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Test Procedures for Change Point in a General Class of Distributions

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Abstract: This paper is concerned with the change point analysis in a general class of distributions. The quasi-Bayes and likelihood ratio test procedures are considered to test the null hypothesis of no change point. Exact and asymptotic behaviors of the two test statistics are derived. To compare the performances of two test procedures, numerical significance levels and powers of tests are tabulated for certain selected values of the parameters. Estimation of the change point based on these two test procedures are also considered. Moreover, the epidemic change point problem is studied as an alternative model for the single change point model. A real data set with epidemic change model is analyzed by two test procedures.

Key words: Bayesian test, Brownian motion, change point, likelihood ratio test, epidemic change point, stochastic integral.

1. Introduction

This paper is concerned with the change point analysis in a general class of distributions. The problem can be described as follows. Given a sequence of independent random variables $X_1, \cdots, X_n$ which are distributed according to the one parameter density function $f_{\theta_i}(x_i)$, $\theta_i \in \Theta$, $i = 1, 2, \cdots, n$, one has to test the null hypothesis $H_0 : \theta_1 = \cdots = \theta_n = \theta_0$, against the alternative hypothesis

$$H_1 : \theta_i = \begin{cases} \theta_0 & i = 1,\ 2, \cdots,\ k_0, \\ \theta_0 + \delta & i = k_0 + 1, \cdots, n. \end{cases}$$

The initial parameter θ_0 may be known or unknown. The change point k_0 ($k_0 = 1, \cdots, n-1$) and the magnitude of change δ are unknown parameters. Without loss of generality, let $\delta \geq 0$. The following regularity conditions are needed.

(i) For every value x, the derivative $g(\theta, x) = \frac{\partial}{\partial\theta} \log f_\theta(x)$ exists for every $\theta \in \Theta$, and $E_{\theta_0}|g(\theta_0, X_1)|^3 < \infty$, so the Fisher information $\sigma^2 = I(\theta_0) = Var_{\theta_0}[g(\theta_0, X_1)]$ exists.

(ii) For any function $h(\cdot)$ with $E_\theta|h(X_1)| < \infty$, we have

$$\frac{d}{d\theta}\int h(x)f_\theta(x)dx = \int h(x)\frac{\partial}{\partial\theta}f_\theta(x)dx,$$

for every $\theta \in \Theta$.

The following example shows that these conditions are satisfied in two rich families of distributions.

Example 1. In the exponential family with density function

$$f_\theta(x) = h(x)\exp\{\varphi_1(\theta)u(x) + \varphi_2(\theta)\},$$

condition (i) is satisfied provided

$$E_{\theta_0}|u(X_1)|^3 < \infty.$$

In the location family

$$f_\theta(x) = f(x-\theta),$$

where the derivative $f'(\cdot)$ exists, condition (i) is satisfied provided

$$\int_{-\infty}^{\infty}\frac{|f'(x)|^3}{[f(x)]^2}dx < \infty.$$

For example, for the logistic distribution $L(\theta, 1)$ with $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$, we have

$$\int_{-\infty}^{\infty}\frac{|f'(x)|^3}{[f(x)]^2}dx = \int_{-\infty}^{\infty}\frac{e^{-x}|1-e^{-x}|^3}{(1+e^{-x})^5}dx < \infty.$$

Condition (ii) is typically satisfied.

Chernoff and Zacks (1964) considered the quasi-Bayesian change point analysis for independent normal observations. Kander and Zacks (1966) (KZ) extended the work of Chernoff and Zacks (1964) to the case of exponential family distributions. The nonparametric methods in change point analysis can be found in Brodsky and Darkhovsky (1993). Broemeling and Gregurich (1996) surveyed the Bayesian estimation of change point via resampling methods. An excellent reference in change point analysis is Csörgő and Horváth (1997). Gupta and Ramanayake (2001) used KZ's quasi-Bayes method to study the epidemic change point in exponential distribution. For more references see Hjort and Koning (2002) and Habibi *et al.* (2005) among the other.

In this note, we consider quasi-Bayes and likelihood ratio test procedures to detect a change in a general class of distributions. This paper is organized as follows. The quasi-Bayes test is studied in Section 2. The exact distribution of the test statistic in some special cases and its asymptotic distribution in general cases are also derived. Section 3 contains the exact and asymptotic distributions of the likelihood ratio test statistic. The performances of the two test procedures are compared in Section 4. Estimation of the change point based on two test procedures is also considered in this section. Section 5 considers the epidemic change point model which is an alternative model for the single change point model. A real data set is also considered in this section. This paper although is extension of an old paper however its approach in presenting the results in term of stochastic integrals is interested. It also considers change point detection in general class of distribution with single and epidemic change point model, a topic which is not considered before.

