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A Two-Parameter Distribution with Increasing and Bathtub Hazard Rate

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Abstract

In this paper a new two-parameter distribution is proposed. This new model provides more flexibility to modeling data with increasing and bathtub hazard rate function. Several statistical and reliability properties of the proposed model are also presented in this paper, such as moments, moment generating function, order statistics and stress-strength reliability. The maximum likelihood estimators for the parameters are discussed as well as a bias corrective approach based on bootstrap techniques. A numerical simulation is carried out to examine the bias and the mean square error of the proposed estimators. Finally, an application using a real data set is presented to illustrate our model.

Keywords *lifetime model; maximum likelihood estimation*

1 Introduction

Mixture models have been playing an important role in distribution theory (Patil and Rao, 1978). In recent years, there has been a renewed interested in proposing new models based on mixture distribution. A simple case can be considered where new models are generated by a two-component mixture

$$f(t|\mathbf{\Lambda_1}, \mathbf{\Lambda_2}, p) = pf_1(t|\mathbf{\Lambda_1}) + (1-p)f_2(t|\mathbf{\Lambda_2})\,, \tag{1}$$

where $0 \leq p \leq 1$ is mixing proportion (MP) and $\mathbf{\Lambda_1}, \mathbf{\Lambda_2}$ are the parameters related to the probability density function (PDF) $f_1(\cdot)$ and $f_2(\cdot)$.

Carta and Ramirez (2007) considered (1) based on a mixture of two Weibull distribution where the MP is a free unknown parameter to be estimated. Ramos and Louzada (2019) proposed a new one parameter distribution based on the mixture of a gamma and an exponential distribution. Ghitany et al. (2011) proposed a weighted Lindley model based on the mixture of two gamma distribution, where the MP is $p = \theta/(\lambda+\theta)$, $\lambda > 0$ and $\theta > 0$. Further, Ghitany et al. (2013) considered $\lambda = 1$ for MP and two Weibull distributions. The obtained model was named as power Lindley distribution. Ramos and Louzada (2016) unified these models by considering the mixture of two generalized gamma distributions. The main reason for the authors considered such MP comes from the fact that the new distributions are generalizations of the well known Lindley distribution. Other generalizations of the Lindley distribution can be view in Ekhosuehi et al. (2018); Yassmen (2019); Kwong and Nadarajah (2019).

On the other hand, many different MPs can be considered instead. For instance, let $p = (\lambda-2\theta)/(\lambda-\theta)$, then $0 \leq p \leq 1$ if $\lambda \geq 0$ and $0 \leq \theta \leq \lambda/2$. Here, we considered this MP based

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on two gamma distribution. The obtained distribution has PDF given by

$$f(t|\theta,\lambda) = \frac{\lambda^{\theta}}{(\lambda-\theta)\Gamma(\theta)} t^{\theta-1}(\lambda+\lambda t-2\theta)e^{-\lambda t}, \qquad t>0, \tag{2}$$

where $0 \leq \theta \leq \lambda/2$ and $\lambda > 0$ are the parameters and $\Gamma(\theta) = \int_0^{\infty} e^{-x}x^{\theta-1}dx$ is the gamma function. Although we have considered two gamma distribution as $f_1(\cdot)$ and $f_2(\cdot)$, the same idea can be extended for other distributions such as Weibull, Lognomal, Exponentied Exponential among others. To the best of our knowledge, this distribution has not yet been presented in the literature. We also have not encountered its respective one parameter particular cases.

Mixture of gamma density functions has been used to describe heterogeneity (see, Mayrose et al. (2005)). In this paper, a significant account of mathematical properties has been presented for the new distribution such as moments, survival properties and its entropy function. For the new distribution the hazard function has increasing or bathtub shape, depending on the values of the parameters. This property plays an important role to describe lifetime data (Chen, 2000; Wang et al., 2002). The stress-strength parameter $R = P(T_2 < T_1)$ where T_1 and T_2 have PDFs given by $f(t|\theta_1,\lambda)$ and $f(t|\theta_2,\lambda)$, respectively. The maximum likelihood estimators (MLEs) of the parameters and its asymptotic properties are discussed. Further, we also present a bias corrective approach for the MLEs based on bootstrap techniques. A simulation study is conducted to examine the performance of the proposed estimators. Finally, use our model to describe a data set related to 30 patients with brain cancer receiving radiotherapy.

The paper is organized as follows. Section 2 introduces our new distribution and its properties such as moments, moment generating function, order statistics, survival properties and stress-strength reliability. Section 3 presents the estimators of the unknown parameters based on MLEs. In Section 4, a simulation study to verify the performance of the MLEs is reported. Section 5 illustrates the relevance of our proposed methodology for a real lifetime data. Section 6 summarizes the present study.

2 Properties

The proposed distribution can be expressed as a two-component mixture

$$f(t|\theta,\lambda) = pf_1(t|\theta,\lambda) + (1-p)f_2(t|\theta,\lambda)\,, \tag{3}$$

where $1-p = \theta/(\lambda-\theta)$ (or $p = (\lambda-2\theta)/(\lambda-\theta)$) and $T_j \sim \text{Gamma}(\theta+j-1,\lambda)$, for $j = 1,2$, that is, $f_j(t|\lambda,\theta)$ is a Gamma density given by

$$f_j(t|\theta,\lambda) = \frac{\lambda^{\theta+j-1}}{\Gamma(\theta+j-1)} t^{\theta+j-2}e^{-\lambda t}.$$

After some algebraic manipulation in (3) the new non-negative random variable has PDF given by (5). Ghitany et al. (2011) followed a similar way but considering that the MP is $p = \theta/(\lambda+\theta)$.

The behavior of the PDF (1) when $t \to 0$ and $t \to \infty$ are, respectively, given by

$$f(0) = \begin{cases} \infty, & \text{if } \theta < 1 \\ \dfrac{(\lambda^2-2\lambda)}{(\lambda-1)}, & \text{if } \theta = 1\ , \\ 0, & \text{if } \theta > 1 \end{cases} \qquad f(\infty) = 0.$$

