B}(T)$ equals to the λ_0 satisfying (4.25). Thus, an iterative algorithm can be applied.

5. Bayesian Predictions and Estimations for Two-Sample Problems

Suppose that the successive failure times of two repairable systems follow the same PLP with intensity function (1.1). Furthermore, for the first system, we assume that the first $(r-1)$ failure times were missing and the remainder failure times (i.e., $Y_{\rm obs}^{\rm ft}$ or $Y_{\rm obs}^{\rm tt}$) have been recorded exactly. We are interested in the following two-sample problems.

ISSUE A2: *How to predict the k-th ($k \geq 1$) failure time y_k of the second system?*

ISSUE B2: *Assume that the number of failures in the time interval* $(0, t_2]$ *for the second system is m but that the exact failure times are unknown, how to predict the k-th ($1 \leq k \leq m$) failure time y_k of the second system?*

ISSUE C2: *What is the probability that at most m failures will occur in* $(0, t_2]$ *for the second system?*

In this section, we utilize a Bayesian approach to address the above three issues.

5.1 Prediction limits for the *k*-th failure time of the 2-nd system

This subsection considers Issue A2. We first note that $2(x_n/\alpha)^\beta \sim \chi^2(2n)$ (Bain, 1978; Theorem 1 of Yu *et al.*, 2008). By replacing x_n by y_k, for the second system, we have $2(y_k/\alpha)^\beta \sim \chi^2(2k)$ or $(y_k/\alpha)^\beta \sim \Gamma(k, 1)$. Thus, the sampling distribution of y_k from a Bayesian viewpoint is given by

$$f(y_k) = f(y_k|\alpha) = f(y_k|\alpha, \beta) = \frac{1}{\Gamma(k)}\beta\alpha^{-k\beta}y_k^{k\beta-1}\exp[-(y_k/\alpha)^\beta]. \tag{5.1}$$

5.1.1 Shape parameter *β* is known

The predictive density of the k-th failure time y_k of the second system is

$$\begin{aligned}
f(y_k|Y_{\text{obs}}) &= \int_0^\infty h(\alpha|Y_{\text{obs}})f(y_k|Y_{\text{obs}}, \alpha)\, d\alpha \\
&= \int_0^\infty h(\alpha|Y_{\text{obs}})f(y_k|\alpha)\, d\alpha \qquad \text{by the independence of } y_k \text{ and } Y_{\text{obs}} \\
&= \frac{\beta\tau^{n\beta}}{B(n,k)} \cdot \frac{y_k^{k\beta-1}}{(\tau^\beta + y_k^\beta)^{n+k}}. \qquad \text{by (2.6) and (5.1)} \qquad (5.2)
\end{aligned}$$

Hence, the Bayes UPL $y_{U,B}^{(\beta)}(k, n, \gamma)$ for y_k with confidence level γ satisfies

$$\begin{aligned}
\gamma &= \int_0^{y_{U,B}^{(\beta)}(k,n,\gamma)} f(y_k|Y_{\text{obs}})\, dy_k \qquad \text{by (5.2)} \\
&= \int_0^{y_{U,B}^{(\beta)}(k,n,\gamma)} \frac{\beta\tau^{n\beta}}{B(n,k)} \cdot \frac{y_k^{k\beta-1}}{(\tau^\beta + y_k^\beta)^{n+k}}\, dy_k \qquad \text{let } y = \frac{ny_k^\beta}{k\tau^\beta} \\
&= \int_0^{n[y_{U,B}^{(\beta)}(k,n,\gamma)]^\beta/(k\tau^\beta)} \frac{1}{B(n,k)}\left(\frac{k}{n}\right)^k y^{k-1}\left(1 + \frac{k}{n}y\right)^{-n-k} dy.
\end{aligned}$$

Note that the integrand in the above integration is exactly the pdf of the $F(2k, 2n)$ distribution; thus $n[y_{U,B}^{(\beta)}(k, n, \gamma)]^{\beta}/(k\tau^{\beta}) = F(2k, 2n; \gamma)$ and

$$y_{U,B}^{(\beta)}(k, n, \gamma) = \tau \left[\frac{k}{n} F(2k, 2n; \gamma) \right]^{1/\beta}. \tag{5.3}$$