2. Quasi-Bayes Test

In this section, following KZ the quasi-Bayes test statistic is derived. Assume that $k_0 = [nt_0]$, for some unknown $t_0 \in (0,1)$. We consider the point t_0 as a random variable with prior density $\pi(t)$, $t \in (0,1)$. First, suppose that θ_0 is known. The marginal likelihoods of the sample under H_0 and H_1 are $\prod_{k=1}^{n} f_{\theta_0}(x_k)$ and

$$\int_0^1 \pi(t) \prod_{k=1}^{[nt]} f_{\theta_0}(x_k) \prod_{k=[nt]+1}^{n} f_{\theta_0+\delta}(x_k)dt,$$

respectively, and so the marginal likelihood ratio function under H_1 to that under H_0 is given by

$$\int_0^1 \pi(t) \exp\{\sum_{k=[nt]+1}^{n} \log f_{\theta_0+\delta}(x_k) - \log f_{\theta_0}(x_k)\}dt.$$

Following KZ, as $\delta \to 0$, then the marginal likelihood ratio can be approximated by

$$\int_0^1 \pi(t) \exp\{\delta \sum_{k=[nt]+1}^{n} g(\theta_0, X_k) + o(\delta)\}dt,$$

and it can be expressed by

$$1 + \delta \int_0^1 \pi(t) \sum_{k=[nt]+1}^{n} g(\theta_0, X_k)dt + o(\delta).$$

Then to test H_0 the corresponding test statistic becomes

$$T_n^{\pi} = \frac{1}{n} \int_0^1 \pi(t) \sum_{k=[nt]+1}^{n} g(\theta_0, X_k)dt.$$

By partitioning $[0,1]$ to n equal subdivisions, it can be shown that

$$T_n^{\pi} = \frac{1}{n}\sum_{k=1}^{n}\Pi(\frac{k-1}{n})g(\theta_0, X_k),$$

where

$$\Pi(t) = \int_0^t \pi(x)dx.$$

KZ derived the test statistic T_n^{π} in exponential families. The test procedure based on T_n^{π} is locally most powerful (see KZ). Under the noninformative prior $\pi(t) = 1$ for $t \in (0,1)$ then the test statistic

$$T_n = \frac{1}{n^2}\sum_{k=1}^{n}(k-1)g(\theta_0, X_k),$$

will be obtained. Habibi *et al.* (2005) studied the behavior of this test statistic.

Example 2. The exact null distribution of T_n can be found in some special cases. In exponential families T_n reduces to KZ test statistic. So the exact null distribution of T_n can be obtained in the normal, exponential, and binomial distributions (see KZ). The exact null distribution of T_n can also be found in the logistic distribution as follows. Without loss of generality, let $\theta_0 = 0$. It is easy to verify that

$$T_n = \frac{1}{n^2}\sum_{i=1}^{n}(i-1)(-2F(X_i)+1),$$

where $F(\cdot)$ is the distribution function of the standard logistic distribution $L(0,1)$. Let $S_n = \sum_{i=1}^{n}(i-1)F(X_i)$ and $g_n(\cdot)$ be the density function of S_n. Then $S_n = \frac{-n^2T_n}{2} + \frac{n(n-1)}{4}$. Variables $(i-1)F(X_i)$ are independent with uniform distribution on $(0, i-1)$. Sadooghi *et al.* (2005) showed that

$$\begin{aligned} g_n(s) \quad = \quad & \frac{(-1)^{n-1}}{(n-1)!(n-2)!}\{(-1)^{n-1}s^{n-2} + \\ & \sum_{k=1}^{n-1}(-1)^{n-1-k}\sum\sum_{1\le j_1<j_2<....<j_k\le n}....\sum[(s-\sum_{l=1}^{k}j_l)_+]^{n-2}\}, \end{aligned}$$

where $x_+ = \max(x,0)$. Now if $h_n(\cdot)$ is the density function of T_n, we have

$$h_n(t) = \frac{n^2}{2}g_n(\frac{-n^2t}{2} + \frac{n(n-1)}{4}).$$

However, since the exact distribution of T_n^π(or T_n) is very complicated in many cases, the asymptotic distribution of T_n^π is considered in Theorem 1. Suppose that $\sigma^2 = I(\theta_0)$, the Fisher information computed at θ_0.

