Journal of Data Science 18 (4), 761–781
October 2020
DOI: 10.6339/JDS.202010_18(4).0011

A Class of Bivariate Semiparametric Families of Distributions

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Abstract

The study of semiparametric families is useful because it provides methods of extending families for adding flexibility in fitting data. The main aim of this paper is to introduce a class of bivariate semiparametric families of distributions. One especial bivariate family of the introduced semiparametric families is discussed in details with its sub-models and different properties. In most of the cases the joint probability distribution, joint distribution and joint hazard functions can be expressed in compact forms. The maximum likelihood and Bayesian estimation are considered for the vector of the unknown parameters. For illustrative purposes a data set has been re-analyzed and the performances are quite satisfactory. A simulation study is performed to see the performances of the estimators.

Keywords *conditional probability; Gompertz distribution; hazard function; joint probability density; maximum likelihood estimation. Pareto distribution; Weibull distribution*

1 Introduction

To mathematically describe any family of distributions, various alternative functions are in common use. These functions include distribution functions, survival functions, densities; hazard functions, reversed hazard functions, cumulative hazard and cumulative reversed hazard functions. When they exist, any of these functions can be obtained from any other.

The distribution function $F(\cdot)$ and the survival function $S(\cdot)$ defined on $(-\infty, \infty)$ as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx, \tag{1}$$

$$S(x) = P(X > x) = \int_{x}^{\infty} f(x)dx, \tag{2}$$

where $f(x)$ is the probability density function for a continuous random variable. The hazard function $h(\cdot)$ and cumulative hazard function $H(\cdot)$ are defined on $(-\infty, \infty)$ respectively, as

$$h(x) = \frac{f(x)}{S(x)}, \tag{3}$$

$$H(x) = -\log S(x). \tag{4}$$

The reversed hazard function $r(\cdot)$ and cumulative reversed hazard $R(\cdot)$ function are defined on $(-\infty, \infty)$ respectively, as

$$r(x) = \frac{f(x)}{F(x)}, \tag{5}$$

$$R(x) = \log F(x). \tag{6}$$

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Received September, 2019; Accepted June, 2020

Now, it can be seen that by using (4) and (6) the distribution and survival functions (1) and (2) can be rewritten as, respectively,

$$F(x) = e^{R(x)} \quad \text{and} \quad S(x) = e^{-H(x)}. \tag{7}$$

Olkin (2007) divided the families of distributions into parametric, non-parametric and semi-parametric. Families of distribution indexed by a real number or by several real numbers are called parametric ones, such as the exponential, Weibull, gamma and lognormal distributions. The non-parametric families are defined by properties that have physically meaningful interpretations; these families of distributions have mostly been studied in the context of reliability theory. The theory that has been developed for these families has thus involved the notation of components and systems, which might be mechanical, electrical, hydraulic or biological systems.

Semiparametric families of distributions which are distinguished by having a parameter that itself a distribution function. These families have a real valued parameter; a possible procedure making use of a semiparametric model is to first select the parameter that is a distribution function. This distribution function is called the underlying distribution. In effect, the choice of an underlying distribution leads to the selection of a parametric model, but with the selection limited to families having the structure of the semiparametric model. The semiparametric families will be discussed in details in the next section.

Analyzing dependent variables is of great importance. For example, In Economic studies; Study the relation between (years of education and personal income, personal income and expenditure and inflation and unemployment), in Biological studies; Study of (blindness in the left and right eye, the age at death of parent and child in a genetic study, the relation between blood pressure and body weight for a patient and the failure time of the left and right kidney) in engineering studies ; analyzing the lifetime of a twine-engine plane, also warranty polices based on failure time and warranty servicing time, as well as, different applications like Shock model, competing risks model, stress model, maintenance model and longevity model.

Bivariate Marshal-Olkin family of great importance for understanding and analyzing the failure time of two variables interacting together, because it takes into consideration all different scenarios of the random variables (i.e. the first random variable is smaller, greater or equal to the second random variable).The main aim of this paper is to introduce a bivariate extension of the semiparametric families of distributions implies in such a way that their marginals follow univariate semiparametric distributions. The proposed bivariate models are shown to have a structure that has a singular part (see Marshall and Olkin, 1967).

The paper is organized as follows: in Section 2, univariate semiparametric families are introduced. Bivariate hazard power parameter (BHPP) family of distributions is defined in Section 3. A new bivariate distributions belongs to BHPP models is discussed in Section 4. Application of BHPP models to a real data set is introduced in Section 5. A simulation study is discussed in Section 6. Other bivariate semiparametric families of distributions are introduced in Section 7. Finally, the conclusion and future work are listed in Section 8.

2 Univariate Semiparametric Families

The study of semiparametric families is useful for two purposes. It provides a new understanding of standard parametric families of distributions that because, the standard families of gamma distributions and Weibull distributions can be thought of as coming from the exponential distribution by way of semiparametric families that added a second parameter, By the same method, it is possible to find a three parameter family that includes both the gamma and Weibull families as special cases . So, it provides methods of extending families for added flexibility in fitting data. In this section some semiparametric families that introduced by Olkin (2007) will be discussed in one dimension and its bivariate extension will be obtained in the next section.

It is important to see that the main criteria for the semiparametric families is that the underlying distribution is a member of the parametric family. And the second criteria, is that once the semiparametric family is used to add a parameter, its reuse may reparameterize the family, but it should fail to again add a new parameter. this is a kind of stability property (Olkin, 2007, p.609).

2.1 Univariate Power Parameter Family (UPP)

Let F_B be a baseline cdf. Suppose that that $F_{\text{UPP}}(\cdot;\alpha)$ is defined in terms of F_B by the formula

$$F_{\text{UPP}}(x;\alpha) = F_B(x^{\alpha}), \quad \alpha > 0. \tag{8}$$

Then α is called a power parameter and $\{F_{\text{UPP}}(\cdot;\alpha)\,\alpha > 0\}$ is an univariate power parameter family with underling distribution F_B.

The corresponding probability density function (pdf) and hazard function are, respectively,

$$f_{\text{UPP}}(x;\alpha) = \alpha x^{\alpha-1} f_B(x^{\alpha}), \tag{9}$$

$$h_{\text{UPP}}(x;\alpha) = \alpha x^{\alpha-1} h_B(x^{\alpha}), \tag{10}$$

where f_B and h_B are a baseline pdf and hazard functions respectively.

The family of Weibull distributions is the prime example of a Power parameter family. For this example the underlying distribution is an exponential distribution. Also Ghitany et al. (2013) introduced a power Lindley distribution, the underlying distribution is a Lindley distribution.

2.2 Univariate Frailty Parameter Family (UFP)

Let S_B be a baseline survival function with cumulative hazard function $H_B = -\log S_B$. Suppose that $S_{\text{UFP}}(\cdot;\alpha)$ is defined in terms of S_B by

$$S_{\text{UFP}}(x;\alpha) = [S_B(x)]^{\alpha} = \exp\{-\alpha H_B(x)\}, \quad \alpha > 0. \tag{11}$$

In this case α is called a frailty parameter and $S_{\text{UFP}}(\cdot;\alpha)$, $\alpha > 0$, is a frailty parameter family, or alternatively, a proportional hazard family with underlying distribution S_B.

The corresponding pdf and hazard function is given respectively, as

$$f_{\text{UFP}}(x;\alpha) = \alpha[S_B(x)]^{\alpha-1} f_B(x), \quad h_{\text{UFP}}(x;\alpha) = \alpha h_B(x), \tag{12}$$

where f_B and h_B are a baseline pdf and hazard functions respectively.

For exponential and Weibull distributions, introducing powers of the survival function does not introduce a new parameter because these families are already proportional hazards families. For a number of other families, however, a new parameter is introduced such as Pareto type I (Olkin, 2007), exponentiated Fréchet (Nadarajah and Kotz, 2003), exponentiated Gumble (Nadarajah, 2006), and extended Lindley Distributions (Bakouch et al., 2012).

2.3 Univariate Resilience Parameter Family (URP)

Let F_B be a baseline distribution function with cumulative reversed hazard function $R_B = \log F_B$ Suppose that $S_{\text{URP}}(\cdot;\alpha)$ is defined in terms of F_B by

$$F_{\text{URP}}\left(x;\alpha\right) = [F_B(x)]^{\alpha} = \exp\left\{\alpha R_B\left(x\right)\right\}, \quad \alpha > 0.$$

In this case α is called a resilience parameter and $F_{\text{URP}}\left(\cdot;\alpha\right), \alpha > 0$ is a resilience parameter family, or alternatively, a proportional reversed hazard family with underlying distribution F_B.

The corresponding pdf and hazard function is given respectively, as

$$f_{\text{URP}}\left(x;\alpha\right) = \alpha[F_B(x)]^{\alpha-1} f_B\left(x\right),$$
$$h_{\text{URP}}\left(x;\alpha\right) = \alpha r_B\left(x\right),$$

where f_B and r_B are a baseline pdf and reversed hazard functions respectively.

There exist number of distributions that produced by adding a resilience power parameter to some failure functions such as generalized exponential distribution (Gupta and Kundu, 1999), Generalized linear failure rate distribution (Sarhan and Kundu, 2009), exponetiated gamma (Nadarajah and Kotz, 2006), exponetiated Kumaraswamy (Lemonte et al., 2013), exponetiated Weibull (Mudholkar and Srivastava, 1993), exponetiated inverted Weibull (Flaih et al., 2012), and so on.