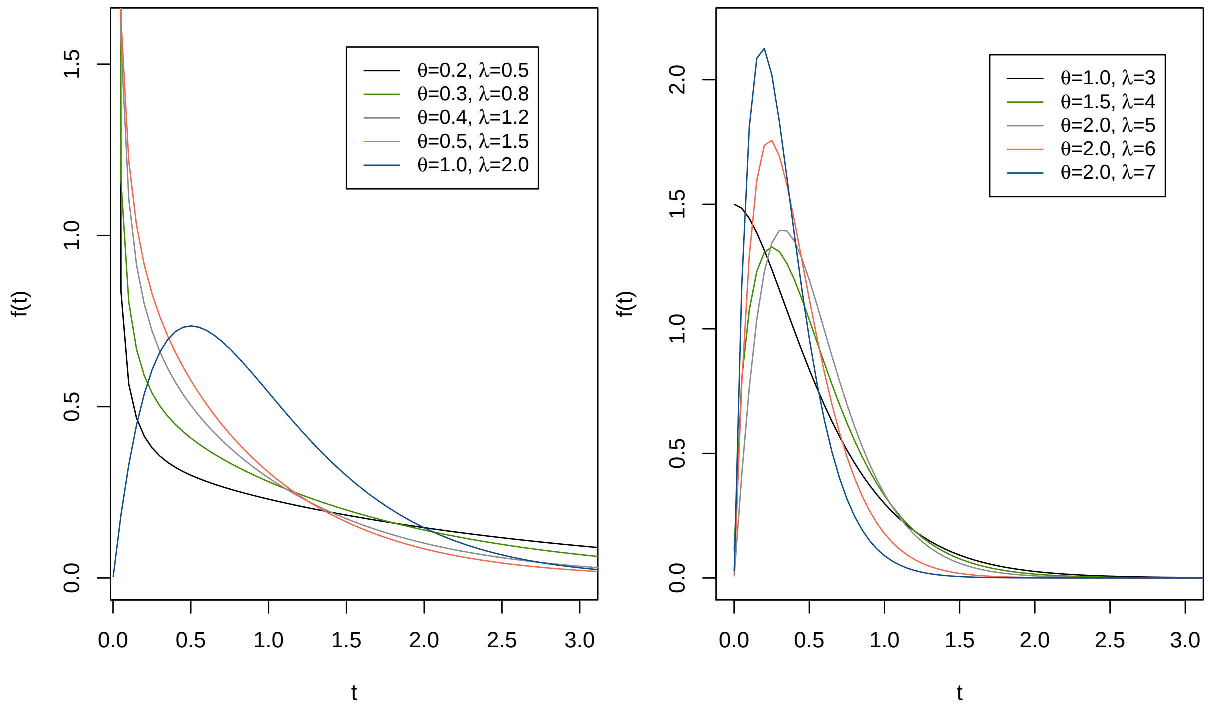

Figure 1: Density function shapes for new distribution considering different values of θ and λ.

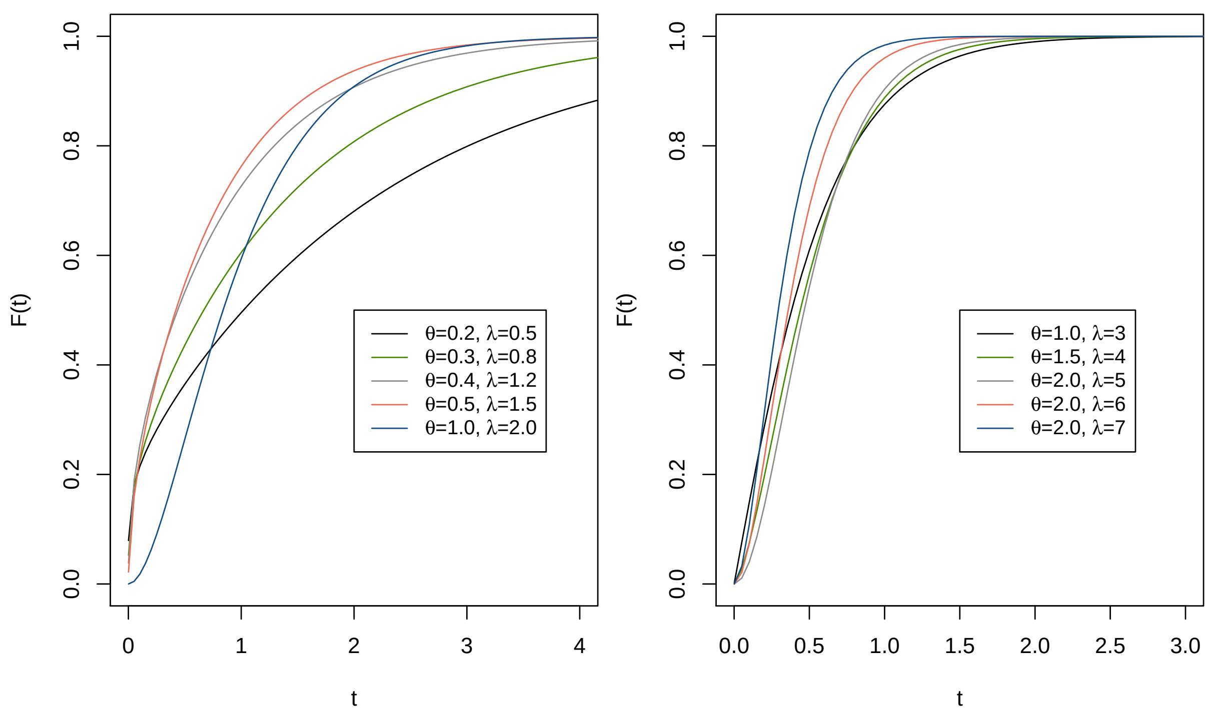

Figure 2: Cumulative density function shapes for new distribution considering different values of θ and fixed λ.

Figure 1 shows the shapes of the density function for different values of θ and λ. The cumulative distribution function from the new distribution is given by

$$F(t|\theta,\lambda) = \frac{\gamma\left(\theta,\lambda t\right)\left(\lambda-\theta\right)-(\lambda t)^{\theta}e^{-\lambda t}}{(\lambda-\theta)\Gamma(\theta)}, \tag{4}$$

where $\gamma(x,y) = \int_0^x w^{y-1}e^{-y}dw$ is the lower incomplete gamma function.

The proposed model may be considered a generalization of the one parameter case given by

$$f(t|\lambda) = \frac{\lambda(\lambda+\lambda t-2)}{(\lambda-1)}e^{-\lambda t}, \qquad t>0 \quad \text{and } \lambda>2, \tag{5}$$

in which is also a new distribution.

2.1 Moments and Moment Generating Function

Moments play an important role in statistical theory, in this section we provide the r-th moment, the mean, variance and the moment generating function for our distribution.