Theorem 1. Assuming regularity conditions (i) and (ii) and under the null hypothesis H_0, we have

$$\sqrt{n}\sigma^{-1}T_n^\pi \xrightarrow{d} N(0, \int_0^1 \Pi^2(t)dt).$$

Proof. Consider the stochastic process $S_n(t)$ as follows:

$$S_n(t) = n^{-1/2}\sigma^{-1}\sum_{i=1}^{[nt]} g(\theta_0, X_i) \quad \text{for } t \in (0,1).$$

Under the null hypothesis, $S_n(\cdot) \xrightarrow{d} W(\cdot)$ in $(D[0,1], d)$ where $W(\cdot)$ is the standard Brownian motion on $[0,1]$ and d is Skorokhod metric (see Billingsley, 1968). The map Λ defined as

$$x(\cdot) \xrightarrow{\Lambda} \int_0^1 \pi(t)x(t)dt,$$

is continuous. The continuity theorem implies that

$$\sqrt{n}\sigma^{-1}T_n^\pi = \int_0^1 \pi(t)(S_n(1) - S_n(t))dt \xrightarrow{d} W(1) - \int_0^1 \pi(t)W(t)dt.$$

Integration by part can be applied to show that

$$W(1) - \int_0^1 \pi(t)W(t)dt \overset{d}{=} \int_0^1 \Pi(t)dW(t) \overset{d}{=} N(0, \int_0^1 \Pi^2(t)dt).$$

Corollary 1. Under (i), (ii) and H_0, then $\sqrt{3n}T_n \xrightarrow{d} N(0, \sigma^2)$.

Remark 1. When the initial parameter θ_0 is unknown, then θ_0 is substituted by $\widehat{\theta}_0$, the maximum likelihood estimate of θ_0 under the null hypothesis, resulting in the following test statistic:

$$\widehat{T}_n^\pi = \frac{1}{n}\sum_{k=1}^{n} \Pi(\frac{k-1}{n})g(\widehat{\theta}_0, X_k).$$

It is easy to show that under the null hypothesis (since $\widehat{\theta}_0 \xrightarrow{p} \theta_0$), then

$$n^{-1/2}\sigma^{-1}\widehat{T}_n^\pi \xrightarrow{d} \int_0^1 \Pi(t)dB(t).$$

Example 3. As a special case of Remark 1, consider a sequence of independent random variables X_i such that

$$X_i = \theta_0 + \delta I(i \geq k_0 + 1) + N_i,$$

where N_i are $i.i.d$ random variables of standard normal $N(0,1)$ distribution. The initial mean θ_0 is unknown and it is replaced by $\overline{X}_n$. Then, the test statistic is given by

$$\widehat{T}_n = \frac{1}{n^2} \sum_{k=1}^{n} (k-1)(X_k - \overline{X}_n).$$

It is seen that

$$\sqrt{n}\widehat{T}_n = \int_0^1 t dB_n(t) \xrightarrow{d} \int_0^1 t dB(t),$$

where $B_n(t) = n^{-1/2} \sum_{k=1}^{[nt]} (X_k - \overline{X}_n)$ converges to $B(\cdot)$, the standard Brownian bridge on $[0,1]$.

Remark 2. Suppose that $X_i = X_{i-1} + \varepsilon_i$, $i = 1, ..., n$, where ε_i are iid normaly distributed with $E(\varepsilon_1) = 0$ and $E(\varepsilon_1^2) = \sigma^2$. Let $H_n(t) = n^{-1/2} \sum_{i=1}^{[nt]} \varepsilon_i$. Then

$$\frac{T_n^\pi}{\sqrt{n}\sigma} = \frac{1}{n} \sum_{k=1}^{n} \Pi(\frac{k-1}{n}) H_n(\frac{k}{n}) \xrightarrow{d} \int_0^1 \Pi(t) W(t) dt.$$

Next, the asymptotic distribution of T_n under the alternative hypothesis is considered. To do so, the following extra condition is assumed.

(iii) $E_{\theta_1} |g(\theta_0, X)|^3 < \infty$.

Let $\mu_{\theta_1} = E_{\theta_1}(g(\theta_0, X_{k_0+1}))$ and $I(\theta_0, \theta_1) = Var_{\theta_1}(g(\theta_0, X_{k_0+1}))$.

Theorem 2. Under the alternative hypothesis H_1, then

$$\sqrt{n}\{T_n^\pi - \mu_{\theta_1} \int_{t_0}^1 \Pi(t) dt\} \xrightarrow{d} \int_0^1 \pi(t) \{W^*(1) - W^*(t)\} dt,$$

where

$$W^*(t) = \begin{cases} \sigma W_1(t) & t \leq t_0, \\ \sigma W_1(t_0) + \sqrt{I(\theta_0, \theta_1)} W_2(t - t_0) & t \geq t_0, \end{cases}$$

which $W_1(\cdot)$ and $W_2(\cdot)$ are two independent standard Brownian motions on $[0,1]$.