Resilience and frailty parameter families have the stability property, i.e., once a resilience or frailty parameter has been introduced; the reintroducing of the same kind of parameter does not extend the family.

2.4 Univariate Hazard Power Parameter Family (UHPP)

According to Equation (7) the survival function and its corresponding cumulative hazard function are related via the formula

$$S\left(x\right) = e^{-H(x)}, \quad \forall x.$$

It follows that if H is a cumulative hazard function, then H^{α} is accumulative hazard function. Thus,

$$S_{\text{UHPP}}\left(x;\alpha\right) = \exp\left\{-\left[H_B\left(x\right)\right]^{\alpha}\right\}, \quad \forall\alpha > 0. \tag{13}$$

Defines a survival function for all $\alpha > 0$, and $S_{\text{UHPP}F}\left(x;\alpha\right)$, $\alpha > 0$ is a semiparametric family. The parameter α is called hazard powerparameter family.The corresponding pdf is given by differentiating (13) as

$$f_{\text{UHPP}}\left(x;\alpha\right) = \alpha h_B\left(\text{x}\right)\left[H_B\left(x\right)\right]^{\alpha-1}\exp\left\{-\left[H_B\left(x\right)\right]^{\alpha}\right\}, \quad \forall\alpha > 0,$$

where $h_B(\cdot)$ and $H_B(\cdot)$ are the baseline hazard and cumulative hazard functions respectively. Accordingly the hazard function for UHPP family is given as

$$h_{\text{UHPP}}\left(x;\alpha\right) = \alpha h_B\left(x\right)\left[H_B\left(x\right)\right]^{\alpha-1}. \tag{14}$$

It is follows if h_B increasing and $\alpha \geq 1$, then $h_{\text{UHPP}F}$ is increasing; if h_B decreasing and $0 < \alpha < 1$, then $h_{\text{UHPP}F}$ is decreasing.

Examples:

1. **Univariate Exponential Distribution**

   Suppose that the underlying distribution is univariate exponential (UE) with scale parameter λ with survival function $S_B(x) = e^{-\lambda x}$ and $H_B(x) = \lambda x$. Then according to (13)-(14) and after adding a hazard power parameter α the survival, density and hazard functions respectively, are

$$S_{\mathrm{UW}}(x;\alpha) = e^{-(\lambda x)^{\alpha}}, \tag{15}$$

$$f_{\mathrm{UW}}(x;\alpha) = \alpha\lambda^{\alpha}x^{\alpha-1}e^{-(\lambda x)^{\alpha}}, \tag{16}$$

$$h_{\mathrm{UW}}(x;\alpha) = \alpha\lambda(\lambda x)^{\alpha-1}. \tag{17}$$

   For $x > 0$ and $\alpha > 0$. (15) is the survival function of an univariate Weibull (UW) distribution with hazard power parameter α and scale parameter λ. it is noted that α can be regarded not only as a hazard power parameter but also as a power parameter as mention above in Section 2.1.

2. **Univariate Gompertz Distribution**

   Suppose that the underlying distribution is univariate Gompertz (UG) distribution with survival function $S_B(x) = \mathrm{e}^{-\xi(\mathrm{e}^{\lambda x}-1)}, \quad \xi, \lambda > 0, x > 0$, according to (13) the survival function of the UG distribution with hazard power parameter α is given as

$$S_{\mathrm{UPHG}}(x) = \mathrm{e}^{-\left[\xi\left(\mathrm{e}^{\lambda x}-1\right)\right]^{\alpha}}, \quad \alpha, \xi, \lambda > 0, x > 0, \tag{18}$$

   and the corresponding pdf and hazard function are given respectively, as

$$f_{\mathrm{UPHG}}(x) = \alpha\lambda\xi\mathrm{e}^{\lambda x}\left[\xi\left(\mathrm{e}^{\lambda x}-1\right)\right]^{\alpha-1}\mathrm{e}^{-\left[\xi\left(\mathrm{e}^{\lambda x}-1\right)\right]^{\alpha}}, \quad \alpha > 0, \quad \xi > 0, \quad \lambda > 0, \quad x > 0,$$

   and

$$h_{\mathrm{UPHG}}(x) = \alpha\lambda\xi\mathrm{e}^{\lambda x}\left[\xi\left(\mathrm{e}^{\lambda x}-1\right)\right]^{\alpha-1}, \quad \alpha > 0, \quad \xi > 0, \quad \lambda > 0, \quad x > 0. \tag{19}$$

   Its observed by (Olkin, 2007) that the hazard function (19) is convex. It is increasing when $\alpha \geq 1$ and when $\alpha < 1$ the hazard function has minimum at $x = [-\log\alpha]/\lambda$.

3. **Univariate Pareto Type I Distribution**

   Suppose the underlying distribution is univariate Pareto (UP) distribution with survival function $S_B(x) = [1+\lambda x]^{-1}$, $\lambda > 0$, $x > 0$. After adding the hazard power parameter α, the survival function for the new distribution is given as

$$S_{\mathrm{UPHP}}(x;\alpha,\lambda) = e^{-[\log(1+\lambda x)]^{\alpha}}, \quad \alpha > 0, \quad \lambda > 0, \quad x > 0. \tag{20}$$

   Accordingly,

$$f_{\mathrm{UPHP}}(x;\alpha,\lambda) = \frac{\lambda\alpha}{(1+\lambda x)}\left[\log(1+\lambda x)\right]^{\alpha-1}e^{-[\log(1+\lambda x)]^{\alpha}}, \tag{21}$$

$$h_{\mathrm{UPHP}}(x;\alpha,\lambda) = \frac{\lambda\alpha}{(1+\lambda x)}\left[\log(1+\lambda x)\right]^{\alpha-1}. \tag{22}$$

   It is noted from (22) that hazard function is decreasing for $\alpha \leq 1$. For $\alpha > 1$ the hazard rate is unimodal with mode at $e^{\alpha-1}/\lambda$.

4. **Univariate Uniform Distribution**
Suppose the underlying distribution is univariate uniform (UU) distribution with $S_B(x) = 1 - x$ and $H_B(x) = -\log(1-x)$. After adding the hazard power parameter α then, the survival function for the new distribution is given as

$$S_{\text{UPHU}}(x;\alpha) = \mathrm{e}^{-[-\log(1-x)]^{\alpha}}, \quad \alpha > 0, \quad 0 < x < 1, \tag{23}$$

Consequently,

$$f_{\text{UPHU}}(x;\alpha) = \frac{\alpha}{(1-x)}\left[-\log(1-x)\right]^{\alpha-1}\mathrm{e}^{-[-\log(1-x)]^{\alpha}}, \tag{24}$$

$$h_{\text{UPHU}}(x;\alpha) = \frac{\alpha}{(1-x)}\left[-\log(1-x)\right]^{\alpha-1}. \tag{25}$$

2.5 Univariate Reversed Hazard Power Parameter Family (URPP)

It's possible to define a reversed hazard power parameter models via the formula $F(x) = \mathrm{e}^{R(x)}$, as following

$$F_{\text{URPP}}(x;\alpha) = \mathrm{e}^{[R(x)]^{\alpha}}, \quad \forall x,$$

where $\alpha > 0$ and $R(x) = \log F$ is a cumulative reversed hazard function.

$$f_{\text{URPP}}(x;\alpha) = \alpha r_B(x)\left[R_B(x)\right]^{\alpha-1}\exp\left\{\left[R_B(x)\right]^{\alpha}\right\}, \quad \forall \alpha > 0,$$

where $r_B(\cdot)$ and $R_B(\cdot)$ are the baseline reversed hazard and cumulative reversed hazard functions respectively. Accordingly the reversed hazard function for URPP family is given as

$$r_{\text{URPP}}(x;\alpha) = \alpha\, r_B(x)\left[R_B(x)\right]^{\alpha-1}.$$

3 Bivariate Hazard Power Parameter (BHPP) Family

Assume the univariate hazard power parameter model is denoted by UHPP(α, Θ) where α is the hazard power parameter and Θ may be a vector of parameters for an underlying distribution. Now suppose that $U_i \sim \text{UHPP}(\alpha_i, \Theta)$, $i = 1, 2, 3$ such that $U_i's$ are mutually independent random variables and define $X_j = \min(U_j, U_3)$, $j = 1, 2$. Such that $X_j's$ are dependent random variables. Hence the joint survival function of the vector (X_1, X_2) denoted by $S_{\text{BHPP}}(x_1, x_2)$ is given as

$$\begin{aligned} S_{\text{BHPP}}(x_1,x_2) &= S_{\text{UHPP}}(x_1;\alpha_1)\, S_{\text{UHPP}}(x_2;\alpha_2)\, S_{\text{UHPP}}(x_3;\alpha_3) \\ &= \exp\left\{-\left[H_B(x_1)\right]^{\alpha_1}\right\}\exp\left\{-\left[H_B(x_2)\right]^{\alpha_2}\right\}\exp\left\{-\left[H_B(x_3)\right]^{\alpha_3}\right\}, \end{aligned} \tag{26}$$

where $x_3 = \max(x_1, x_2)$.