Proposition 1. For the random variable T with new distribution, the r-th moment is given by

$$\mu_r = E[T^r] = \frac{\theta(\theta+1)\ldots(\theta+r-1)(\lambda-\theta+r)}{\lambda^r(\lambda-\theta)}\,, \quad \text{for} \quad r\in\mathbb{N}. \tag{6}$$

Proof. Note that if $X \sim \text{Gamma}(\theta,\lambda)$ distribution then the r-th is given by

$$E[X^r;\theta,\lambda] = \frac{\Gamma(\theta+r)}{\lambda^r\Gamma(\theta)} = \frac{\theta(\theta+1)\ldots(\theta+r-1)}{\lambda^r}\,, \quad \text{for } r\in\mathbb{N}.$$

Since the proposed model can be expressed as a two-component mixture, we have

$$\begin{aligned}\mu_r = E[T^r] &= \int_0^\infty t^r f(t|\theta,\lambda)dt = pE[X^r;\theta,\lambda] + (1-p)E[X^r;\theta+1,\lambda]\\ &= \left(\frac{\lambda-2\theta}{\lambda-\theta}\right)\frac{\Gamma(\theta+r)}{\lambda^r\Gamma(\theta)} + \frac{\theta}{(\lambda-\theta)}\frac{\Gamma(\theta+1+r)}{\lambda^r\Gamma(\theta+1)}\\ &= \frac{(\lambda-\theta+r)\Gamma(\theta+r)}{\lambda^r(\lambda-\theta)\Gamma(\theta)}\\ &= \frac{\theta(\theta+1)\ldots(\theta+r-1)(\lambda-\theta+r)}{\lambda^r(\lambda-\theta)}.\end{aligned}$$

□

Proposition 2. The r-th central moment for the random variable T is given by

$$\begin{aligned}M_r = E[T-\mu]^r &= \sum_{i=0}^{r}\binom{r}{i}(-\mu)^{r-i}E[T^i]\\ &= \sum_{i=0}^{r}\binom{r}{i}\left(-\frac{(\lambda-\theta+1)\theta}{\lambda(\lambda-\theta)}\right)^{r-i}\left(\frac{\theta(\theta+1)\ldots(\theta+i-1)(\lambda-\theta+i)}{\lambda^i(\lambda-\theta)}\right).\end{aligned} \tag{7}$$

Proof. The result follows directly from the Proposition 1. □

Proposition 3. A random variable T with PDF (5) has the mean and variance given by

$$\mu = \frac{(\lambda - \theta + 1)\theta}{\lambda(\lambda - \theta)} \quad \text{and} \quad \sigma^2 = \frac{\theta\left((\lambda - \theta + 2)(\theta + 1) - \theta(\lambda - \theta + 1)^2\right)}{\lambda^2(\lambda - \theta)^2}. \tag{8}$$

Proof. From (6) and considering $r = 1$, it follows that $\mu_1 = \mu$. The second result follows from (7) considering $r = 2$ and with some algebra the proof is completed. □

Proposition 4. For the random variable T, the Moment Generating Function is given by

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\theta}\left(1 + \frac{\theta t}{(\lambda - \theta)(\lambda - t)}\right). \tag{9}$$

Proof. Indeed, by definition $M_X(t) = E(e^{tX})$ we have

$$M_X(t) = p\int_0^{\infty} e^{tX_1} f_1(x)dx + (1-p)\int_0^{\infty} e^{tX_2} f_2(x)dx = pM_{X_1}(t) + (1-p)M_{X_2}(t),$$

for $p = \frac{\lambda - 2\theta}{\lambda - \theta}$ and $1 - p = \frac{\theta}{\lambda - \theta}$.

From $X_1 \sim \Gamma(\theta, \lambda)$ and $X_2 \sim \Gamma(\theta + 1, \lambda)$ we have $M_{X_1}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\theta}$ and $M_{X_2}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\theta+1}$, respectively, for $t < \lambda$. Thus,

$$M_X(t) = \frac{\lambda - 2\theta}{\lambda - \theta}\left(\frac{\lambda}{\lambda - t}\right)^{\theta} + \frac{\theta}{\lambda - \theta}\left(\frac{\lambda}{\lambda - t}\right)^{\theta+1},$$

and after some algebric computations the proof is completed.

□

2.2 Order Statistics

Let $X_1, X_2, \ldots X_n$ be a random sample from (5) and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the the corresponding order statistics. It is well known that the probability density function and the cumulative distribution function of the of $r-$th order statistic say $X_{r:n}$, $1 \leq r \leq n$ are given by

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x) \\ &= \frac{n!}{(r-1)!(n-r)!}\sum_{u=0}^{n-r}(-1)^u \binom{n-r}{u} [F(x)]^{r-1+u} f(x) \end{aligned} \tag{10}$$

and

$$F_{r:n}(x) = \sum_{l=k}^{n}\binom{n}{l}[F(x)]^l[1 - F(x)]^{n-l} = \sum_{l=k}^{n}\sum_{u=0}^{n-r}(-1)^u\binom{n}{l}\binom{n-r}{u}[F(x)]^{l+u}, \tag{11}$$

respectively, for $k = 1, 2, \ldots, n$. It follows from (10) and (11) that

$$\begin{aligned} f_{r:n}(x) =& \frac{n!\lambda^{\theta} t^{\theta-1}(\lambda + \lambda t - 2\theta)e^{-\lambda t}}{(r-1)!(n-r)!(\lambda - \theta)\Gamma(\theta)}\sum_{u=0}^{n-r}(-1)^u\binom{n-r}{u} \\ &\times \left(\frac{\gamma(\theta, \lambda t)(\lambda - \theta) - (\lambda t)^{\theta} e^{-\lambda t}}{(\lambda - \theta)\Gamma(\theta)}\right)^{r-1+u} \end{aligned}$$

and

$$F_{r:n}(x) = \sum_{l=k}^{n}\sum_{u=0}^{n-r}(-1)^u \left(\begin{array}{c} n \\ l \end{array} \right) \left(\begin{array}{c} n-r \\ u \end{array} \right) \left(\frac{\gamma\left(\theta,\lambda t\right)\left(\lambda-\theta\right)-(\lambda t)^{\theta}e^{-\lambda t}}{(\lambda-\theta)\Gamma(\theta)}\right)^{l+u}.$$

2.3 Survival Properties

The survival function of T representing the probability of an observation does not fail until a specified time t is given by

$$S(t|\theta,\lambda) = \frac{\Gamma\left(\theta,\lambda t\right)\left(\lambda-\theta\right)+(\lambda t)^{\theta}e^{-\lambda t}}{(\lambda-\theta)\Gamma(\theta)}, \tag{12}$$

where $\Gamma(y,x) = \int_x^{\infty} w^{y-1}e^{-w}dw$ is the uper incomplete gamma function. The hazard function of T is given by

$$h(t|\theta,\lambda) = \frac{f(t|\theta,\lambda)}{S(t|\theta,\lambda)} = \frac{\lambda^{\theta}t^{\theta-1}(\lambda+\lambda t-2\theta)e^{-\lambda t}}{\Gamma\left(\theta,\lambda t\right)\left(\lambda-\theta\right)+(\lambda t)^{\theta}e^{-\lambda t}}. \tag{13}$$

This model has increasing and bathtub hazard rate. The following Lemma is useful to prove such result.

Lemma 1. Glaser (1980): Let T be a non-negative continuous random variable with twice differentiable PDF $f(t)$ and hazard rate function $h(t)$ and $\eta(t) = -\frac{\partial}{\partial t}\log f(t)$.