Proof. Let $S_n^*(t) = n^{-1/2} \sum_{k=1}^{[nt]} Y_k$, where

$$Y_k = g(\theta_0, X_k) - E_{H_1}(g(\theta_0, X_k)).$$

Note that $\{Y_k, k = 1, \cdots, k_0\}$ and $\{Y_k, k = k_0+1, \cdots, n\}$ are two independent of *iid* random variables. Then there exist two independent standard Brownian motions on $[0,1]$ namely $W_1(\cdot)$ and $W_2(\cdot)$ such that

$$\begin{cases} n^{-1/2}\sum_{k=1}^{[nt]} Y_k \xrightarrow{d} \sigma W_1(t), & \text{for } t \leq t_0, \\ n^{-1/2}\sum_{k=[nt_0]+1}^{[nt]} Y_k \xrightarrow{d} \sqrt{I(\theta_0,\theta_1)}W_2(t-t_0), & \text{for } t \geq t_0, \end{cases}$$

Notice that $S_n^*(\cdot) \xrightarrow{d} W^*(\cdot)$ (see Theorem 2) in $(D(0,1), d)$ where d is the Skorokhod metric. Notice that

$$\sqrt{n}\{T_n^\pi - E(T_n^\pi)\} = \int_0^1 \pi(t)\{S_n^*(1) - S_n^*(t)\}dt.$$

The continuity theorem implies that

$$\int_0^1 \pi(t)\{S_n^*(1) - S_n^*(t)\}dt \xrightarrow{d} \int_0^1 \pi(t)\{W^*(1) - W^*(t)\}dt.$$

Therefore, $\sqrt{n}\{T_n^\pi - E(T_n^\pi)\} \xrightarrow{d} \int_0^1 \pi(t)\{W^*(1) - W^*(t)\}dt$. Notice that $E(T_n^\pi) = \frac{\mu_{\theta_1}}{n}\sum_{k=[nt_0]+1}^{n} \Pi(\frac{k-1}{n})$ which converges to $\mu_{\theta_1}\int_{t_0}^1 \Pi(t)dt$ and

$$E(T_n^\pi) - \mu_{\theta_1}\int_{t_0}^1 \Pi(t)dt = O(n^{-1}).$$

Write $A = \sqrt{n}\{T_n^\pi - E(T_n^\pi)\}$. Then

$$\begin{aligned} A &= \sqrt{n}\{T_n^\pi - \mu_{\theta_1}\int_{t_0}^1 \Pi(t)dt\} + \sqrt{n}\{\mu_{\theta_1}\int_{t_0}^1 \Pi(t)dt - E(T_n^\pi)\} \\ &= \sqrt{n}\{T_n^\pi - \mu_{\theta_1}\int_{t_0}^1 \Pi(t)dt\} + O(n^{-1/2}). \end{aligned}$$

This completes the proof.

Corollary 2. Under (i), (ii), (iii) and H_1 then

$$\sqrt{n}(T_n - \frac{1-t_0^2}{2}\mu_{\theta_1}) \xrightarrow{d} N(0, \frac{\sigma_{t_0}^2(\theta_0,\theta_1)}{3}),$$

where $\sigma_{t_0}^2(\theta_0,\theta_1) = t_0^3 I(\theta_0) + (1-t_0^3)I(\theta_0,\theta_1)$.

Although, deriving Corollary 2 from Theorem 1 is straightforward, but we present a proof briefly. Let $Y_i = (i-1)g(\theta_0, X_i)$. Then

$$0 \leq \lim_{n\to\infty} \frac{C_n}{D_n} \leq \lim_{n\to\infty} O(n^{-1/6}) = 0,$$

at which $C_n = \sqrt[3]{\sum_{i=1}^n E|Y_i - E(Y_i)|^3}$ and $D_n = \sqrt{\sum_{i=1}^n Var(Y_i)}$. Therefore by Lyapunov's theorem $\frac{\sum_{i=1}^n \{Y_i - EY_i\}}{\sqrt{\sum_{i=1}^n Var(Y_i)}} \xrightarrow{d} N(0,1)$. Replacing EY_i and $Var(Y_i)$ gives the result.

Corollary 3. The approximate power of test in size α based on T_n is given by

$$\beta_\alpha(\delta) = 1 - \Phi(\frac{\sigma z_\alpha - \frac{\sqrt{3n}}{2}(1-t_0^2)\mu_{\theta_1}}{\sigma_{t_0}(\theta_0, \theta_1)}).$$

For example, when $X_i = \theta_0 + \delta I(i \geq k_0 + 1) + N_i$, $N_i \overset{i.i.d}{\sim} N(0,1)$, we have $\mu_{\theta_1} = \delta$, $\sigma_{t_0}(\theta_0, \theta_1) = 1$ and then $\beta_\alpha(\delta) = \Phi\{-z_\alpha + \frac{\sqrt{3n}}{2}(1-t_0^2)\delta\}$. It can be shown that as $\delta \to 0$, then

$$\frac{\mu_{\theta_1}}{\delta} \to I(\theta_0) \quad \text{and} \quad \frac{I(\theta_0, \theta_1) - I(\theta_0)}{\delta} \to E_{\theta_0}(g^3(\theta_0, X_1)).$$