The joint survival function of BHPP model can be stretching in the following form

$$S_{\text{BHPP}}(x_1,x_2) = \begin{cases} S_1(x_1,x_2) & x_1 < x_2, \\ S_2(x_1,x_2) & x_1 > x_2, \\ S_3(x) & x_1 = x_2 = x, \end{cases} \tag{27}$$

where

$$\begin{aligned} S_1(x_1,x_2) &= \exp\{-\left[H_B(x_1)\right]^{\alpha_1} - \left[H_B(x_2)\right]^{\alpha_2} - \left[H_B(x_2)\right]^{\alpha_3}\}, \\ S_2(x_1,x_2) &= \exp\{-\left[H_B(x_1)\right]^{\alpha_1} - \left[H_B(x_1)\right]^{\alpha_3} - \left[H_B(x_2)\right]^{\alpha_2}\}, \\ S_3(x) &= \exp\{-\left[H_B(x)\right]^{\alpha_1} - \left[H_B(x)\right]^{\alpha_2} - \left[H_B(x)\right]^{\alpha_3}\}. \end{aligned}$$

Accordingly, the joint pdf of BHPP model can be obtained as

$$f_{\mathrm{BHPP}}(x_1,x_2)=\begin{cases} f_1(x_1,x_2) & x_1<x_2,\\ f_2(x_1,x_2) & x_1<x_2,\\ f_3(x) & x_1=x_2=x,\end{cases} \tag{28}$$

where

$$\begin{aligned} f_1(x_1,x_2)=&\{\alpha_2 h_B(x_2)[H_B(x_2)]^{\alpha_2-1}+\alpha_3 h_B(x_2)[H_B(x_2)]^{\alpha_3-1}\}\\ &\cdot \alpha_1 h_B(x_1)[H_B(x_1)]^{\alpha_1-1}S_1(x_1,x_2),\\ f_2(x_1,x_2)=&\{\alpha_1 h_B(x_1)[H_B(x_1)]^{\alpha_1-1}+\alpha_3 h_B(x_1)[H_B(x_1)]^{\alpha_3-1}\}\\ &\cdot \alpha_2 h_B(x_2)[H_B(x_2)]^{\alpha_2-1}S_2(x_1,x_2),\\ f_3(x)=&\alpha_3 h_B(x)[H_B(x)]^{\alpha_3-1}S_3(x). \end{aligned}$$

The joint distribution function of (X_1,X_2) is given by

$$F_{\mathrm{BHPP}}(x_1,x_2)=\begin{cases} F_1(x_1,x_2) & x_1<x_2,\\ F_2(x_1,x_2) & x_1>x_2,\\ F_3(x) & x_1=x_2=x,\end{cases}$$

where

$$\begin{aligned} F_1(x_1,x_2)&=F_{\mathrm{UHPP}}(x_1;\alpha_{13})-F_{\mathrm{UHPP}}(x_1;\alpha_1)\left[1-F_{\mathrm{UHPP}}(x_2;\alpha_{23})\right],\\ F_2(x_1,x_2)&=F_{\mathrm{UHPP}}(x_2;\alpha_{23})-F_{\mathrm{UHPP}}(x_2;\alpha_2)\left[1-F_{\mathrm{UHPP}}(x_1;\alpha_{13})\right],\\ F_3(x)&=1-F_{\mathrm{UHPP}}(x;\alpha_{123}), \end{aligned}$$

where $\alpha_{ij}=\alpha_i+\alpha_j, i\neq j$.

The joint hazard function of the dependent variables (X_1,X_2) is obtained as follows

$$h_{\mathrm{BHPP}}(x_1,x_2)=\begin{cases} h_1(x_1,x_2) & x_1<x_2,\\ h_2(x_1,x_2) & x_1>x_2,\\ h_3(x) & x_1=x_2=x.\end{cases}$$

where

$$\begin{aligned} h_1(x_1,x_2)&=h_{\mathrm{UHPP}}(x_1;\alpha_1)\{h_{\mathrm{UHPP}}(x_2;\alpha_2)+h_{\mathrm{UHPP}}(x_2;\alpha_3)\}\\ &=\alpha_1 h_B(x_1)[H_B(x_1)]^{\alpha_1-1}\left\{\alpha_2 h_B(x_2)[H_B(x_2)]^{\alpha_2-1}+\alpha_3 h_B(x_2)[H_B(x_2)]^{\alpha_3-1}\right\},\\ h_2(x_1,x_2)&=h_{\mathrm{UHPP}}(x_2;\alpha_2)\{h_{\mathrm{UHPP}}(x_1;\alpha_1)+h_{\mathrm{UHPP}}(x_1;\alpha_3)\}\\ &=\alpha_2 h_B(x_2)[H_B(x_2)]^{\alpha_2-1}\left\{\alpha_1 h_B(x_1)[H_B(x_1)]^{\alpha_1-1}+\alpha_3 h_B(x_1)[H_B(x_1)]^{\alpha_3-1}\right\},\\ h_3(x)&=h_{\mathrm{UHPP}}(x;\alpha_3)=\alpha_3 h_B(x)[H_B(x)]^{\alpha_3-1}. \end{aligned}$$

The marginal survival and densities of X_1 and X_2 is given respectively, as follows

$$\begin{aligned} S_{X_i}(x_i)=&\exp\{-[H_B(x_i)]^{\alpha_i}-[H_B(x_i)]^{\alpha_3}\},\quad i=1,2,\\ f_{X_i}(x_i)=&\{\alpha_i h_B(x_i)[H_B(x_i)]^{\alpha_i-1}+\alpha_3 h_B(x_i)[H_B(x_i)]^{\alpha_3-1}\}\\ &\cdot\exp\{-[H_B(x_i)]^{\alpha_i}-[H_B(x_i)]^{\alpha_3}\},\quad i=1,2 \end{aligned}$$

Further, for the BHPP family the conditional density of X_{1i} given $X_{2j} = x_{2j}$ is given by

$$f_{X_i/X_j}(x_1, x_2) = \begin{cases} f_{i/j}^{(1)}(x_i/x_j) & \text{if } x_i < x_j, \\ f_{i/j}^{(2)}(x_i/x_j) & \text{if } x_j > x_i, \\ f_{i/j}^{(3)}(x_i/x_j) & \text{if } x_i = x_j, \end{cases}$$

where

$$\begin{aligned} f_{i/j}^{(1)}(x_i/x_j) =& \alpha_1 h_B(x_i)[H_B(x_i)]^{\alpha_1 - 1} \exp\{-[H_B(x_i)]^{\alpha_1}\}, \\ f_{i/j}^{(2)}(x_i/x_j) =& \alpha_2 h_B(x_j)[H_B(x_j)]^{\alpha_2 - 1} \exp\{-[H_B(x_j)]^{\alpha_2}\}, \\ f_{i/j}^{(3)}(x_i/x_j) =& \left\{ \frac{\alpha_3 h_B(x_i)[H_B(x_i)]^{\alpha_3 - 1}}{[\alpha_2 h_B(x_j)[H_B(x_j)]^{\alpha_2 - 1} + \alpha_3 h_B(x_j)[H_B(x_j)]^{\alpha_3 - 1}]} \right\} \\ & \cdot \exp\{-[H_B(x_i)]^{\alpha_1} - [H_B(x_i)]^{\alpha_2} - [H_B(x_i)]^{\alpha_3} + [H_B(x_j)]^{\alpha_2} + [H_B(x_j)]^{\alpha_3}\}. \end{aligned}$$

3.1 Maximum Likelihood Estimation for BHPP Models

Assume that $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ be a complete random sample from BHPP $(\alpha_1, \alpha_2, \alpha_3)$ family of distributions whose pdf and survival function are given in (28) and (27). Consider the following notation

$$I_1 = \{i; x_{1i} < x_{2i}\}, \quad I_2 = \{i; x_{1i} > x_{2i}\}, \quad I_3 = \{x_{1i} = x_{2i} = x_i\}, \quad I = I_1 \cup I_2 \cup I_3,$$

$$|I_1| = n_1, \quad |I_2| = n_2, \quad |I_3| = n_3, \quad \text{and} \quad n_1 + n_2 + n_3 = n.$$

The log-likelihood function of the sample of size n from BHPP $(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\begin{aligned} l(\underline{\alpha}) = & \, n_1 \log \alpha_1 + n_2 \log \alpha_2 + n_3 \log \alpha_3 \\ & + (\alpha_1 - 1) \sum_{I_1} \log [H_B(x_{1i})] + (\alpha_2 - 1) \sum_{I_2} \log [H_B(x_{2i})] + (\alpha_3 - 1) \sum_{I_3} \log [H_B(x_i)] \\ & - \sum_{I} [H_B(x_{1i})]^{\alpha_1} + [H_B(x_{1i})]^{\alpha_1} + [H_B(x_i)]^{\alpha_1} \\ & - \sum_{I} [H_B(x_{2i})]^{\alpha_2} + [H_B(x_{2i})]^{\alpha_2} + [H_B(x_{2i})]^{\alpha_2} \\ & - \sum_{I} [H_B(x_{2i})]^{\alpha_3} + [H_B(x_{1i})]^{\alpha_3} + [H_B(x_i)]^{\alpha_3} \\ & + \sum_{I} \log [h_B(x_{1i})] + \log [h_B(x_{2i})] + \log [h_B(x_i)] \\ & + \sum_{I_1} \Phi(x_{2i}; \alpha_2, \alpha_3) + \sum_{I_2} \Phi(x_{1i}; \alpha_1, \alpha_3), \end{aligned}$$

where $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$,

$$\Phi(x_{ki}; \alpha_k, \alpha_3) = \log \left[\alpha_k h_B(x_{ki}) [H_B(x_{ki})]^{\alpha_k - 1} + \alpha_3 h_B(x_{ki}) [H_B(x_{ki})]^{\alpha_3 - 1} \right], \quad k = 1, 2.$$