5.1.2 Shape parameter β is unknown

Note that the independency of y_k and Y_{obs}, then, the predictive density of y_k is

$$\begin{aligned} f(y_k|Y_{\text{obs}}) &= \int_0^\infty \int_0^\infty h(\alpha, \beta|Y_{\text{obs}}) f(y_k|Y_{\text{obs}}, \alpha, \beta)\, d\alpha d\beta \\ &= \int_0^\infty \int_0^\infty h(\alpha, \beta|Y_{\text{obs}}) f(y_k|\alpha, \beta)\, d\alpha d\beta \qquad \text{by (2.10) and (5.1)} \\ &= \frac{\Gamma(n+k)}{c\,\Gamma(k)} \cdot \int_0^\infty \beta^{n-r}(u x_r^{r-1})^{\beta} \frac{y_k^{k\beta-1}}{(\tau^{\beta} + y_k^{\beta})^{n+k}}\, d\beta. \end{aligned} \tag{5.4}$$

Thus, the Bayes UPL $y_{U,B}(k, n, r, \gamma)$ for y_k satisfies

$$\begin{aligned} \gamma &= \int_0^{y_{U,B}(k,n,r,\gamma)} f(y_k|Y_{\text{obs}})\, dy_k \\ &= \frac{z^{n-r}}{B(n,k)\Gamma(n-r)} \int_0^\infty \beta^{n-r}(u x_r^{r-1})^{\beta} \int_0^{y_{U,B}(k,n,\gamma)} \frac{y_k^{k\beta-1}}{(\tau^{\beta} + y_k^{\beta})^{n+k}}\, dy_k d\beta. \end{aligned}$$

It is easy to show that

$$y_{U,B}(k, n, r, \gamma) = \tau V^{z/(n-r+1)}, \tag{5.5}$$

where V is the solution to the following equation

$$\begin{aligned} \gamma &= \int_0^\infty F(nk^{-1}V^{\beta z/(n-r+1)}|2k, 2n) \cdot \frac{z^{n-r}}{\Gamma(n-r)} \beta^{n-r-1} e^{-\beta z}\, d\beta \\ &= \int_0^\infty F(nk^{-1}V^{x/(n-r+1)}|2k, 2n) \cdot \chi^2(x|2n-2r)\, dx \end{aligned} \tag{5.6}$$

with $F(\cdot|m, n)$ and $\chi^2(\cdot|n)$ being the cdf of $F(m, n)$ and the density of $\chi^2(n)$, respectively.

5.2 Prediction limits for y_k given $N(t_2) = m$

For Issue B2, we first need to find the conditional density $f(y_k|N(t_2)=m)$. Setting $r=1$ and replacing n, t, and x_i by m, t_2 and y_i respectively in (2.2), we obtain

$$f(y_1,\cdots,y_m;\ N(t_2)=m)=\beta^m\alpha^{-m\beta}\left(\prod_{i=1}^{m}y_i^{\beta-1}\right)\exp[-(t_2/\alpha)^\beta].$$

By integrating out the variables $y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_m$ from the above joint density, we have

$$f(y_k;\ N(t_2)=m)=\frac{\beta y_k^{k\beta-1}[t_2^\beta-y_k^\beta]^{m-k}}{(k-1)!(m-k)!\alpha^{m\beta}}\cdot\exp[-(t_2/\alpha)^\beta],$$

and

$$\begin{aligned} f(y_k|N(t_2)=m) &= \frac{f(y_k;\ N(t_2)=m)}{\Pr\{N(t_2)=m\}} \\ &= \frac{1}{B(k,m-k+1)}\beta y_k^{k\beta-1}t_2^{-m\beta}[t_2^\beta-y_k^\beta]^{m-k}, \quad y_k<t_2. \end{aligned} \tag{5.7}$$

It is noteworthy that (5.7) does not depend on α.