1. Let $\eta'(t)$ be $\eta'(t) > 0$ ($\eta'(t) < 0$), $\forall t$, then $h(t)$ is increasing (decreasing).
2. Suppose that exists $t_0 > 0$ such that, $\eta'(t) < 0$, $\forall t \in (0,t_0)$, $\eta'(t) = 0$ and $\eta'(t) > 0$, $\forall t \in (t_0, 0)$, then if

$$\lim_{t\to 0^+} f(t) = \infty,$$

   $h(t)$ has bathtub.

Theorem 1. The hazard function (13) is bathtub if $\theta < 1$ and increasing if $\theta \geq 1$.

Proof. Let

$$\eta(t) = -\frac{\theta-1}{t} - \frac{1}{(1+t-2\theta/\lambda)} + \lambda\,, \tag{14}$$

it follows that

$$\eta'(t) = \frac{\theta-1}{t^2} + \frac{1}{\left(1+t-2\theta/\lambda\right)^2}. \tag{15}$$

From (15) we observe that, if $\theta \geq 1$ then $\eta'(t) > 0$ and consequently the hazard function is increasing $\forall t > 0$. By other hand, if $\theta < 1$ the function $\eta(t)$ has a global minimum given by $t_0 = \frac{(\lambda-2\theta)(1-\theta+\sqrt{1-\theta})}{\theta\lambda}$, that is, $\eta(t)$ has a bathtub shape and since

$$\lim_{t\to 0^+} f(t) = \infty,$$

that implies that, $h(t)$ has a bathtub shape. □

Figure 3 gives examples from the shapes of the density function for different values of θ and λ.

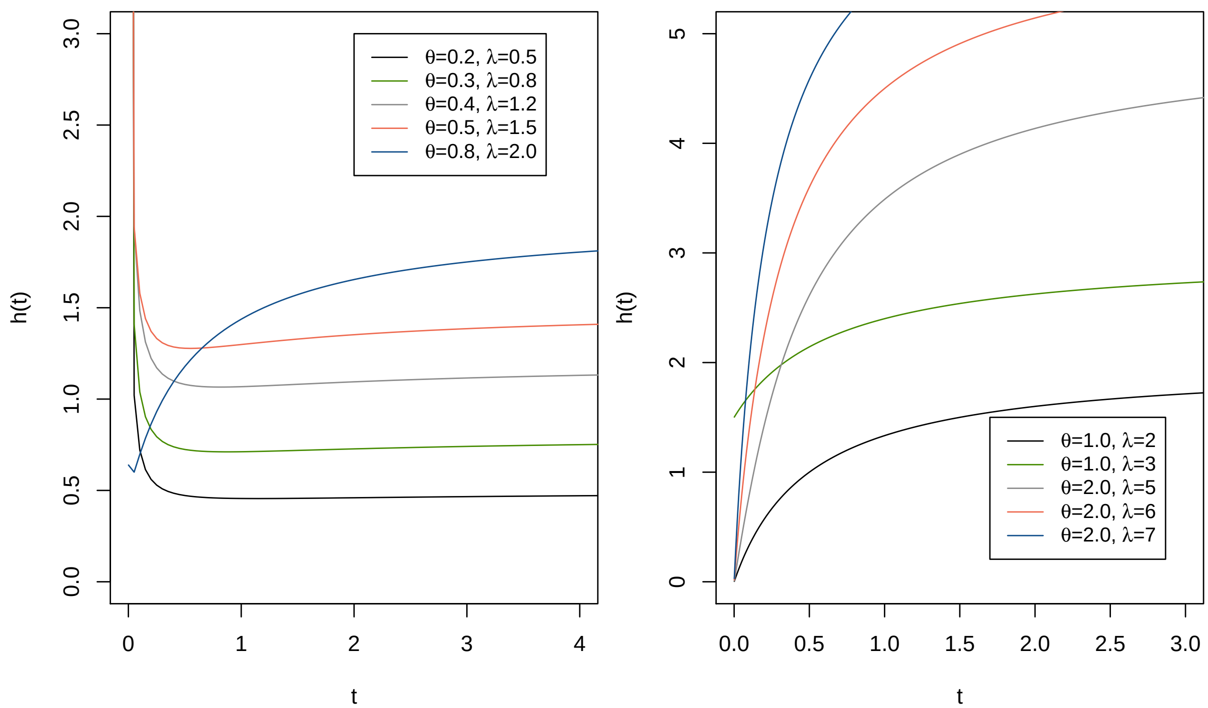

Figure 3: Hazard function shapes for the new distribution considering different values of θ and fixed λ.

2.4 Stress-strength reliability

In reliability analysis, the stress-strength reliability usually referred to as the stress-strength model has obtained wide attention in the literature, including quality control, engineering statistics, reliability, medicine, psychology, biostatistics, stochastic precedence, and probabilistic mechanical design (see Kotz and Pensky (2003) for a comprehensive review).

Suppose T_2 represents the 'stress' which is applied to a certain device and T_1 represents the strength to sustain the stress, then the stress-strength reliability is computed as $R = P(T_2 < T_1)$,
Proposition 5. Suppose that T_1 and T_2 have PDFs given by $f(t|\theta_1, \lambda)$ and $f(t|\theta_2, \lambda)$ distributions, respectively. If the random variables are independent then the stress-strength reliability is given by

$$\begin{aligned} R =& P\{T_2 < T_1\} = \frac{p_1 p_2}{2^{(\theta_1+\theta_2)}\theta_2 B(\theta_1, \theta_2)}\, {}_1F_2\left(1, \theta_1+\theta_2; 1+\theta_2; \frac{1}{2}\right) \\ &+ \frac{q_1 p_2}{2^{(\theta_1+\theta_2+1)}\theta_2 B(\theta_1+1,\ \theta_2)}\, {}_1F_2\left(1, \theta_1+\theta_2+1; 1+\theta_2; \frac{1}{2}\right) \\ &+ \frac{p_1 q_2}{2^{(\theta_1+\theta_2+1)}(\theta_2+1) B(\theta_1,\ \theta_2+1)}\, {}_1F_2\left(1, \theta_1+\theta_2+1; 2+\theta_2; \frac{1}{2}\right) \\ &+ \frac{q_1 q_2}{2^{(\theta_1+\theta_2+2)}(\theta_2+1) B(\theta_1+1,\ \theta_2+1)}\, {}_1F_2\left(1, \theta_1+\theta_2+2; 2+\theta_2; \frac{1}{2}\right), \end{aligned}$$

where $p_1 = \frac{\lambda-2\theta_1}{\lambda-\theta_1}$, $q_1 = \frac{\theta_1}{\lambda-\theta_1}$, $p_2 = \frac{\lambda-2\theta_2}{\lambda-\theta_2}$, $q_2 = \frac{\theta_2}{\lambda-\theta_2}$. $B(a_1,\, a_2)$ is the beta function defined as

$B(a_1,\, a_2) = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)}$ and

$$ {}_pF_q(\alpha_1, \ldots,\, \alpha_k; \beta_1, \ldots,\, \beta_l, x) = \sum_{i=0}^{\infty} \frac{(\alpha_1)_i \ldots (\alpha_k)_i\, x^i}{(\beta_1)_i \ldots (\beta_k)_i\, i!}, $$

is the generalized hypergeometric function.