Then $\sigma_{t_0}^2(\theta_0, \theta_1)$ can be approximated by

$$I(\theta_0) + \delta(1 - t_0^3)E_{\theta_0}(g^3(\theta_0, X_1)).$$

In the special case when $g(\theta_0, x) = g^*(x - \theta_0)$ for some odd function g^*, and when the null distribution of X_1 is symmetric under θ_0, then $E_{\theta_0}(g^3(\theta_0, X_1)) = 0$, and then the power is approximatly $\Phi(-z_\alpha + \frac{\sqrt{3n}}{2}(1-t_0^2)\sigma\delta)$. For example, when X_i are normal with variance 1, then $\sigma = 1$ and the power is approximately $\Phi\{-z_\alpha + \frac{\sqrt{3n}}{2}(1-t_0^2)\delta\}$.

Remark 3. We can estimate the location of change point using the quasi-Bayesian test. To see this in details, we consider the special case $X_i = \theta_0 + \delta I(i \geq k_0 + 1) + N_i$, where N_i are *iid* random variables from $N(0,1)$ distribution and $\delta > 0$. The change point estimator $\widehat{k}_n$ based on quasi-Bayes test is given by

$$\widehat{k}_n = \text{argmin}\ U_k,$$

where $U_k = \frac{1}{n}\sum_{i=1}^k \Pi(\frac{i-1}{n})(X_i - \overline{X})$ for $k = 1, ..., n-1$. Define $g(t) = \int_0^t \Pi(x)dx$. Since $U_k - E(U_k) = \frac{1}{n}\sum_{i=1}^k \Pi(\frac{i-1}{n})(N_i - \overline{N})$, and

$$n^{-1/2} \sup_{0<t<1} |\sum_{i=1}^{[nt]} \Pi(\frac{i-1}{n})(N_i - \overline{N})| \xrightarrow{d} \sup_{0<t<1} |\int_0^t \Pi(x)dB(x)|,$$

then, we get

$$\sup_{0<t<1} |U_{[nt]} - E(U_{[nt]})| = o_p(1),$$

that is $U_{[nt]}$ is pretty close to its mean function $E(U_{[nt]})$. Then to study limiting behavior of $U_{[nt]}$, it is enough to study the limiting behavior of $E(U_{[nt]})$. It is easy to see that $E(U_{[n\cdot]}) \to U(\cdot)$, where

$$U(t) = \begin{cases} -\delta(1-t_0)g(t) & t \le t_0, \\ -\delta(1-t_0)g(t_0) + \delta t_0(g(t) - g(t_0)) & t > t_0. \end{cases}$$

This shows the consistency of $\widehat{t}_n = \frac{\widehat{k}_n}{n}$, that is $\widehat{t}_n \xrightarrow{p} t_0$, as $n \to \infty$ (see Bai, 1994).

3. Likelihood Ratio Test

Here, the likelihood ratio test is considered to test the null hypothesis of no change point. First, assume θ_0 is known. The likelihood ratio function under H_1 to that under H_0 is given by

$$\max_{1\le k\le n-1} \prod_{i=k+1}^{n} \frac{f_{\theta_0+\delta}(x_i)}{f_{\theta_0}(x_i)}.$$

It is easy to verify that as $\delta \to 0^+$, then the likelihood ratio function can be approximated by

$$1 + \delta \max_{1\le k\le n-1} \sum_{i=k+1}^{n} g(\theta_0, X_i) + o(\delta),$$

(see Section 2). One would reject H_0 whenever the observed value of T_n^* is large, where

$$T_n^* = \frac{1}{n} \max_{1\le k\le n-1} \sum_{i=k+1}^{n} g(\theta_0, X_i).$$

One can show that under the null hypothesis H_0, as $n \to \infty$, then

$$\frac{\sqrt{n}T_n^*}{\sigma} = \sup_{0<t<1} n^{-1/2} \sum_{i=[nt]+1}^{n} g(\theta_0, X_i) \xrightarrow{d}$$

$$\sup_{0<t<1} \{W(1) - W(t)\} \stackrel{d}{=} \sup_{0<t<1} W(1-t) = \sup_{0<t<1} W(t) \stackrel{d}{=} |N|,$$

where N is distributed as standard normal distribution (see Billingsley, 1968).

Remark 4. The likelihood ratio test statistics T_n^* is larger than the quasi-Bayes test statistic T_n^π. To see this, note that

$$T_n^\pi = \frac{1}{n} \int_0^1 \pi(t) \sum_{k=[nt]+1}^{n} g(\theta_0, X_k) dt \le \frac{1}{n} \sup_{0<t<1} \sum_{k=[nt]+1}^{n} g(\theta_0, X_k) = T_n^*.$$

This shows that the critical values of likelihood ratio test are larger than the values for the quasi-Bayes test.