5.2.1 Shape parameter β is known

Since y_k is independent of the observed data Y_{obs} for the first system, we also have

$$f(y_k|N(t_2)=m,\ Y_{\text{obs}})=f(y_k|N(t_2)=m).$$

Given $N(t_2)=m$, the Bayes UPL $y_{U,B}^{(\beta)}(k,m,\gamma)$ for y_k with level γ satisfies

$$\begin{aligned} \gamma &= \int_0^{y_{U,B}^{(\beta)}(k,m,\gamma)} f(y_k|N(t_2)=m,\ Y_{\text{obs}})\,dy_k \\ &= \frac{1}{B(k,m-k+1)}\int_0^{[y_{U,B}^{(\beta)}(k,m,\gamma)/t_2]^\beta} x^{k-1}(1-x)^{m-k}\,dx. \end{aligned}$$

Hence, $[y_{U,B}^{(\beta)}(k,m,\gamma)/t_2]^\beta$ is equal to the γ percentage point of the $\mathcal{B}eta(k,m-k+1)$ distribution. Similar to (4.6), we obtain

$$y_{U,B}^{(\beta)}(k,m,\gamma)=t_2\left[1+\frac{m-k+1}{k}F^{-1}(2k,2m-2k+2;\ \gamma)\right]^{-1/\beta}. \tag{5.8}$$

5.2.2 Shape parameter β is unknown

From (2.10) and (5.7), we have

$$
\begin{aligned}
& f(y_k|N(t_2)=m,\ Y_{\text{obs}}) \\
=& \int_0^\infty \int_0^\infty f(y_k|N(t_2)=m)\cdot h(\alpha,\beta|Y_{\text{obs}})\,d\alpha a\beta \\
=& \frac{z^{n-r}}{B(k,m-k+1)\Gamma(n-r)} \int_0^\infty \beta^{n-r} e^{-\beta z} t_2^{-k\beta} y_k^{k\beta-1} [1-(y_k/t_2)^\beta]^{m-k}\,d\beta \\
=& \frac{\Gamma(n-r+1) z^{n-r}}{B(k,m-k+1)\Gamma(n-r)} \sum_{i=0}^{m-k} (-1)^i \binom{m-k}{i} y_k^{-1} [z+(k+i)\ln(t_2/y_k)]^{-(n-r+1)}.
\end{aligned}
$$

The Bayes UPL $y_{U,B}(k,m,n,r,\gamma)$ for y_k with level γ satisfies

$$
\begin{aligned}
\gamma =& \int_0^{y_{U,B}(k,m,n,r,\gamma)} f(y_k|N(t_2)=m,\ Y_{\text{obs}})\,dy_k \\
=& \frac{1}{B(k,m-k+1)} \sum_{i=0}^{m-k} \frac{(-1)^i}{k+i} \binom{m-k}{i} \left[1+\frac{k+i}{z}\ln\frac{t_2}{y_{U,B}(k,m,n,r,\gamma)}\right]^{-(n-r)}.
\end{aligned}
$$

Therefore,

$$
y_{U,B}(k,m,n,r,\gamma) = t_2 e^{-zW}, \tag{5.9}
$$

where W can be determined by

$$
\gamma = \frac{1}{B(k,m-k+1)} \sum_{i=0}^{m-k} \frac{(-1)^i}{k+i} \binom{m-k}{i} [1+(k+i)W]^{-(n-r)}. \tag{5.10}
$$

In particular, when $k=m$ we have $W=(\gamma^{-\frac{1}{n-r}}-1)/m$ and

$$
y_{U,B}(m,m,n,r,\gamma) = t_2 \exp\left[-\frac{z}{m}\left(\gamma^{-\frac{1}{n-r}}-1\right)\right]. \tag{5.11}
$$

5.3 Estimating the probability of $N(t_2) \le m$

For Issue C2, the probability that at most m (a pre-specified value) failures will occur in $(0,\, t_2]$ for the second system is given by

$$
\gamma_m = \Pr\{N(t_2)\le m|Y_{\text{obs}}\} = \sum_{k=0}^{m} \Pr\{N(t_2)=k|Y_{\text{obs}}\}. \tag{5.12}
$$

Note that

$$
\Pr\{N(t_2)=k\} = \frac{(t_2/\alpha)^{k\beta}}{k!}\exp[-(t_2/\alpha)^\beta].
$$

When β is known, we have

$$\begin{aligned}\Pr\{N(t_2)=k|Y_{\rm obs}\} &= \int_0^\infty \Pr\{N(t_2)=k|\alpha\}\cdot h(\alpha|Y_{\rm obs})\,d\alpha\\ &= \int_0^\infty \Pr\{N(t_2)=k\}\cdot h(\alpha|Y_{\rm obs})\,d\alpha\\ &= \binom{n+k-1}{k} q^k(1-q)^n.\end{aligned}$$