Proof. Since T_1 and T_2 have PDFs given by $f(t|\theta_1, \lambda)$ and $f(t|\theta_2, \lambda)$ and the distributions are expressed as the two-component mixture given (3), respectively, by

$$ f_{T_1}(t) = p_1 f(t; \theta_1,\, \lambda) + q_1 f(t; \theta_1 + 1,\, \lambda) $$

and

$$ F_{T_2}(t) = p_2 F(t; \theta_2,\, \lambda) + q_2 F(t; \theta_2 + 1,\, \lambda), $$

where $f(t)$ and $F(t)$ are density and cumulative functions of two-parameters Gamma distribution. Therefore

$$ \begin{aligned} R = P\{T_2 < T_1\} &= \int_0^\infty F_{T_2}(t) f_{T_1}(t) dt \\ &= \int_0^\infty \Big(p_2 F(t; \theta_2,\, \lambda) + q_2 F(t; \theta_2 + 1,\, \lambda)\Big)\Big(p_1 f(t; \theta_1, \lambda) + q_1 f(t; \theta_1 + 1,\, \lambda)\Big) dt \\ &= p_1 p_2 \int_0^\infty F(t; \theta_2, \lambda) f(t; \theta_1, \lambda) dt + q_1 p_2 \int_0^\infty F(t; \theta_2, \lambda) f(t; \theta_1 + 1, \lambda) dt \\ &\quad + p_1 q_2 \int_0^\infty F(t; \theta_2 + 1, \lambda) f(t; \theta_1, \lambda) dt + +q_1 q_2 \int_0^\infty F(t; \theta_2 + 1, \lambda) f(t; \theta_1 + 1, \lambda) dt. \end{aligned} \tag{16} $$

By replacing the densities and cumulative functions $f(\cdot)$ and $F(\cdot)$ on (16) we have

$$ \begin{aligned} R =& \frac{p_1 p_2 \lambda^{\theta_1}}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^\infty t^{\theta_1 - 1} \gamma(\theta_2, \lambda t) e^{-\lambda t} dt + \frac{q_1 p_2 \lambda^{\theta_1 + 1}}{\Gamma(\theta_1 + 1)\Gamma(\theta_2)} \int_0^\infty t^{\theta_1} \gamma(\theta_2, \lambda t) e^{-\lambda t} dt \\ &+ \frac{p_1 q_2 \lambda^{\theta_1}}{\Gamma(\theta_1)\Gamma(\theta_2 + 1)} \int_0^\infty t^{\theta_1 - 1} \gamma(\theta_2 + 1, \lambda t) e^{-\lambda t} dt + \frac{q_1 q_2 \lambda^{\theta_1 + 1}}{\Gamma(\theta_1 + 1)\Gamma(\theta_2 + 1)} \times \\ &\times \int_0^\infty t^{\theta_1} \gamma(\theta_2 + 1, \lambda t) e^{-\lambda t} dt. \end{aligned} \tag{17} $$

Nadarajah (2003) shows that

$$ \frac{b^{\mathrm{a}_1}}{\Gamma(\mathrm{a}_1)\Gamma(\mathrm{a}_2)} \int_0^\infty t^{\mathrm{a}_1 - 1} \gamma(a_2, bt) e^{-\mathrm{b}t} dt = \frac{2^{-(\mathrm{a}_1 + \mathrm{a}_2)}}{\mathrm{a}_2 B(\mathrm{a}_1,\, \mathrm{a}_2)}\, {}_1F_2\left(1, a_1 + a_2; 1 + a_2; \frac{1}{2}\right) \tag{18} $$

Thus, from (17) and (18) the proof is completed.

□

3 Inference

In this section we present the maximum likelihood estimator of the parameters θ and λ of the proposed distribution as well as a bias corrective approach.

3.1 Maximum Likelihood Estimation

Let $T_1, \ldots, T_n$ be a random sample such that T has PDF given in (1). In this case, the likelihood function from (5) is given by

$$L(\boldsymbol{\Theta}; \boldsymbol{t}) = \frac{\lambda^{n\theta}}{(\lambda-\theta)^n \Gamma(\theta)^n} \left\{ \prod_{i=1}^n t_i^{\theta-1} \right\} \prod_{i=1}^n (\lambda + \lambda t_i - 2\theta) \exp\left\{ -\lambda \sum_{i=1}^n t_i \right\}. \tag{19}$$

The log-likelihood function $l(\boldsymbol{\Theta}; \boldsymbol{t}) = \log L(\boldsymbol{\Theta}; \boldsymbol{t})$ is given by

$$\begin{aligned} l(\boldsymbol{\Theta}; \boldsymbol{t}) = n\theta \log\lambda - n\log(\lambda-\theta) - n\log\Gamma(\theta) - \lambda\sum_{i=1}^n t_i + (\theta-1)\sum_{i=1}^n \log(t_i) \\ + \sum_{i=1}^n \log\left(\lambda + \lambda t_i - 2\theta\right). \end{aligned} \tag{20}$$

From the expressions $\frac{\partial}{\partial\theta} l(\boldsymbol{\Theta}; \boldsymbol{t}) = 0$, $\frac{\partial}{\partial\lambda} l(\boldsymbol{\Theta}; \boldsymbol{t}) = 0$, we get the likelihood equations

$$n\log(\lambda) + \sum_{i=1}^n \log(t_i) + \frac{n}{\lambda-\theta} - n\psi(\theta) - \sum_{i=1}^n \frac{2}{\lambda + \lambda t_i - 2\theta} = 0$$

and

$$\frac{n\theta}{\lambda} - \sum_{i=1}^n t_i + \frac{n}{\lambda-\theta} + \sum_{i=1}^n \frac{1+t_i}{\lambda + \lambda t_i - 2\theta} = 0,$$

where $\psi(k) = \frac{\partial}{\partial k}\log\Gamma(k) = \frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function.