Remark 5. When the initial parameter θ_0 is unknown, again θ_0 is substituted by $\widehat{\theta}_0$, the maximum likelihood estimate of θ_0 under the null hypothesis, resulting in the following test statistic:

$$\widehat{T_n^*} = \frac{1}{n} \max_{1 \le k \le n-1} \sum_{i=k+1}^{n} g(\widehat{\theta}_0, X_i).$$

It is easy to show that under some mild conditions then

$$\frac{\sqrt{n}\widehat{T_n^*}}{\sigma} \xrightarrow{d} \sup_{0<t<1} B(t).$$

Example 4. To see Remark 5, consider the special case $X_i = \theta_0 + \delta I(i \ge k_0 + 1) + N_i$, where $N_i \overset{i.i.d}{\sim} N(0,1)$. Since θ_0 is unknown it is estimated by $\overline{X}_n$ and the test statistic is given by

$$\widehat{T_n^*} = \frac{1}{n} \max_{1 \le k \le n-1} v_k,$$

where $v_k = \sum_{i=k+1}^{n}(X_i - \overline{X}_n)$, $k = 1, \cdot\cdot\cdot, n-1$. Under the null hypothesis

$$n^{-1/2} \sum_{i=[n\cdot]+1}^{n} (X_i - \overline{X}_n) = -n^{-1/2} \sum_{i=1}^{[n\cdot]} (X_i - \overline{X}_n) \xrightarrow{d} -B(\cdot),$$

where $B(\cdot)$ is the standard Brownian bridge on $[0, 1]$. Since $B(\cdot) \overset{d}{=} -B(\cdot)$, the continuity theorem implies that $\sqrt{n}\widehat{T_n^*} \xrightarrow{d} \sup_{0<t<1} B(t)$.

Under the null hypothesis, random vector $\mathbf{v} = (v_1, ..., v_{n-1})$ has a multivariate normal distribution $N_{n-1}(\mathbf{0}, \mathbf{\Sigma})$ with

$$\mathbf{\Sigma}_{ij} = \min(i, j) - \frac{ij}{n},$$

(see Hawkins, 1977). Under H_1, then $\mathbf{v} \sim N_{n-1}(\delta\mu, \mathbf{\Sigma})$, where $\mu = (\mu_1, \cdot\cdot\cdot, \mu_n)$ with

$$\mu_k = \begin{cases} (1 - \frac{k_0}{n})k & k = 1, \cdot\cdot\cdot, k_0, \\ k_0(1 - \frac{k}{n}) & k = k_0 + 1, \cdot\cdot\cdot, n. \end{cases}$$

The exact distribution of $\widehat{T_n^*}$ is the distribution of maximum of a multivariate normal. Then the α-*th* quantile of $\widehat{T_n^*}$ is the α-*th* equi-quantile of a multivariate normal distribution which is considered by Genz (1992).

Remark 6. The change point estimator $\widehat{k}_n$ based on the likelihood ratio test when $\delta > 0$ is given by

$$\widehat{k}_n = \text{argmin } V_k, \ k = 1, ..., n-1,$$

where $V_k = \frac{v_k}{n}$. Consider model $X_i = \theta_0 + \delta I(i \geq k_0 + 1) + N_i$, where $N_i \overset{i.i.d}{\sim} N(0, 1)$. It is easy to verify that

$$\sup_{0<t<1} |V_{[nt]} - E(V_{[nt]})| = o_p(1) \text{ and } E(V_{[n\cdot]}) \to \delta V(\cdot),$$

where

$$V(t) = \begin{cases} (1-t_0)t & t \leq t_0, \\ t_0(1-t) & t > t_0. \end{cases}$$

This fact suggests plotting V_k for $k = 1, 2, ..., n-1$. The first point $\widehat{k}_n$ at which V_k attains its minimum is the likelihood ratio change point estimator.

4. Comparisons

In this section, we compare the performance of the quasi-Bayes and likelihood ratio tests by studying their significance levels and powers. The significance levels of quasi-Bayes and likelihood ratio tests are α_n and α_n^* respectively, where

$$\alpha_n = P_{\theta_0}(\frac{\sqrt{3n}T_n}{\sigma} \geq z_\alpha) \quad \text{and} \quad \alpha_n^* = P_{\theta_0}(\frac{\sqrt{n}T_n^*}{\sigma} \geq z_\alpha).$$

Table 1: Estimated significance level of quasi-Bayes test

$\alpha \backslash n$	25	50	75	100	125	150	175	200
0.01	0.0089	0.0095	0.0097	0.0099	0.0101	0.01	0.01	0.01
0.025	0.0231	0.0245	0.0248	0.0251	0.0251	0.025	0.025	0.025
0.05	0.048	0.049	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.097	0.098	0.098	0.099	0.1	0.1	0.1	0.1