Accordingly, the likelihood equations can be written as

$$\frac{n_1}{\hat{\alpha}_1} + \sum_{I_1} \log\left[H_B\left(x_{1i}\right)\right] + \sum_{I_2} \Psi(x_{1i}; \alpha_1, \alpha_3)$$
$$= \sum_{I} \log\left[H_B\left(x_{1i}\right)\right] \{H_B\left(x_{1i}\right)^{\hat{\alpha}_1} + H_B\left(x_{1i}\right)^{\hat{\alpha}_1} + \log\left[H_B\left(x_i\right)\right]\left[H_B\left(x_i\right)\right]^{\hat{\alpha}_1},$$
$$\frac{n_2}{\hat{\alpha}_2} + \sum_{I_2} \log\left[H_B\left(x_{2i}\right)\right] + \sum_{I_1} \Psi(x_{2i}; \alpha_2, \alpha_3)$$
$$= \sum_{I} \log\left[H_B\left(x_{2i}\right)\right] \{H_B\left(x_{2i}\right)^{\hat{\alpha}_2} + H_B\left(x_{2i}\right)^{\hat{\alpha}_2} + \log\left[H_B\left(x_i\right)\right]\left[H_B\left(x_i\right)\right]^{\hat{\alpha}_2},$$
$$\frac{n_3}{\hat{\alpha}_3} + \sum_{I_3} \log\left[H_B\left(x_i\right)\right] + \sum_{I_1 \cup I_2} \xi\left(x_{2i}; \alpha_2, \alpha_3\right) + \xi\left(x_{1i}; \alpha_1, \alpha_3\right)$$
$$= \sum_{I} \log\left[H_B\left(x_{2i}\right)\right]\left[H_B\left(x_{2i}\right)\right]^{\hat{\alpha}_3} + \log\left[H_B\left(x_{1i}\right)\right]\left[H_B\left(x_{1i}\right)\right]^{\hat{\alpha}_3} + \log\left[H_B\left(x_i\right)\right]\left[H_B\left(x_i\right)\right]^{\hat{\alpha}_3},$$

where

$$\xi\left(x_{ki}; \alpha_k, \alpha_3\right) = \frac{\left[H_B\left(x_{ki}\right)\right]^{\alpha_3 - 1}\left[1 + \alpha_3 \log\left[H_B\left(x_{ki}\right)\right]\right]}{\alpha_k\left[H_B\left(x_{ki}\right)\right]^{\alpha_k - 1} + \alpha_3\left[H_B\left(x_{ki}\right)\right]^{\alpha_3 - 1}}, \quad k = 1, 2,$$

and

$$\Psi\left(x_{ki}; \alpha_k, \alpha_3\right) = \frac{\left[H_B\left(x_{ki}\right)\right]^{\alpha_k - 1}\left[1 + \alpha_k \log\left[H_B\left(x_{ki}\right)\right]\right]}{\alpha_k\left[H_B\left(x_{ki}\right)\right]^{\alpha_k - 1} + \alpha_3\left[H_B\left(x_{ki}\right)\right]^{\alpha_3 - 1}}, \quad k = 1, 2.$$

The second derivatives are given as follows

$$\frac{\partial^2 l\left(\underline{\alpha}\right)}{\partial \alpha_1^2} = -\frac{n_1}{\alpha_1^2} + \sum_{I_2} \eta\left(x_{1i}; \alpha_1, \alpha_3\right) - \sum_{I} \left(\log\left[H_B\left(x_{1i}\right)\right]\right)^2 \left\{H_B\left(x_{1i}\right)^{\hat{\alpha}_1} + H_B\left(x_{1i}\right)^{\hat{\alpha}_1}\right\}$$
$$+ \left(\log\left[H_B\left(x_i\right)\right]\right)^2 \left[H_B\left(x_i\right)\right]^{\hat{\alpha}_1},$$
$$\frac{\partial^2 l\left(\underline{\alpha}\right)}{\partial \alpha_2^2} = -\frac{n_2}{\alpha_2^2} + \sum_{I_1} \eta(x_{2i}; \alpha_2, \alpha_3) - \sum_{I} (\log[H_B(x_{2i})])^2 \{H_B(x_{2i})^{\hat{\alpha}_2} + H_B(x_{2i})^{\hat{\alpha}_2}\}$$
$$+ (\log[H_B(x_i)])^2 [H_B(x_i)]^{\hat{\alpha}_1},$$
$$\frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_3^2} = -\frac{n_3}{\alpha_3^2} + \sum_{I_1 \cup I_2} \delta(x_{2i}; \alpha_2, \alpha_3) + \delta(x_{1i}; \alpha_1, \alpha_3) \sum_{I} (\log[H_B(x_{2i})])^2 [H_B(x_{2i})]^{\hat{\alpha}_3}$$
$$+ (\log[H_B(x_{1i})])^2 [H_B(x_{1i})]^{\hat{\alpha}_3} + (\log[H_B(x_i)])^2 [H_B(x_i)]^{\hat{\alpha}_1},$$
$$\frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_1 \partial \alpha_3} = \sum_{I_2} \epsilon(x_{1i}; \alpha_1, \alpha_3) \quad \text{and} \quad \frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_2 \partial \alpha_3} = \sum_{I_1} \epsilon(x_{2i}; \alpha_2, \alpha_3),$$

where

$$\begin{aligned}
\mathrm{A}\left(x_{ki};\alpha_k,\alpha_3\right) &= \alpha_k\left[H_B\left(x_{ki}\right)\right]^{\alpha_k-1}+\alpha_3\left[H_B\left(x_{ki}\right)\right]^{\alpha_3-1}, \quad k=1,2,\\
\mathrm{B}\left(x_{ki};\alpha_k\right) &= \left[H_B\left(x_{ki}\right)\right]^{\alpha_k-1}\left[1+\alpha_k\log\left[H_B\left(x_{ki}\right)\right]\right], \quad k=1,2,\\
\mathrm{C}\left(x_{ki};\alpha_3\right) &= \left[H_B\left(x_{ki}\right)\right]^{\alpha_3-1}\left[1+\alpha_3\log\left[H_B\left(x_{ki}\right)\right]\right], \quad k=1,2,\\
\mathrm{E}\left(x_{ki};\alpha_k\right) &= \left[H_B\left(x_{ki}\right)\right]^{\alpha_k-1}\log\left[H_B\left(x_{ki}\right)\right]\left[2+\log\left[H_B\left(x_{ki}\right)\right]\right], \quad k=1,2,\\
\mathrm{G}\left(x_{ki};\alpha_k\right) &= \left[H_B\left(x_{ki}\right)\right]^{\alpha_3-1}\log\left[H_B\left(x_{ki}\right)\right]\left[2+\log\left[H_B\left(x_{ki}\right)\right]\right], \quad k=1,2,\\
\eta\left(x_{ki};\alpha_k,\alpha_3\right) &= \frac{\mathrm{A}\left(x_{ki};\alpha_k,\alpha_3\right).\mathrm{E}\left(x_{ki};\alpha_k\right)-\left[\mathrm{B}\left(x_{ki};\alpha_k\right)\right]^2}{\left[\mathrm{A}\left(x_{ki};\alpha_k,\alpha_3\right)\right]^2}, \quad k=1,2,\\
\delta\left(x_{ki};\alpha_k,\alpha_3\right) &= \frac{\left[\mathrm{A}\left(x_{ki};\alpha_k,\alpha_3\right)\right]^2\mathrm{G}\left(x_{ki};\alpha_k\right)-\left[\mathrm{C}\left(x_{ki};\alpha_k\right)\right]^2}{\left[\mathrm{A}\left(x_{ki};\alpha_k,\alpha_3\right)\right]^2}, \quad k=1,2,\\
\epsilon\left(x_{ki};\alpha_k,\alpha_3\right) &= -\frac{\mathrm{B}\left(x_{ki};\alpha_k\right)\ \mathrm{C}\left(x_{ki};\alpha_k\right)}{\left[\mathrm{A}\left(x_{ki};\alpha_k,\alpha_3\right)\right]^2}, \quad k=1,2.
\end{aligned}$$

The asymptotic variance-covariance matrix of $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\alpha}_3$ is obtained by inverting the Fisher information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood estimates. Accordingly; the asymptotic variance-covariance matrix can be written as follows

$$F^{-1}=\left.\begin{bmatrix} I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{bmatrix}^{-1}\right|_{\alpha=\hat{\alpha},}$$

where $I_{ij}=-\left.\frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_i\partial\alpha_j}\right|_{\alpha=\hat{\alpha}.}$

Now, the asymptotic normality results will be stated to obtain the asymptotic confidence intervals ofα_1, $\alpha_2$and α_3. Under particular regularity conditions it can be stated as follow

$$\sqrt{n}\left[(\hat{\alpha}_1-\alpha_1),(\hat{\alpha}_2-\alpha_2),(\hat{\alpha}_3-\alpha_3)\right]\rightarrow N_3(0,F^{-1}) \quad \text{as} \quad n\rightarrow\infty,$$

where F^{-1} is the variance-covariance matrix, $\hat{\alpha}=(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3)$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3)$. Since α is unknown, then $F^{-1}(\alpha)$ is estimated by $F^{-1}(\hat{\alpha})$; the asymptotic variance-covariance matrix that defined above and this can be used to obtain the asymptotic confidence intervals of α_1,α_2 and α_3.