Substituting this into (5.12), we obtain

$$\gamma_m^{(\beta)} = (1-q)^n \sum_{k=0}^m \binom{n+k-1}{k} q^k, \tag{5.13}$$

where

$$q \,\hat{=}\, t_2^\beta/[\tau^\beta + t_2^\beta]. \tag{5.14}$$

When β is unknown, we have

$$\begin{aligned}\gamma_m &= \sum_{k=0}^m \Pr\{N(t_2)=k|Y_{\rm obs}\}\\ &= \sum_{k=0}^m \int_0^\infty\int_0^\infty \Pr\{N(t_2)=k\}\cdot h(\alpha,\beta|Y_{\rm obs})\,d\alpha d\beta\\ &= \frac{z^{n-r}}{\Gamma(n-r)}\int_0^\infty \beta^{n-r-1}e^{-\beta z}\gamma_m^{(\beta)}\,d\beta, \end{aligned}\tag{5.15}$$

where $\gamma_m^{(\beta)}$ is given in (5.13).

6. Numerical Examples

In this section, two real examples from an engine development program and a repairable system are used to illustrate the proposed methodologies.

6.1 Engine failure data

Zhou & Weng (1992, p.51–52) reported a total of 40 failures for an engine undergoing development testing in the time interval (0, 8063]. The data are given by $*, *, *$, 171, 234, 274, 377, 530, 533, 941, 1074, 1188, 1248, 2298, 2347, 2347, 2381, 2456, 2456, 2500, 2913, 3022, 3038, 3728, 3873, 4724, 5147, 5179, 5587, 5626, 6824, 6983, 7106, 7106, 7568, 7568, 7593, 7642, 7928, 8063 hours. Here, the exact failure times for the first three failures are unknown and are denoted by $*$. Since these data are failure-truncated, we have $n = 40$, $\tau = x_{40} = 8063$ and $r = 4$. Yu, Tian & Tang (2008) performed a goodness-of-fit test which fails to reject the hypothesis that the data come from a PLP the the 0.05 significant level with the MLE of β being $\hat{\beta} = 0.6761$.

We first consider the case that β is known by letting $\beta = \hat{\beta} = 0.6761$. From (3.3) and (3.4), the Bayes point estimator of α is $\tilde{\alpha}(\beta) = 36.0680$ and the two-sided 95% Bayes CI for α is $[22.5067, 56.6112]$.

When β is unknown, from (3.7) and (3.8), the Bayes point estimate of β equals $\tilde{\beta} = 0.6578$ while the two-sided 95% Bayes CI for β is $[0.4607, 0.8894]$. It is easy to obtain $z = 54.73$ from (2.11). To investigate the posterior density of β, we first generate $m = 50,000$ i.i.d. samples of β according to (3.6). Figure 1(a) shows that the posterior density of β is quite symmetric and should be well approximated by a normal distribution. We also generate $m = 50,000$ i.i.d. posterior samples $\{X^{(\ell)}\}_{\ell=1}^m$ of X based on (3.9). For each given pair $(\beta^{(\ell)}, X^{(\ell)})$, we then calculate $\alpha^{(\ell)} = \tau\,(X^{(\ell)})^{-1/\beta^{(\ell)}}$ and $\{\alpha^{(\ell)}\}_{\ell=1}^m$ are the i.i.d. posterior samples of α. Based on these i.i.d. posterior samples, the Bayes point estimator for α is given by $\tilde{\alpha} = 39.1501$ while the two-sided 95% Bayes CI for α is $[2.5117, 135.9328]$. Figure 1(b) shows that the posterior density of α is obviously skewed.