Table 2: Estimated significance level of likelihood ratio test

$\alpha \backslash n$	25	50	75	100	125	150	175	200	250	300
0.01	0.008	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.01	0.01
0.025	0.019	0.022	0.024	0.024	0.024	0.024	0.024	0.024	0.025	0.025
0.05	0.045	0.045	0.045	0.047	0.048	0.048	0.049	0.049	0.05	0.05
0.1	0.093	0.095	0.095	0.096	0.096	0.096	0.097	0.098	0.1	0.1

Table 3: Values of $\beta_\alpha(\delta)$ and $\widehat{\beta}^*_\alpha(\delta)$

k_0	$\beta_\alpha(\delta_1)$	$\widehat{\beta}^*_\alpha(\delta_1)$	$\beta_\alpha(\delta_2)$	$\widehat{\beta}^*_\alpha(\delta_2)$
1	0.617	0.0795	0.9995	0.988
3	0.6075	0.072	0.9995	0.9825
5	0.5965	0.076	0.9992	0.976
7	0.579	0.0725	0.9992	0.971
9	0.567	0.0735	0.996	0.958
11	0.562	0.079	0.995	0.9575
13	0.566	0.0695	0.9855	0.932
15	0.569	0.0705	0.98	0.9225
17	0.5585	0.0745	0.968	0.897
19	0.547	0.073	0.9535	0.878
21	0.5245	0.076	0.9225	0.838
23	0.5285	0.065	0.9055	0.785
25	0.5445	0.0545	0.8795	0.766
27	0.5115	0.062	0.825	0.692
29	0.517	0.0515	0.809	0.6415
31	0.501	0.0615	0.7535	0.5235
33	0.4835	0.0645	0.706	0.456
35	0.4645	0.0465	0.6565	0.3695
37	0.503	0.0535	0.614	0.3125
39	0.476	0.043	0.5795	0.244
41	0.4885	0.05	0.5355	0.174
43	0.4885	0.0455	0.5195	0.1205
45	0.481	0.0465	0.5065	0.1035
47	0.4785	0.0315	0.4745	0.0595
49	0.466	0.046	0.4835	0.049

4.1 Rate of convergence of α_n and α_n^*

In what follows, we compare the rate of convergence of α_n and α_n^* to α in the case of logistic distribution. For a given n, we compute α_n and α_n^* using a Monte Carlo experiment with $R = 20000$ repetitions. Let $\widehat{\alpha}_{nR}$ ($\widehat{\alpha}^*_{nR}$) be the number of times that the null hypothesis H_0 of no change is rejected based on the quasi-Bayes test (likelihood ratio test) over R. The SLLN guarantees that $\widehat{\alpha}_{nR}$ ($\widehat{\alpha}^*_{nR}$) (see Tables 1, 2) is pretty close to $\alpha_n(\alpha_n^*)$. The rates of convergence of α_n and α_n^* to α seem good although it seems α_n converges to α a little faster.

4.2 Approximated power of two tests

Here, we compare the powers of two test procedures in the logistic observations $L(\delta, 1)$. The power of quasi-Bayes test $\beta_\alpha(\delta)$ (see Corollary 2) are given in Table 3 for $\alpha = 0.05$ and $k_0 = 1, 3, ..., 49$. Table 3 also contains the power of likelihood ratio test $\widehat{\beta}^*_\alpha(\delta)$ which is estimated using a Monte Carlo simulation study with $R = 20000$ repetitions. In order to keep the table in reasonable size, only the case of sample size $n = 50$ and magnitude of changes $(\delta_1, \delta_2) = (0.09, 1)$ with a significance level $\alpha = 0.05$ is reported. It is seen from the Table 3 that the power of quasi-Bayes test is larger than the power of likelihood ratio test in all cells. The power of likelihood ratio test is too small for $\delta_1 = 0.09$. Higher powers for two tests are achieved if k_0 occurs in the beginning of the sequence.

5. Epidemic Change Point

The epidemic change point model is an alternative for the single change point model. Yao (1993) published a survey of the available test procedures together with their comparisons. Brodsky and Darkhovsy (1993) constructed estimators for change points and studied their properties. In this section, the epidemic change point is considered in a general class of distributions. Epidemic change point analysis has many applications in practice and studying it in a general class of distribution is an interested topic. Consider a sequence of independent random variables $X_1, \cdots, X_n$ whose density functions are $f_{\theta_i}(x_i)$, $\theta_i \in \Theta$, $i = 1, \cdots, n$, one has to test the null hypothesis $H_0 : \theta_1 = \cdots = \theta_n = \theta_0$, against the alternative hypothesis

$$H_1 : \theta_i = \begin{cases} \theta_0 & i = 1,\ 2, \cdots,\ k_0, \\ \theta_0 + \delta & i = k_0 + 1, \cdots, k_1, \\ \theta_0 & i = k_1 + 1, \cdots, n. \end{cases}$$