3.2 Bayesian Estimation

In this section, the Bayesian estimation for the BHPP models parameters is considered under the assumption that the random variables α_1, α_2 and α_3 are independently distributed with gamma prior distributions as follows

$$\pi_i\left(\alpha_i\right)=\frac{b_i^{a_i}}{\Gamma(a_i)}\alpha_i^{a_i-1}e^{-b_i\alpha_i}, \quad i=1,2,3, \quad \alpha_i>0.$$

The joint prior density of α_1, α_2 and α_3 can be written as

$$\pi_0\left(\Theta\right)=\prod_{i=1}^{3}\frac{b_i^{a_i}}{\Gamma(a_i)}\alpha_i^{a_i-1}\mathrm{e}^{-b_i\alpha_i}.$$

Hyper-Parameters Determination The hyper-parameters involved in priors can be easily evaluated, if the prior mean and variance are considered to be known. The prior mean and prior variance will be obtained from the maximum likelihood estimates of $(\alpha_1, \alpha_2, \alpha_3)$ by equating the mean and variance of $(\hat{\alpha}_1^j, \hat{\alpha}_2^j, \hat{\alpha}_3^j)$ with the mean and variance of the considered priors (gamma prior), where $j = 1, 2, \ldots, k$ and k is the number of random samples generated from the model. Thus, on equating the mean and variance of $(\hat{\alpha}_1^j, \hat{\alpha}_2^j, \hat{\alpha}_3^j)$ with the mean and variance of gamma priors, gets

$$\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}_1^j = \frac{a_1}{b_1} \quad \text{and} \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}_1^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}_1^j\right)^2 = \frac{a_1}{b_1^2}.$$

Now on solving the above two equations, the estimated hyper-parameters can be written as

$$a_1 = \frac{\left(\frac{1}{k}\sum\limits_{j=1}^{k}\hat{\alpha}_1^j\right)^2}{\frac{1}{k-1}\sum\limits_{j=1}^{k}\left(\hat{\alpha}_1^j - \frac{1}{k}\sum\limits_{j=1}^{k}\hat{\alpha}_1^j\right)^2} \quad \text{and} \quad b_1 = \frac{\frac{1}{k}\sum\limits_{j=1}^{k}\hat{\alpha}_1^j}{\frac{1}{k-1}\sum\limits_{j=1}^{k}\left(\hat{\alpha}_1^j - \frac{1}{k}\sum\limits_{j=1}^{k}\hat{\alpha}_1^j\right)^2}.$$

Similar procedure for determining the hyper parameters (a_2, b_2, a_3, b_3) can be used for α_2 and α_3. Since

$$f(D,\theta) = \pi_0(\theta)\, L(D \mid \theta) \quad \text{and} \quad f(D) = \int f(D \mid \theta) d\theta = \int \pi_0(\theta)\, L(D \mid \theta) d\theta,$$

hence, the joint posterior density function of $\theta = (\alpha_1, \alpha_2, \alpha_3)$ given the data D, denoted by $\pi_1(\theta \mid D)$ can be written as

$$\pi_1(\theta \mid D) = \frac{\pi_0(\theta)\, L(D \mid \theta)}{\int \pi_0(\theta)\, L(D \mid \theta) d\theta},$$

where $D = \{(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})\}$ and $L(D \mid \theta)$ is the likelihood function.

Therefore, the Bayes estimates of the unknown parameters $\theta = (\alpha_1, \alpha_2, \alpha_3)$ under square error loss function (SEL) can be calculated through the following equations as follows

$$\tilde{\theta}_i = \mathrm{E}(\theta_i \mid D) = \int\int\int \theta_i \pi_1(\theta_i \mid D) d\theta_1 d\theta_2 d\theta_3. \tag{29}$$

Obviously, the *3* integrals given by (29) cannot be obtained in a closed form. In this case, the MCMC technique to generate samples from the posterior distributions is used and then compute the Bayes estimators for the individual parameters. A wide variety of MCMC schemes are available, and it can be difficult to choose among them. An important sub-class of MCMC methods is Gibbs sampling and more general Metropolis within Gibbs samplers. The advantage of using the MCMC method over the MLE method is that we can always obtain a reasonable interval estimate of the parameters by constructing the probability intervals based on empirical posterior distribution. To generate samples from the proposed family, the Metropolis-Hastings (M-H) method (Metropolis et al. (1953) with normal proposal distribution) is used.

Thus, the following steps of M-H algorithm are performed to draw samples from the posterior density and in turn compute the Bayes estimates (BEs) of $\theta = (\alpha_1, \alpha_2, \alpha_3)$.

Set initial values $\theta^{(0)}$ M = burn-in. For $i = 1, \ldots, N$ repeat the following steps:

1. Set $\theta = \theta^{(i-1)}$.

2. Generate a new candidate parameter values ω from $N_3(\log(\theta), S_\theta)$.
3. Set $\theta' = \exp(\omega)$.
4. Calculate $A = \min\left(1, \frac{\pi\left(\theta' \mid \underline{x}\right)}{\pi(\theta \mid \underline{x})}\right)$.
5. Update $\theta^{(i)} = \theta'$ with probability A; otherwise set $\theta^{(i)} = \theta$.

The approximate BEs of $\theta^{(i)} = \left(\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}\right)'$, $i = 1, \ldots, N$ with respect to squared error loss (SEL) function is given by

$$\tilde{\theta} = \frac{1}{N - M} \sum_{i=M+1}^{N} \theta^{(i)},$$

where $\tilde{\theta}$ is BEs under SEL and M is the burn-in-period (that is, a number of iterations before the stationary distribution is achieved).

4 A New Bivariate Distributions Belongs to BHPP Models: Case Studies

4.1 Bivariate Weibull Distribution

Using Equations (15)-(17) in Equations (26)-(28), a new bivariate Weibull distribution denoted by BW$(\alpha_1, \alpha_2, \alpha_3, \lambda)$ can be defined by the joint survival function

$$S_{\text{BW}}(x_1, x_2) = \exp\{-(\lambda x_1)^{\alpha_1}\} \exp\{-(\lambda x_2)^{\alpha_2}\} \exp\{-(\lambda x_3)^{\alpha_3}\},$$

where $x_3 = \max(x_1, x_2)$.

The joint survival function of BW model can be stretching in the following form

$$S_{\text{BW}}(x_1, x_2) = \begin{cases} S_1(x_1, x_2) & x_1 < x_2, \\ S_2(x_1, x_2) & x_1 > x_2, \\ S_3(x) & x_1 = x_2 = x. \end{cases}$$

where

$$\begin{aligned} S_1(x_1, x_2) &= \exp\{-(\lambda x_1)^{\alpha_1} - (\lambda x_2)^{\alpha_2} - (\lambda x_2)^{\alpha_3}\}, \\ S_2(x_1, x_2) &= \exp\{-(\lambda x_1)^{\alpha_1} - (\lambda x_1)^{\alpha_3} - (\lambda x_2)^{\alpha_2}\}, \\ S_3(x) &= \exp\{-(\lambda x)^{\alpha_1} - (\lambda x)^{\alpha_2} - (\lambda x)^{\alpha_3}\}. \end{aligned}$$

Accordingly, the joint pdf of BW model can be obtained as

$$f_{\text{BW}}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & x_1 < x_2, \\ f_2(x_1, x_2) & x_1 < x_2, \\ f_3(x) & x_1 = x_2 = x. \end{cases} \tag{30}$$

where

$$\begin{aligned} f_1(x_1, x_2) =& \alpha_1 \lambda^{\alpha_1} x_1^{\alpha_1 - 1} \alpha_2 \lambda^{\alpha_2} x_2^{\alpha_2 - 1} + \alpha_3 \lambda^{\alpha_3} x_2^{\alpha_3 - 1} S_1(x_1, x_2), \\ f_2(x_1, x_2) =& \alpha_2 \lambda^{\alpha_2} x_2^{\alpha_2 - 1} \alpha_1 \lambda^{\alpha_1} x_1^{\alpha_1 - 1} + \alpha_3 \lambda^{\alpha_3} x_1^{\alpha_3 - 1} S_2(x_1, x_2), \\ f_3(x) =& \alpha_3 \lambda^{\alpha_3} x^{\alpha_3 - 1} S_3(x). \end{aligned}$$

4.2 Bivariate Power Hazard Gompertz (BPHG) Distribution

Using Equations (18)-(19) in Equations (26)-(28), a new bivariate Gompertz distribution denoted by $BPHG(\alpha_1, \alpha_2, \alpha_3, \lambda, \xi)$ can be defined by the joint survival function

$$S_{BPHG}(x_1, x_2) = \exp\left\{-[\xi(e^{\lambda x_1}-1)]^{\alpha_1} - [\xi(e^{\lambda x_2}-1)]^{\alpha_2} - [\xi(e^{\lambda x_3}-1)]^{\alpha_3}\right\},$$

where $x_3 = \max(x_1, x_2)$.