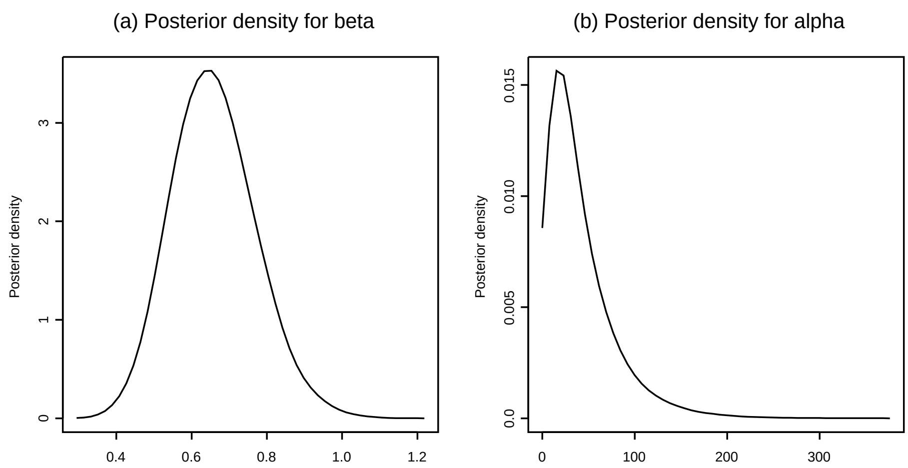

Figure 1: The posterior densities of β and α generated by the conditional sampling method introduced in Section 3.1 with $m = 50,000$ for the engine failure data.

Finally, from (3.12) and (3.13), we readily obtain the pseudo Bayes point estimate of $\lambda(x_n)$ as 0.00326, and the two-sided 95% Bayes CI for $\lambda(x_n)$ as $[0.00195, 0.00490]$.

6.2 A repairable system failure data

The following failure data from a prototype of a repairable system is given in ReliaSoft Corporation (2005). Briefly, a total of 12 failures for a system tested for $t = 6500$ hours are given by: (80), (175), (265), (400), 590, 1100, 1650, 2010, 2400, 3380, 5100, 6400. ReliaSoft Corporation (2005) showed that the above life data followed a PLP. For illustrative purpose, we assume that the exact failure times for the first four failures are unknown. Since these data are time-truncated, we have $t = 6500$, $n = 12$ and $r = 5$. The MLE of β is $\hat{\beta} = (n - r + 1)/z = 0.4389$.

We first consider the case that β is known by letting $\beta = \hat{\beta} = 0.4389$. From (3.3) and (3.4), the Bayes point estimator for α is $\tilde{\alpha}(\beta) = 31.7042$ while the two-sided 95% Bayes CI for α is $[7.3213, 101.7389]$. From (3.15) and (3.16), the Bayes point estimator of $\lambda(t)$ is $\tilde{\lambda}(\beta) = 0.00081$, and a two-sided 95% Bayes CI of $\lambda(t)$ is $[0.00042, 0.00133]$.

When β is unknown, we first generate $m = 50,000$ i.i.d. posterior samples $\{\beta^{(\ell)}\}_{\ell=1}^{m}$ of β according to (3.6). Thus, the Bayes point estimator for β is $\tilde{\beta} = 0.3838$ while the two-sided 95% Bayes CI for β is $[0.1542, 0.7151]$. Figure 2(a) plots the posterior density of β. Next, for each given $\beta^{(\ell)}$, we generate $\lambda_t^{(\ell)}$ based on (3.20) for $\ell = 1, \cdots, m$. Using these i.i.d. posterior samples of $\lambda(t)$, the Bayes point estimator for $\lambda(t)$ is $\tilde{\lambda}_t = 0.000708$ and the two-sided 95% Bayes CI for $\lambda(t)$ is $[0.000226, 0.00155]$. Figure 2(b) shows that the posterior density of $\lambda(t)$ is skewed.