Under mild conditions (see Migon et al. (2014)) the ML estimates are asymptotically normal distributed with a bivariate normal distribution given by

$$(\hat{\theta}, \hat{\lambda}) \sim N_2[(\theta, \lambda), I^{-1}((\theta, \lambda))] \text{ for } n \to \infty,$$

where the elements of the Fisher information matrix given by

$$h_{11}(\theta, \lambda) = \frac{n}{(\lambda-\theta)^2} + n\psi'(\theta) - 4nE\left[\frac{1}{(\lambda + \lambda t_i - 2\theta)^2}\right],$$

$$h_{12}(\theta, \lambda) = h_{21}(\theta, \lambda) = -\frac{n}{\lambda} - \frac{n}{(\lambda-\theta)^2} - 2nE\left[\frac{1+t}{(\lambda + \lambda t_i - 2\theta)^2}\right],$$

$$h_{22}(\theta, \lambda) = \frac{n(\theta+1)}{\lambda^2} - \frac{n}{(\lambda-\theta)^2} + nE\left[\left(\frac{1+t}{\lambda + \lambda t_i - 2\theta}\right)^2\right],$$

and $\psi'(k) = \frac{\partial}{\partial^2 k}\log\Gamma(k)$ is the trigamma function.

3.2 Bootstrap resampling method

In this section, we considered an corrective approach to reduce the bias of the MLEs. For our proposed model closed-form expressions for the bias are not possible since the higher-order derivatives do not have closed-form expression. To overcome this problem we consider the bootstrap resampling method proposed by Efron (1992) is a power alternative to reduce the bias of the

MLEs specially in cases where the it is difficult to derive the analitical expression of the bias (see Cox and Snell (1968)). This method consists in generating pseudo-samples from the original to estimate the bias of the MLEs. Further, bias-corrected MLEs are obtained by subtraction of the estimated bias in relation to the original MLEs.

Here, we follow the same steps as describe in Reath et al. (2018). Let $\mathbf{t} = (t_1, \ldots, t_n)^\top$ be a sample with size n randomly selected from the random variable T and has the distribution function $F = F_\eta(t)$. Also, let the parameter η be a function of F given by $\eta = t(F)$ and $\hat{\eta}$ be an estimator of η based on $\mathbf{t}$, that is, $\hat{\eta} = s(\mathbf{t})$. The pseudo-samples $\mathbf{t}^* = (t_1^*, \ldots, t_n^*)^\top$ are obtained by resampling with replacement the original sample $\mathbf{t}$. The bootstrap replicates of $\hat{\eta}$ is calculated, where $\hat{\eta}^* = s(\mathbf{t}^*)$ and the empirical cdf (ecdf) of $\hat{\eta}^*$ is used to estimate $F_{\hat{\eta}}$ (cdf of $\hat{\eta}$). Let $B_F(\hat{\eta}, \eta)$ be the bias of the estimator $\hat{\eta} = s(\mathbf{t})$ given by

$$B_F(\hat{\eta}, \eta) = E_F[\hat{\eta}, \eta] = E_F[s(\mathbf{t})] - \eta(F).$$

Note that the expectation is obtained in respect to F. The bootstrap estimators of the bias were obtained by replacing F with $F_{\hat{\eta}}$, where F is generated from the original sample. Therefore, the bootstrap bias estimate is

$$\hat{B}_{F_{\hat{\eta}}}(\hat{\eta}, \eta) = E_{F_{\hat{\eta}}}[\hat{\eta^*}] - \hat{\eta}.$$

By taking M bootstrap samples $(\mathbf{t}^{*(1)}, \mathbf{t}^{*(2)}, \ldots, \mathbf{t}^{*(M)})$ that are generated independently from the original sample $\mathbf{t}$ and the respective bootstrap estimates $(\hat{\eta}^{*(1)}, \hat{\eta}^{*(2)}, \ldots, \hat{\eta}^{*(M)})$ are calculated, then we can easily obtain the approximately bootstrap expectations $E_{F_{\hat{\eta}}}[\hat{\eta^*}]$ by

$$\hat{\eta}^{*(\cdot)} = \frac{1}{M}\sum_{i=1}^{M} \hat{\eta}^{*(i)}.$$

Hence, the obtained bias estimate based on M replications of $\hat{\eta}$ is $\hat{B}_F(\hat{\eta}, \eta) = \hat{\eta}^{*(\cdot)} - \hat{\eta}$, this implies that the bias corrected estimators obtained through by bootstrap resampling method can be obtained by

$$\eta^B = \hat{\eta} - \hat{B}_F(\hat{\eta}, \eta) = 2\hat{\eta} - \hat{\eta}^{*(\cdot)}.$$

For the proposed model, we have η^B denoted by $\hat{\theta}_{BOOT} = (\hat{\phi}_{BOOT}, \hat{\lambda}_{BOOT})^\top$.

4 Simulation Analysis

In this section, a simulation study is presented to compare the efficiency of the maximum likelihood method with the bias correction approach in the presence of complete and censored data. These comparisons are performed by computing the Bias and the mean square errors (MSE) given by

$$\text{Bias}(\Theta_i) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\Theta}_{i,j} - \Theta_i)\ , \quad \text{MSE}(\Theta_i) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\Theta}_{i,j} - \Theta_i)^2, \quad \text{for } i = 1, 2,$$

where N is the number of estimates and $\Theta = (\theta, \lambda)$. The data set is generated as follows:

1. Generate $U_i \sim \text{Uniform}(0, 1), i = 1, \ldots, n$;
2. Generate $X_i \sim \text{Gamma}(\theta, \lambda), i = 1, \ldots, n$;
3. Generate $Y_i \sim \text{Gamma}(\theta + 1, \lambda), i = 1, \ldots, n$;