Or, the joint survival function of BPHG model can be written as

$$S_{BPHG}(x_1, x_2) = \begin{cases} S_1(x_1, x_2) & x_1 < x_2, \\ S_2(x_1, x_2) & x_1 > x_2, \\ S_3(x) & x_1 = x_2 = x, \end{cases}$$

where

$$\begin{aligned} S_1(x_1, x_2) &= \exp\left\{-\left[\xi\left(e^{\lambda x_1}-1\right)\right]^{\alpha_1} - \left[\xi\left(e^{\lambda x_2}-1\right)\right]^{\alpha_2} - [\xi(e^{\lambda x_2}-1)]^{\alpha_3}\right\}, \\ S_2(x_1, x_2) &= \exp\left\{-\left[\xi\left(e^{\lambda x_1}-1\right)\right]^{\alpha_1} - \left[\xi\left(e^{\lambda x_2}-1\right)\right]^{\alpha_2} - [\xi(e^{\lambda x_1}-1)]^{\alpha_3}\right\}, \\ S_3(x) &= \exp\left\{-\left[\xi\left(e^{\lambda x}-1\right)\right]^{\alpha_1} - \left[\xi\left(e^{\lambda x}-1\right)\right]^{\alpha_2} - [\xi(e^{\lambda x}-1)]^{\alpha_3}\right\}. \end{aligned}$$

The joint pdf of BPHG model can be written as

$$f_{BPHG}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & x_1 < x_2, \\ f_2(x_1, x_2) & x_1 < x_2, \\ f_3(x) & x_1 = x_2 = x, \end{cases}$$

where

$$\begin{aligned} f_1(x_1, x_2) =& (\lambda\xi)^2 \alpha_1 \alpha_2 \mathrm{e}^{\lambda x_1} \mathrm{e}^{\lambda x_2} \left[\xi(\mathrm{e}^{\lambda x_1}-1)\right]^{\alpha_1-1} \left[\xi\left(\mathrm{e}^{\lambda x_2}-1\right)\right]^{\alpha_2-1} \\ &+ \alpha_3 \left[\xi(\mathrm{e}^{\lambda x_2}-1)\right]^{\alpha_3-1} S_1(x_1, x_2), \\ f_2(x_1, x_2) =& (\lambda\xi)^2 \alpha_2 \mathrm{e}^{\lambda x_1} \mathrm{e}^{\lambda x_2} [\xi(\mathrm{e}^{\lambda x_2}-1)]^{\alpha_2-1} \{\alpha_1 \left[\xi\left(\mathrm{e}^{\lambda x_1}-1\right)\right]^{\alpha_1-1} \\ &+ \alpha_3 [\xi(\mathrm{e}^{\lambda x_1}-1)]^{\alpha_3-1} S_2(x_1, x_2), \\ f_3(x) =& \lambda\xi\alpha_3 \mathrm{e}^{\lambda x} [\xi(\mathrm{e}^{\lambda x}-1)]^{\alpha_3-1} S_3(x_1, x_2). \end{aligned}$$

4.3 Bivariate Power Hazard Pareto (BPHP) Distribution

Using (20)-(22) in (26)-(28), a BPHP distribution denoted by $BPHP(\alpha_1, \alpha_2, \alpha_3, \lambda)$ can be introduced by the joint survival function

$$S_{BPHP}(x_1, x_2) = \exp\{-[\log(1+\lambda x_1)]^{\alpha_1}\} \exp\{-[\log(1+\lambda x_2)]^{\alpha_2}\} \exp\{-[\log(1+\lambda x_3)]^{\alpha_3}\},$$

where $x_3 = \max(x_1, x_2)$.

That is,

$$S_{BPHP}(x_1, x_2) = \begin{cases} S_1(x_1, x_2) & x_1 < x_2, \\ S_2(x_1, x_2) & x_1 > x_2, \\ S_3(x) & x_1 = x_2 = x, \end{cases}$$

where

$$\begin{aligned}
S_1(x_1,x_2) &= \exp\left\{-\left[\log(1+\lambda x_1)\right]^{\alpha_1} - \left[\log(1+\lambda x_2)\right]^{\alpha_2} - \left[\log(1+\lambda x_2)\right]^{\alpha_3}\right\},\\
S_2(x_1,x_2) &= \exp\left\{-\left[\log(1+\lambda x_1)\right]^{\alpha_1} - \left[\log(1+\lambda x_1)\right]^{\alpha_3} - \left[\log(1+\lambda x_2)\right]^{\alpha_2}\right\},\\
S_3(x) &= \exp\left\{-\left[\log(1+\lambda x)\right]^{\alpha_1} - \left[\log(1+\lambda x)\right]^{\alpha_2} - \left[\log(1+\lambda x)\right]^{\alpha_3}\right\}.
\end{aligned}$$

The joint pdf of BHPP model can be written as

$$f_{\text{BHPP}}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & x_1 < x_2,\\ f_2(x_1,x_2) & x_1 < x_2,\\ f_3(x) & x_1 = x_2 = x,\end{cases}$$

where

$$\begin{aligned}
f_1(x_1,x_2) =& \frac{\lambda^2\alpha_1\alpha_2}{(1+\lambda x_1)(1+\lambda x_2)}\left[\log(1+\lambda x_1)\right]^{\alpha_1-1}\left[\log(1+\lambda x_2)\right]^{\alpha_2-1}\\
&+\alpha_3\left[\log(1+\lambda x_2)\right]^{\alpha_3-1}S_1(x_1,x_2),\\
f_2(x_1,x_2) =& \frac{\lambda^2\alpha_1\alpha_2}{(1+\lambda x_1)(1+\lambda x_2)}\left[\log(1+\lambda x_2)\right]^{\alpha_2-1}\left[\log(1+\lambda x_1)\right]^{\alpha_1-1}\\
&+\alpha_3\left[\log(1+\lambda x_1)\right]^{\alpha_3-1}S_2(x_1,x_2),\\
f_3(x) =& \frac{\lambda\alpha_3}{(1+\lambda x)}\left[\log(1+\lambda x)\right]^{\alpha_3-1}S_3(x).
\end{aligned}$$

4.4 Bivariate Power Hazard Uniform (BPHU) Distribution

Using (23)-(25) in (26)-(28), a BPHU distribution denoted by $BPHU(\alpha_1,\alpha_2,\alpha_3)$ can be introduced by the joint survival function

$$S_{BPHU}(x_1,x_2) = \exp\left\{-[-\log(1-x_1)]^{\alpha_1} - [-\log(1-x_2)]^{\alpha_2} - [-\log(1-x_3)]^{\alpha_3}\right\},$$

where $x_3 = \max(x_1,x_2)$,

$$S_{BPHU}(x_1,x_2) = \begin{cases} S_1(x_1,x_2), & x_1 < x_2,\\ S_2(x_1,x_2), & x_1 > x_2,\\ S_3(x), & x_1 = x_2 = x,\end{cases}$$

where

$$\begin{aligned}
S_1(x_1,x_2) &= \exp\left\{-\left[-\log(1-x_1)\right]^{\alpha_1} - \left[-\log(1-x_2)\right]^{\alpha_2} - \left[-\log(1-x_2)\right]^{\alpha_3}\right\},\\
S_2(x_1,x_2) &= \exp\left\{-\left[-\log(1-x_1)\right]^{\alpha_1} - \left[-\log(1-x_2)\right]^{\alpha_2} - \left[-\log(1-x_1)\right]^{\alpha_3}\right\},\\
S_3(x) &= \exp\left\{-\left[-\log(1-x)\right]^{\alpha_1} - \left[-\log(1-x)\right]^{\alpha_2} - \left[-\log(1-x)\right]^{\alpha_3}\right\}.
\end{aligned}$$

The joint pdf of BPHU model can be written as

$$f_{BPHU}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & x_1 < x_2,\\ f_2(x_1,x_2) & x_1 < x_2,\\ f_3(x) & x_1 = x_2 = x,\end{cases}$$

where

$$f_1(x_1, x_2) = \frac{\alpha_1\alpha_2}{(1-x_1)(1-x_2)}\left[-\log(1-x_1)\right]^{\alpha_1-1}\left[-\log(1-x_2)\right]^{\alpha_2-1} + \alpha_3[-\log(1-x_2)]^{\alpha_3-1}S_1(x_1, x_2),$$

$$f_2(x_1, x_2) = \frac{\alpha_1\alpha_2}{(1-x_1)(1-x_2)}\left[-\log(1-x_1)\right]^{\alpha_1-1}\left[-\log(1-x_1)\right]^{\alpha_1-1} + \alpha_3[-\log(1-x_1)]^{\alpha_3-1}S_2(x_1, x_2),$$

$$f_3(x) = \frac{\alpha_3}{(1-x)}\left[-\log(1-x)\right]^{\alpha_3-1}S_3(x).$$

5 Application of BHPP Models to Real Data Set

To see how the BHPP models work in practices, one data set will be reanalyzed in this section. The data set has been obtained from Meintanis (2007). The data represent the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. In this case all possibilities are open, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

These data were analyzed by Meintanis (2007), who considered the Marshall-Olkin bivariate exponential distribution, and by many authors such as Kundu and Dey (2009), Kundu and Gupta (2009), Muhammed (2016), Muhammed (2017), Muhammed (2019), and Muhammed (2020). Here, these data will be fitted to three BHPP models namely: (i) bivariate Weibull (BW), (ii) bivariate hazard power parameter Gompertz (BHPG) and (iii) bivariate hazard power parameter Pareto (BHPP) distributions. Note that both BW and BHPP are four parameters models but BHPG is a five parameter model.The main aim is to see, how the different BHPP models and the MLE works in practice.