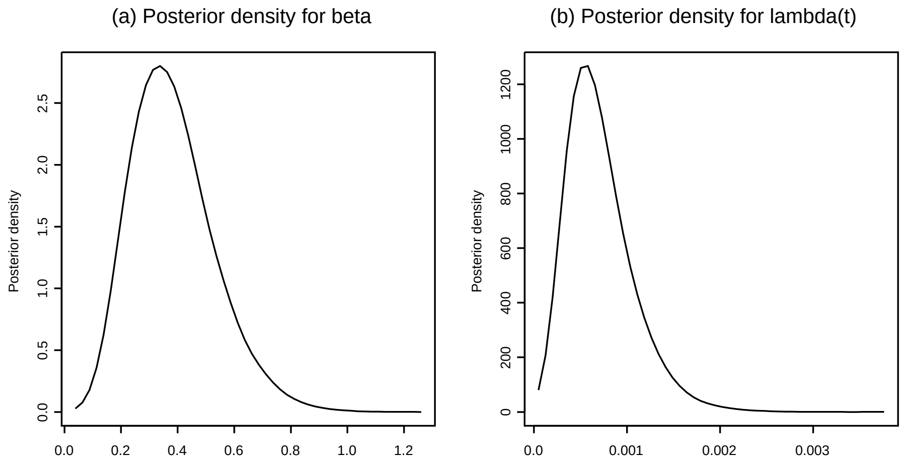

Figure 2: The posterior densities of β and $\lambda(t)$ generated by the conditional sampling method introduced in Section 3.3 with $m = 50,000$ for the repairable system failure data.

Now assume that we wish to obtain 0.95 Bayesian upper prediction limits on x_{12+k} for $k = 1$. When β is known (say, $\beta = 0.4389$), using (4.6), we have

$x^{(\beta)}_{U,B}(12, 1, 0.95) = 11479.9$. When β is unknown, we obtained $y_\gamma = 2.4925$ by solving (4.10). Thus, from (4.9), we get $x_{U,B}(12, 1, r, 0.95) = 14630.13$.

Suppose that we are interested in the probability γ_k that at most k failures will occur in the future time period $(\tau,\ T] = (6500,\ 7500]$. (i) When β is known (say, $\beta = 0.4389$), using (4.16) and (4.17), we have $\gamma_0 = 0.4706$, $\gamma_1 = 0.8144$, $\gamma_2 = 0.9505$, $\gamma_3 = 0.9891$, $\gamma_4 = 0.9979$, $\gamma_5 = 0.99965$, and $\gamma_6 = 0.99995$. (ii) When β is unknown, from (4.18) we obtain $\gamma_0 = 0.53246$, $\gamma_1 = 0.84436$, $\gamma_2 = 0.95668$, $\gamma_3 = 0.98912$, $\gamma_4 = 0.99743$, $\gamma_5 = 0.99941$ and $\gamma_6 = 0.99987$. Figure 3 shows the desired probabilities for known and unknown β.

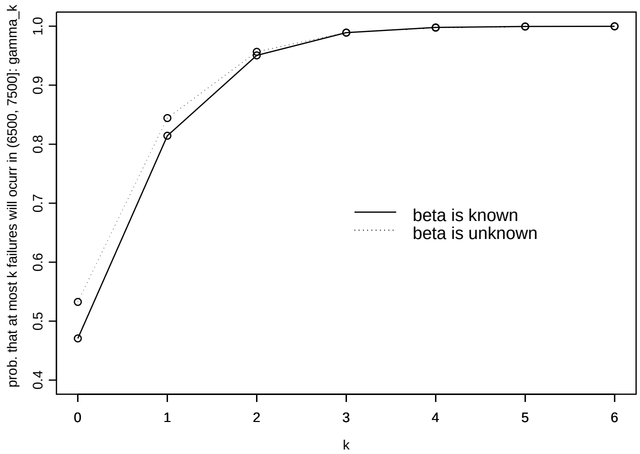

Figure 3: Comparison of the probabilities γ_k that at most k failures will occur in the time interval $(6500, 7500]$ for known $\beta = 0.4389$ and unknown β.

7. Conclusion

In this article, we consider Bayesian estimation and prediction methods for the PLP in the presence of left-truncated data. Bayesian point and credible interval estimates for the parameters of interest are derived. Bayesian prediction limits of future failure times are available for both failure- and time-truncated cases. We also show how five single-sample and three two-sample issues are addressed by the proposed Bayesian method.

It is interesting to note that (4.6) does not depend on r. In other words, when β is known, the Bayes prediction limit of x_{n+k} remains the same regardless of the absence of the first $(r-1)$ failure times (i.e., $\{x_i\}_{i=1}^{r-1}$). This result is not surprising because the conditional distribution of x_{n+k} given $x_1, \cdots, x_{n+k-1}$ depends only on x_{n+k-1} but not $x_1, \cdots, x_{n+k-2}$ (cf. (1.2)).