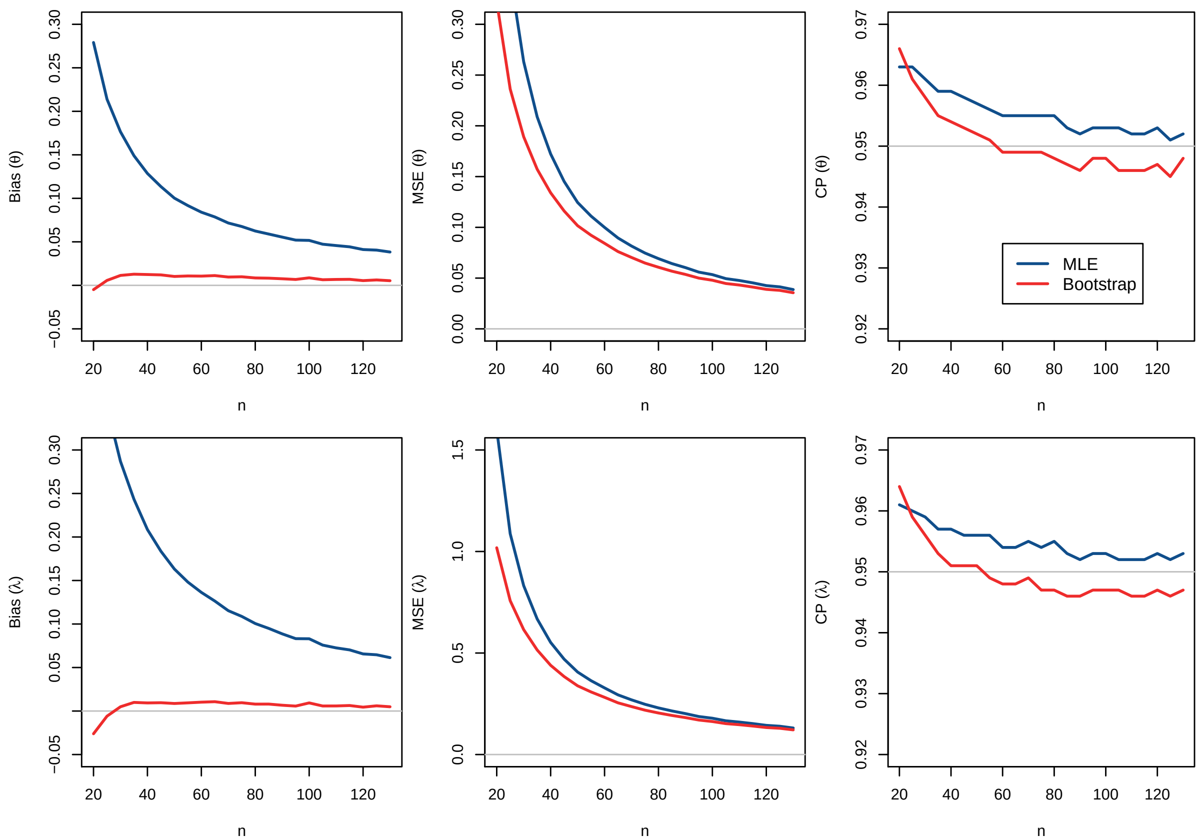

Figure 4: Bias, MSEs related from the estimates of $\theta = 1.5$ and $\lambda = 3$ for N simulated samples under the MLEs and the Bias corrected MLES.

4. If $U_i \leq p = (\lambda - 2\theta)/(\lambda - \theta)$, then set $T_i = X_i$, otherwise, set $T_i = Y_i, i = 1, \ldots, n$.

The simulation study is performed under the assumption of $n = (20, 35, \ldots,\ 130)$, $(\theta, \lambda) = ((1.5, 3), (0.5, 3))$ and $N = 50,000$. It is important to point out that, the results of this simulation study were similar for different choices of θ and λ. Following Reath et al. (2018) we considered $M = 1,000$ to compute the bootstrap method. The programs can be obtained, upon request. Figures 4–5 present the Bias, the MSE and the coverage probability with a 95% confidence level of the estimates obtained through the MLEs and the Bias corrected MLES for different samples of size.

From the obtained results, we can conclude that as there is an increase of n both Bias and MSE tend to zero, i.e., the estimators are asymptotic efficiency. Moreover, the coverage probability of the confidence levels tend to the nominal value assumed 0.95. Therefore, the MLE showed to be a good estimator for the parameters of the proposed distribution.

5 An Application

In this section, the proposed distribution is fully applied in an important data set. The data have been presented by the Medical Research Council Working Party on Misonidazole in Gliomas (MRC Working Party on Misonidazole in Gliomas, 1983). Table 1 consists of the time in day (divided per 1000) of 30 patients with brain cancer receiving radiotherapy alone or radiotherapy

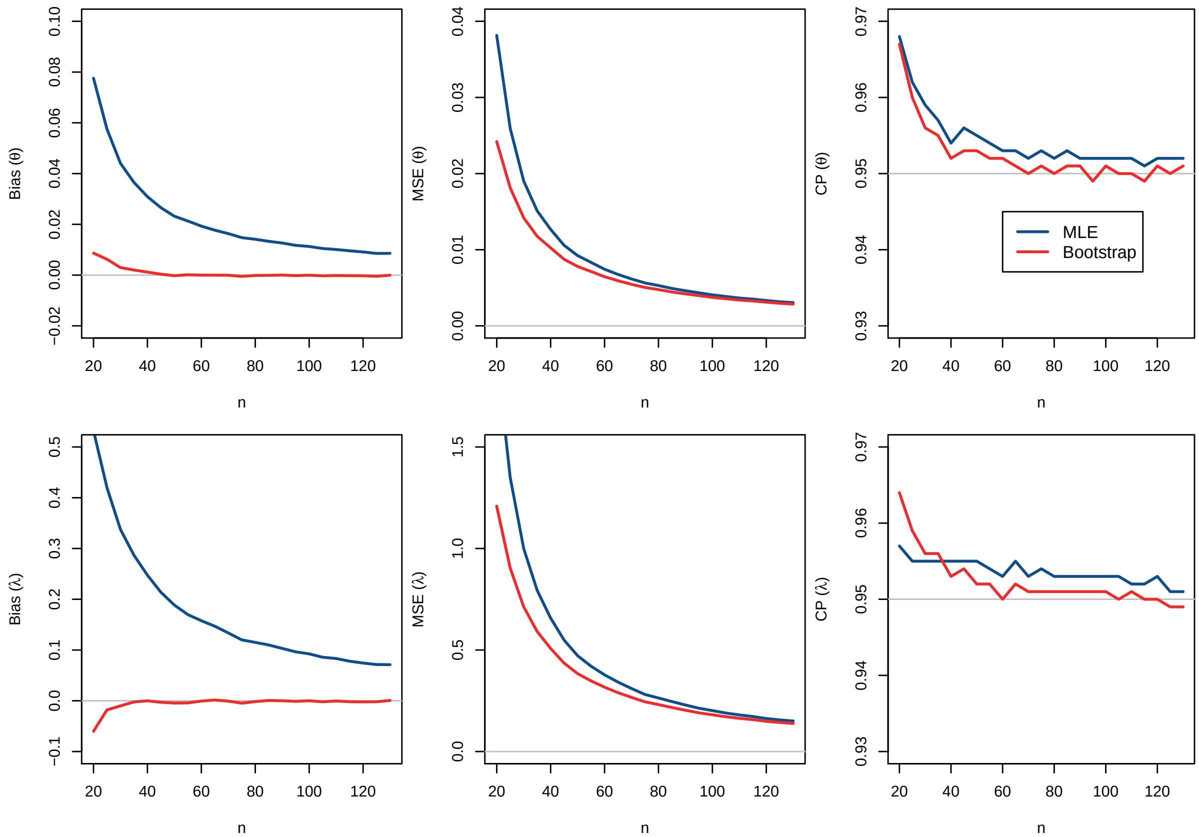

Figure 5: Bias, MSEs related from the estimates of $\theta = 0.5$ and $\lambda = 3$ for N simulated samples under the MLEs and the Bias corrected MLES.

Table 1: Data set related to 30 patients with brain cancer receiving radiotherapy alone or radiotherapy plus the radiosensitiser misonidazol, extracted from MRC Working Party on Misonidazole in Gliomas (1983).