First, suppose that θ_0 is known. Let $k_0 = [nt_0]$ and $k_1 = [nt_1]$ for some $0 < t_0 <$

$t_1 < 1$. Denote prior of (t_0, t_1) by $\pi(t_0, t_1)$, where

$$\pi(t_0, t_1) = \pi_1(t_1|t_0)\pi_0(t_0),\ 0 < t_0 < t_1 < 1.$$

Similar to Section 2, the quasi-Bayes test will reject H_0, when $T_n^{\pi e}$ is large, where

$$T_n^{\pi e} = \frac{1}{n}\int_0^1 \int_t^1 \{L_n(s) - L_n(t)\}\pi(t, s)dsdt,$$

at which $L_n(t) = \sum_{i=1}^{[nt]} g(\theta_0, X_i)$. The Donsker theorem implies that $n^{-1/2}\{L_n(s) - L_n(t)\} \xrightarrow{d} \sigma\{W(s) - W(t)\}$. Then the continuity theorem implies that

$$\sqrt{n}T_n^{\pi e} \xrightarrow{d} \sigma \int_0^1 \int_t^1 (W(s) - W(t))\pi_1(s|t)\pi_0(t)dsdt.$$

The likelihood ratio test statistic is

$$T_n^{*e} = \frac{1}{n} \sup_{0<t<s<1} \{L_n(s) - L_n(t)\}.$$

It is easy to see that

$$\sqrt{n}T_n^{*e} \xrightarrow{d} \sigma \sup_{0<t<s<1} \{W(s) - W(t)\}.$$

Remark 7. When θ_0 is unknown, it is estimated by $\widehat{\theta}_0$. The above asymptotic distributions of quasi-Bayes and likelihood ratio statistics are held by replacing $W(\cdot)$ with $B(\cdot)$, the standard Brownian bridge on $[0, 1]$.

6. Stanford Heart Transplant Data

The data set is (taken from Kalbfleisch and Prentice, 1980) contains 35 patients with known age groups. The average survival time of the patients were indexed by age group. There can be doubts about the exsitence of an epidemic change in the sequence. To check this possibility, we performed the two test procedures for this data set. The p-values of quasi-Bayesian and likelihood ratio tests are 0.0235 and 0.0552, respectively. We can reject the null hypothesis of no change, in favor of an epidemic change for this data set. The ML estimators of two change points are 29 and 48 years, respectively.

References

Bai, J. (1994). Least square estimation of a shift in linear processes. *Journal of Time Series Analysis* **5**, 453-472.

Billingsley, P. (1968).*Convergence of Probability Measures.* Wiley.

Brodsky, B. E., and Darkhovsy, B. S., (1993). *Nonparametric Methods in Change Point Problems.* Kluwer Academic Press.

Broemeling, L. D. and Gregurich, M. A. (1996). On a Bayesian approach for the shift point problem. *Commun. Statist. Theory Methods.* **25**, 2267-2279.

Chernoff, H. and Zacks, S. (1964). Estimating the current mean of a normal distribution which is subjected to changes in time. *Ann. Math. Statist.* **35.** 999-1018.

Csörgő, M. and Horváth, L., (1997).*Limit Theorems in Change-Point Analysis.* Wiley.

Genz, A. (1992). Numerical computation of multivariate normal probabilities. *Journal of Computational and Graphical Statistics*, **1**, 141-150.

Gupta, A. K. and Ramanayake, A. (2001). Change points with linear trend for the exponential distribution. *J. Statist. Plann. Inference* **93**, 181-195.

Habibi, R., Sadooghi-Alvandi, S. M., and Nematollahi, A. R., (2005). Change point detection in general class of distribution. *Commun. Statist. Theory and Method.* **34**, 1935-1938.

Hawkins, D. M. (1977). Testing a Sequence of Observations for a Shift in Location. *J. Amer, Statist. Asso.* **72**, 180-186.

Hjort, N. L. and Koning, A. (2002). Tests for constancy of model parameters over time. *Nonparametric Statistics* **14**, 113-132.

Kalbfleisch, J. D. and Prentice, R. L. (1980). *The statistical analysis of failure time data.* New York.

Kander, Z. and Zacks, S., (1966). Test procedures for possible changes in parameters of statistical distributions occurring at unknown time points. *Ann. Math. Statist.* **37.** 1196-1210.

Sadooghi-Alvandi, S. M., Nematollahi, A. R., and Habibi, R., (2005). On the distribution of the sum of independent uniform random variables. Tech. Report. Shiraz University.

Yao, Q., (1993). Tests for change points with epidemic alternatives, *Biometrika* **80,** 179-191.

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