Before trying to analyze the data using the BHPP models, first fit the marginals (UHPP) models to X_1, X_2 separately. The UHPP models are (i) univariate Weibull(UW), (ii) univariate hazard power parameter Gompertz (UHPG) and (iii) univariate hazard power parameter Pareto (UHPP). Table 1 shows the MLEs, the Kolmogorov-Smirnov ((KS) distances between the fitted distribution and the empirical distribution function for X_1 and X_2 with correspondence p-value, the Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). That gives an indication that the BHPP models may be used to analyze this data set.

Now, the data will fitted to the three BHPP models defined above, the MLEs and the standard error (SE) will be calculated for each bivariate model. To compare these model with each other or with any other bivariate models that represent this data the AIC, BIC, HQIC and CAIC are calculated for the three PHPP models as shown in Table 2. The BW model provides a better fit than the other tested models, because it has the smallest value among AIC, BIC, HQIC and CAIC and standard error.

6 Simulation Study

In this section, the results of a Monte Carlo simulation study testing the performance of MLE and Bayesian estimation for the BHPP model Parameters will be introduced in general and

Table 1: MLEs, KS, AIC, BIC, HQIC and CAIC for UHPP Models for the football data.

UHPP Model		α	λ	ξ	K-S	p-value	AIC	BIC	HQIC	CAIC
UW	X_1	2.124	0.022	-	0.084	0.957	329.7	332.9	330.8	330.1
	X_2	1.421	0.028	-	0.106	0.804	330.5	333.7	331.6	330.8
UHPG	X_1	1.130	0.035	0.227	0.101	0.841	329.9	334.7	331.6	330.6
	X_2	1.065	0.0184	0.975	0.097	0.880	331.1	335.9	332.8	331.8
UHPP	X_1	3.157	0.038	-	0.097	0.879	333.2	336.4	334.3	333.5
	X_2	2.100	0.050	-	0.128	0.579	333.8	337.0	334.9	334.1

Table 2: MLEs, SE, AIC, BIC, HQIC and CAIC for the BHPP Models for the Football Data.

BHPP Model	Estimates (Standard Error)					AIC	BIC	HQIC	CAIC
	α_1	α_2	α_3	λ	ξ				
BW	0.680	0.765	1.603	0.021	-	582.4	588.8	584.7	583.6
	(0.235)	(0.155)	(0.310)	(0.001)	-				
BHPG	0.575	0.658	1.281	0.010	1.637	583.7	591.8	586.6	585.7
	(0.221)	(0.169)	(0.385)	(0.010)	(2.201)				
BHPP	0.971	1.050	2.455	0.037	-	587.7	595.7	590.5	589.6
	(0.220)	(0.217)	(0.455)	(0.003)	-				

especially for BW model which defined by Equation (30) and denoted by BW$(\alpha_1, \alpha_2, \alpha_3, \lambda)$ and belongs to the BHPP models

The evaluation of the MLE and the Bayes estimation was performed based on the following quantities for each sample size: the Average Estimates (AE), the Mean Squared Error (MSE), Bias and confidence interval length (CL) are estimated from $R = 10000$ replications for $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ and $\hat{\lambda}$ the sample size has been considered at $n = 20, 30, 40, 50, 70$ and 100, and some values for the parameters $\alpha_1, \alpha_2, \alpha_3$ and λ have been considered.

The algorithm to generate from BHPP Models goes as follows.

Step 1. Generate U_1, U_2 and U_3 from U$(0, 1)$.

Step 2. Compute

$$Z_1 = H_B^{-1}([-\log U_1]^{1/\alpha_1}), \quad Z_2 = H_B^{-1}([-\log U_2]^{1/\alpha_2}) \quad \text{and} \quad Z_3 = H_B^{-1}([-\log U_3]^{1/\alpha_3}).$$

Step 3. Obtain $X_1 = \min(Z_1, Z_3)$ and $X_2 = \min(Z_2, Z_3)$.

Step 4. Define the indicator functions as

$$\delta_{1i} = \begin{cases} 1 & x_{1i} < x_{1i}, \\ 0 & \text{otherwise}, \end{cases} \quad \delta_{2i} = \begin{cases} 1 & x_{1i} > x_{1i}, \\ 0 & \text{otherwise}, \end{cases} \quad \delta_{3i} = \begin{cases} 1 & x_{1i} = x_{1i}, \\ 0 & \text{otherwise}, \end{cases}$$

Step 5. The corresponding sample size n must satisfy $n = n_1 + n_2 + n_3$ such that

$$n_1 = \sum_{i=1}^{n} \delta_{1i}, \quad n_2 = \sum_{i=1}^{n} \delta_{2i} \quad \text{and} \quad n_3 = \sum_{i=1}^{n} \delta_{3i}.$$

For different choices of sample sizes, 10000 data sets were generated. The results are summarized in Table 3 and 4. The estimates are work well and MSE and RAB decreases as the sample

Table 3: MLE and Bayes estimates for BW model parameters with $\alpha_1 = 1.75$, $\alpha_2 = 1.5$, $\alpha_3 = 2.5$, $\lambda = 1.5$.

		MLE				Bayesian Esitimation			
n		AE	Bias	MSE	CL	AE	Bias	MSE	CL
20	α_1	1.300	−0.450	0.394	1.712	1.532	−0.219	0.185	1.451
	α_2	1.544	0.044	0.265	2.012	1.380	−0.120	0.116	1.251
	α_3	1.981	−0.519	0.635	2.370	2.238	−0.263	0.267	1.746
	λ	1.073	−0.427	0.188	0.312	1.139	−0.361	0.185	0.917
30	α_1	1.255	−0.496	0.371	1.389	1.409	−0.341	0.233	1.340
	α_2	1.515	0.015	0.174	1.632	1.387	−0.113	0.107	1.203
	α_3	1.894	−0.606	0.601	1.891	2.160	−0.341	0.303	1.696
	λ	1.077	−0.423	0.182	0.198	1.144	−0.356	0.180	0.905
40	α_1	1.240	−0.510	0.346	1.149	1.393	−0.357	0.211	1.132
	α_2	1.487	−0.013	0.112	1.310	1.371	−0.129	0.083	1.013
	α_3	1.898	−0.602	0.533	1.621	2.129	−0.371	0.232	1.203
	λ	1.086	−0.414	0.175	0.251	1.140	−0.360	0.172	0.810
50	α_1	1.209	−0.542	0.363	1.036	1.441	−0.309	0.169	1.061
	α_2	1.518	0.018	0.098	1.224	1.421	−0.079	0.066	0.960
	α_3	1.792	−0.708	0.633	1.420	2.238	−0.262	0.175	1.280
	λ	1.086	−0.404	0.174	0.413	1.143	−0.357	0.170	0.813
70	α_1	1.204	−0.546	0.341	0.805	1.470	−0.280	0.115	0.746
	α_2	1.484	−0.016	0.068	1.018	1.413	−0.087	0.041	0.719
	α_3	1.767	−0.733	0.627	1.176	2.214	−0.286	0.135	0.905
	λ	1.106	−0.394	0.168	0.435	1.158	−0.343	0.161	0.816
100	α_1	1.199	−0.551	0.334	0.687	1.530	−0.220	0.071	0.591
	α_2	1.495	−0.005	0.047	0.845	1.422	−0.078	0.028	0.576
	α_3	1.746	−0.754	0.629	0.967	2.275	−0.225	0.071	0.565
	λ	1.145	−0.355	0.152	0.635	1.176	−0.324	0.145	0.787

size increases. For increasing sample size the MSEs of the considered parameters decreases. As expected, for small sample sizes, the results corresponding to Bayesian procedure are better than those corresponding to non- Bayesian procedure in the sense of MSE, bias and CL.

7 Other Bivariate Semiparametric Families of Distributions

7.1 Bivariate Power Parameter Family (BPP)

Assuming that U_1, U_2 and U_3 are mutually independent random variables such that $U_1 \sim \text{UPP}(\alpha_1)$, $U_2 \sim \text{UPP}(\alpha_2)$ and $U_3 \sim \text{UPP}(\alpha_3)$. Define $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$ then by using Equations (8) and (9), the bivariate Power parameter family of distributions denoted by $\text{BPP}(\alpha_1, \alpha_2, \alpha_3)$ is defined by the joint cdf as follows

$$F_{\text{BPP}}\,(x_1, x_2) = F_B\,(x_1^{\alpha_1})\; F_B\,(x_2^{\alpha_2})\, F_B(x_3^{\alpha_3}),$$

where $x_3 = \min(x_1, x_2)$.

Table 4: MLE and Bayes estimates for BW model parameters with $\alpha_1 = 1.75$, $\alpha_2 = 2$, $\alpha_3 = 2.5$, $\lambda = 3$.