1.084	0.022	0.040	0.025	0.487	0.696	0.887	0.336	0.213	0.361
0.244	0.799	0.180	0.488	0.121	0.210	0.575	0.258	0.273	1.098
0.819	0.014	0.734	0.225	0.152	0.207	0.943	0.581	0.371	0.085

plus the radiosensitiser misonidazole.

The results obtained from the proposed model are compared with other two-parameter distributions such as Weibull, Gamma, Lognormal and the Exponentiated Exponential (EE) distributions (Gupta and Kundu, 2001).

The discrimination among the models are conducted by different discrimination criteria based on log likelihood function. Such discrimination criterion methods are respectively:

- Akaike information criterion AIC $= -2l(\hat{\boldsymbol{\theta}}; \boldsymbol{x}) + 2k$;
- Corrected Akaike information criterion AICC $=$ AIC $+(2\,k\,(k+1))/(n-k-1)$;
- Hannan-Quinn information criterion HQIC $= -2\,l(\hat{\boldsymbol{\theta}}; \boldsymbol{x}) + 2\,k\,\log\left(\log(n)\right)$;
- Consistent Akaike information criterion CAIC $= -2\,l(\hat{\boldsymbol{\theta}}; \boldsymbol{x}) + k\left(\log(n) + 1\right)$,

where k is the number of parameters to be fitted and $\hat{\boldsymbol{\theta}}$ is the MLEs of $\boldsymbol{\theta}$,

Table 2: Results of AIC, AICc, HQIC, CAIC criteria and the p-value for the KS test for the compared distributions considering the 30 patients with brain cancer receiving radiotherapy alone or radiotherapy plus the radiosensitiser misonidazol.

Test	Proposed	Weibull	Gamma	Lognormal	EE
AIC	**10.398**	10.456	10.931	17.719	11.040
AICc	**6.843**	6.901	7.376	14.163	7.484
CAIC	**15.201**	15.259	15.734	22.521	15.842
HQIC	**11.295**	11.353	11.828	18.615	11.936
P-value	**0.9770**	0.9624	0.9599	0.4455	0.9430

Table 3: MLE, Standard-error and 95% confidence intervals for θ and λ.

$\boldsymbol{\theta}$	MLE	S. error	$CI_{95\%}(\boldsymbol{\theta})$
θ	1.0878	0.0696	(0.5707; 1.6048)
λ	3.6215	0.7081	(1.9722; 5.2708)

The best model is the one which provides the minimum values of these criteria. The Kolmogorov-Smirnov (KS) test is also considered in order to check the goodness of the fit for the models. This procedure depends on the KS statistic $D_n = \sup_x |F_n(x) - F(x;\boldsymbol{\theta})|$, where $\sup x$ is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function and $F(x;\boldsymbol{\theta})$ is the cdf of the fitted distribution. Considering a significance level of 5%, if the data comes from $F(x;\boldsymbol{\theta})$ (null hypothesis), then hypothesis is rejected if the p-value is smaller than 0.05.

Table 2 presents the results of AIC, AICc, HQIC, CAIC criteria, for the compared distributions.

Table 3 displays the MLEs, standard-error and 95% confidence intervals for θ and λ.

In Figure 6, we have the survival function adjusted by the compared distributions and the non-parametric survival function.

Comparing the empirical survival function with the adjusted models we observe a goodness of the fit for the proposed model, which is confirmed from different discrimination criterion methods as the new distribution has the minimum value for all statistics and the largest for the P-value. Consequently, we conclude that the data related to 30 patients with brain cancer receiving radiotherapy alone or radiotherapy plus the radiosensitizer misonidazol can be described by our new distribution.

6 Concluding Remarks

In this paper, we introduce a new two-parameter distribution. Further, its mathematical properties were studied in detail. The hazard function of this distribution showed increasing and bathtub hazard rate, which is uncommon for simple two-parameters distribution. The MLEs for the parameters, its asymptotic properties and a bias corrective approach based on bootstrap techniques were discussed. The simulation study showed that as the samples size increase, both Bias and MSE tend to zero, i.e., the estimators are asymptotic efficiency. Finally, the practical importance of our model was reported in a real application, the goodness of fit for the proposed data set showed that our model returned better fitting in comparison with other well-known

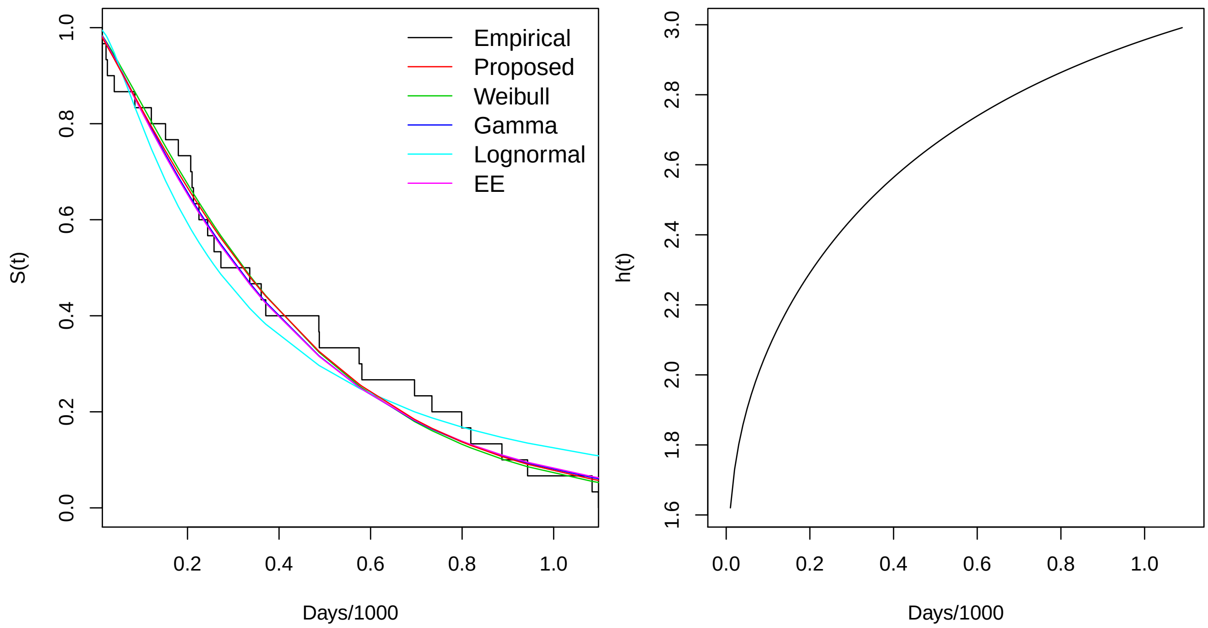

Figure 6: Survival function adjusted by the compared distributions and a non-parametric method considering the data sets related to 30 patients with brain cancer receiving radiotherapy alone or radiotherapy plus the radiosensitiser misonidazol.

distributions.

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