Furthermore, when β is known, the Bayes estimates of α (see (3.3) and (3.4)), $\lambda(t)$ (see (3.15) and (3.16)), $\Pr\{N(\tau, T) \le k\}$ (cf. (4.14)), $\Pr\{\lambda(T) \le \lambda_0\}$ (cf.

(4.24)), as well as the Bayes prediction limit of y_k (cf. (5.3)), remain the same regardless of the absence of the first $(r-1)$ failure times. In other words, these posterior estimates and prediction depend on the data only through n and τ, and do not depend on r and x_r.

In addition, we notice that the Bayesian UPL of x_{n+k} under the noninformative prior (2.5) is identical to the classical UPL of x_{n+k} by comparing (4.6) with (3.18) in Yu, Tian and Tang (2008). Most importantly, the prediction limit for x_{n+k} is available only for the failure-truncated case under the classical approach while the prediction limits for x_{n+k} are available for both the failure- and time-truncated cases under Bayesian approach.

By comparing (4.9) with (3.18) and (3.19) in Yu, Tian and Tang (2008), we find that the Bayesian UPL for x_{n+k} under the joint noninformative prior (2.9) is identical to the classical UPL for x_{n+k}. Again, the Bayes prediction limits for x_{n+k} are available for both failure- and time-truncated cases.

Acknowledgments

The research of JW Yu was fully supported by a grant (09C390) from Scientific Research Fund of Hunan Provincial Education Department.

References

Bain, L. J. (1978). *Statistical Analysis of Reliability and Life-Testing Models.* Marcel Dekker, New York.

Box, G. E. P. and Tiao, G. C. (1973). *Bayesian Inference in Statistical Analysis.* Reading, Addison-Wesley, Mass.

Calabria, R., Guida, M. and Pulcini, G. (1990). Bayes estimation of prediction intervals for a power law process. *Communications in Statistics–Theory and Methods* **19**, 3023-3035.

Crow, L. H. (1974). Reliability analysis for complex repairable systems. In *Reliability and Biometry — Statistical Analysis of Life Length* (F. Proschan and R. J. Serfling, eds.), 379-410. SIAM, Philadelphia.

Crow, L. H. and Basu, A. P. (1988). Reliability growth estimation with missing data - II. *IEEE Proceedings Annual Reliability and Maintainability Symposium*, 248-253.

Crowder, M. J., Kimber, A. C., Smith, R. L. and Sweeting, T. J. (1991). *Statistical Analysis of Reliability.* Chapman and Hall, London.

Duane, J. T. (1964). Learning curve approach to reliability monitoring. *IEEE Transactions on Aerospace* **2**, 563-566.

Guida, M., Calabria, R. and Pulcini, G. (1989). Bayes inference for a nonhomogeneous Poisson process with power intensity law. *IEEE Transactions on Reliability* **38**, 603-609.

Higgins, J. J. and Tsokos, C. P. (1981). A quasi-Bayes estimate of the failure intensity of a reliability growth model. *IEEE Transactions on Reliability* **30**, 471-475.

ReliaSoft Corporation (2005). *Reliability Growth and Repairable System Analysis Reference.*

Rigdon, S. E. (2002). Properties of the Duane plot for repairable systems. *Quality and Reliability Engineering International* **18**, 1-4.

Sen, A. (2002). Bayesian Estimation and Prediction of the Intensity of the Power Law Process. *Journal of Statistical Computation and Simulation* **72**, 613-631.

Yu, J. W., Tian, G. L. and Tang, M. L. (2008). Statistical inference and prediction for the Weibull process with incomplete observations. *Computational Statistics and Data Analysis* **52**, 1587-1603.

Zhou, Y. Q. and Weng, Z. Xi (1992). *Reliability Growth.* Science Press, Beijing.

Received February 20, 2010; accepted July 31, 2010.

Guo-Liang Tian
Department of Statistics and Actuarial Science
The University of Hong Kong
Pokfulam Road, Hong Kong, P. R. China
gltian@hku.hk

Man-Lai Tang
Department of Mathematics
Hong Kong Baptist University
Kowloon Tong, Hong Kong, P. R. China
mltang@math.hkbu.edu.hk

Jun-Wu Yu
School of Mathematics and Computational Science
Hunan University of Science and Technology
Xiangtan, Hunan 411201, P. R. China
jwyu.my265@yahoo.com.cn