		MLE				Bayesian Esitimation			
n		AE	Bias	MSE	CL	AE	Bias	MSE	CL
20	α_1	1.640	−0.110	0.347	2.270	1.616	−0.134	0.130	1.315
	α_2	1.413	-0.587	0.544	1.752	1.697	-0.303	0.251	1.567
	α_3	2.316	-0.184	0.547	2.809	2.290	-0.210	0.222	1.655
	λ	2.209	-0.792	0.652	0.621	2.246	-0.754	0.588	0.545
30	α_1	1.602	-0.148	0.195	1.630	1.647	-0.103	0.077	1.010
	α_2	1.372	-0.628	0.521	1.398	1.761	-0.239	0.158	1.247
	α_3	2.231	-0.269	0.358	2.096	2.252	-0.248	0.153	1.185
	λ	2.226	-0.774	0.648	0.862	2.262	-0.738	0.586	0.796
40	α_1	1.598	-0.152	0.151	1.401	1.653	-0.097	0.054	0.827
	α_2	1.362	-0.638	0.506	1.231	1.732	-0.268	0.136	0.991
	α_3	2.151	-0.349	0.323	1.756	2.214	-0.286	0.143	0.967
	λ	2.151	-0.349	0.323	1.756	2.214	-0.286	0.143	0.967
50	α_1	2.151	-0.349	0.323	1.756	2.214	-0.286	0.143	0.967
	α_2	1.341	-0.659	0.513	1.098	1.716	-0.285	0.119	0.761
	α_3	2.131	-0.369	0.297	1.571	2.260	-0.241	0.093	0.738
	λ	2.276	-0.724	0.629	1.269	2.314	-0.686	0.571	1.247
70	α_1	1.575	-0.176	0.102	1.044	1.632	-0.118	0.049	0.731
	α_2	1.337	-0.663	0.491	0.893	1.718	-0.282	0.111	0.699
	α_3	2.010	-0.400	0.273	1.319	2.280	-0.220	0.093	0.823
	λ	2.342	-0.659	0.609	1.641	2.362	-0.638	0.503	1.214
100	α_1	1.553	-0.197	0.091	0.897	1.623	-0.127	0.041	0.620
	α_2	1.314	-0.686	0.474	0.231	1.717	-0.283	0.083	0.209
	α_3	2.072	-0.428	0.264	1.116	2.289	-0.211	0.091	0.847
	λ	2.394	-0.606	0.533	1.594	2.382	-0.618	0.412	0.686

$$F_{\mathrm{BPP}}\left(x_1,x_2\right)=\begin{cases}F_1\left(x_1,x_2\right) & x_1<x_2,\\ F_2\left(x_1,x_2\right) & x_1>x_2,\\ F_3\left(x\right) & x_1=x_2=x,\end{cases}$$

where

$$\begin{aligned}F_1\left(x_1,x_2\right)=&F_B\left(x_1^{\alpha_1}\right)F_B\left(x_1^{\alpha_3}\right)F_B(x_2^{\alpha_2}),\\ F_2\left(x_1,x_2\right)=&F_B\left(x_1^{\alpha_1}\right)F_B\left(x_2^{\alpha_2}\right)F_B(x_2^{\alpha_3}),\\ F_3\left(x\right)=&F_B\left(x^{\alpha_1}\right)F_B\left(x^{\alpha_2}\right)F_B\left(x^{\alpha_3}\right).\end{aligned}$$

7.2 Bivariate Frailty Parameter Family (BFP)

Assuming that U_1, U_2 and U_3 are mutually independent random variables such that $U_1 \sim \mathrm{UFP}(\alpha_1)$, $U_2 \sim \mathrm{UFP}(\alpha_2)$ and $U_3 \sim \mathrm{UFP}(\alpha_3)$. Define $X_1 = \min(U_1, U_3)$ and $X_2 = \min(U_2, U_3)$ then by using Equations (11) and (12), the bivariate frailty parameter family of distributions (or bivariate proportional hazard models) denoted by BFP$(\alpha_1, \alpha_2, \alpha_3)$ is defined by the joint survival and density functions respectively, as follows

$$S_{\mathrm{BFP}}(x_1, x_2) = [S_B(x_1)]^{\alpha_1}[S_B(x_2)]^{\alpha_2}[S_B(x_3)]^{\alpha_3},$$

such that $x_3 = \max(x_1, x_2)$, or

$$S_{\mathrm{BFP}}(x_1, x_2) = \exp\{-\alpha_1 H_B(x_1) - \alpha_2 H_B(x_2) - \alpha_3 H_B(x_3)\},$$

and

$$f_{\mathrm{BFP}}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & x_1 < x_2, \\ f_2(x_1, x_2) & x_1 > x_2, \\ f_3(x) & x_1 = x_2 = x, \end{cases}$$

where

$$\begin{aligned} f_1(x_1, x_2) =& \alpha_1(\alpha_2 + \alpha_3) f_B(x_1) f_B(x_2) [S_B(x_1)]^{\alpha_1 - 1}[S_B(x_1)]^{\alpha_2 + \alpha_3 - 1}, \\ f_2(x_1, x_2) =& (\alpha_1 + \alpha_3)\alpha_2 f_B(x_1) f_B(x_2) [S_B(x_1)]^{\alpha_1 + \alpha_3 - 1}[S_B(x_1)]^{\alpha_2 - 1}, \\ f_3(x) =& \alpha_3 f_B(x) [S_B(x_1)]^{\alpha_1 + \alpha_2 + \alpha_3 - 1}. \end{aligned}$$

And it will be discussed in details in a separate paper.

7.3 Bivariate Resilience Parameter Family (BRP)

Based on the same bivariate idea in the previous sections and by using maximization process a bivariate resilience parameter family of distributions is introduced by Kundu and Gupta (2010) with the joint cdf and pdf respectively, as follows

$$F_{BRP}(x_1, x_2) = [F_B(x_1)]^{\alpha_1}[F_B(x_2)]^{\alpha_2}[F_B(x_3)]^{\alpha_3} \quad \text{such that} \quad x_3 = \min(x_1, x_2),$$

$$f_{BRP}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2, \\ f_2(x_1, x_2), & x_1 > x_2, \\ f_3(x), & x_1 = x_2 = x, \end{cases}$$

where

$$\begin{aligned} f_1(x_1, x_2) =& (\alpha_1 + \alpha_3)\alpha_2 f_B(x_1) f_B(x_2) [F_B(x_1)]^{\alpha_1 + \alpha_3 - 1}[F_B(x_2)]^{\alpha_2 - 1}, \\ f_2(x_1, x_2) =& \alpha_1(\alpha_2 + \alpha_3) f_B(x_1) f_B(x_2) [F_B(x_1)]^{\alpha_1 - 1}[F_B(x_2)]^{\alpha_2 + \alpha_3 - 1}, \\ f_3(x) =& \alpha_3 f_B(x) [F_B(x)]^{\alpha_1 + \alpha_2 + \alpha_3 - 1}. \end{aligned}$$

And they called this family as bivariate proportional reversed hazard family of distribution. Muhammed (2013) discussed some properties of this family and estimated the unknown parameters of some distributions belong to this family under different censoring schemes.

7.4 Bivariate Reversed Hazard Power Parameter Family (BRPP)

Assume $U_1 \sim \text{URPP}(\alpha_1)$, $U_2 \sim \text{URPP}(\alpha_2)$ and $U_3 \sim \text{URPP}(\alpha_3)$ and $U's$ are independent random variables. Let $X_1 = \max(U_1, U_3)$ and $X_2 = \max(U_2, U_3)$.

Then, (X_1, X_2) constitute a BRPP class of distribution with the following cdf and pdf

$$F_{\text{BRPP}}(x_1, x_2) = \exp\{[R_B(x_1)]^{\alpha_1}\} \exp\{[R_B(x_2)]^{\alpha_2}\} \exp\{[R_B(x_3)]^{\alpha_3}\},$$

where $x_3 = \min(x_1, x_2)$.

The joint cdf of BHPP model can be stretching in the following form

$$F_{\text{BRPP}}(x_1, x_2) = \begin{cases} F_1(x_1, x_2) & x_1 < x_2, \\ F_2(x_1, x_2) & x_1 > x_2, \\ F_3(x) & x_1 = x_2 = x, \end{cases}$$

$$\begin{aligned} F_1(x_1, x_2) &= \exp\{-[R_B(x_1)]^{\alpha_{13}} - [R_B(x_2)]^{\alpha_2}\}, \\ F_2(x_1, x_2) &= \exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_{23}}\}, \\ F_3(x) &= \exp\{-[R_B(x)]^{\alpha_{123}}\}, \end{aligned}$$

where

$$\begin{aligned} [R_B(x_i)]^{\alpha_{i3}} &= [R_B(x_i)]^{\alpha_i} + [R_B(x_i)]^{\alpha_3}, \quad i = 1, 2, \\ [R_B(x)]^{\alpha_{123}} &= [R_B(x)]^{\alpha_1} + [R_B(x)]^{\alpha_2} + [R_B(x)]^{\alpha_3}. \end{aligned}$$

And it will be considered in details in a future work.

8 Conclusion

In this paper, a review of some univariate semi parametric families of distributions such as power parameter, frailty parameter, resilience parameter, hazard power parameter and reversed hazard power parameter is discussed in details. Moreover, proposed bivariate extensions for these families are introduced based on Marshal-Olkin idea. The bivariate hazard power parameter family is discussed with its main properties and the MLE and Bayesian estimation are also considered for the shape parameters. A simulation study and a real data set are considered. As a future work, other bivariate extensions for these families will be introduced based on other copulas soon as possible.

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