

The Odd Inverse Pareto-Weibull Distribution

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Abstract

The odd inverse Pareto-Weibull distribution is introduced as a new lifetime distribution based on the inverse Pareto and the T-X family. Some mathematical properties of the new distribution are studied. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is derived. The importance and flexibility of the proposed model are assessed using a real data.

Keywords *inverse Pareto distribution; moments; maximum likelihood; odd inverse Pareto; order statistics*

1 Introduction

In statistical inference, parametric modeling of data is the main source of obtaining a new family of distributions in hope of adding flexibility for existing ones. So many authors introduced different new families which are modification, generalization or extension of old ones. Some of well-known families are: Weibull-G (Bourguignon et al., 2014), gamma-G (Zografos and Balakrishnan, 2009), beta-G (Eugene et al., 2002), logistic-G (Torabi and Hedesh, 2012), Kumaraswamy-G (Cordeiro et al., 2017), Lomax-G (Cordeiro et al., 2014), odd Lomax-G (Cordeiro et al., 2019) and exponentiated Weibull-H (Cordeiro et al., 2017) families.

Recently, Aldahlan et al. (2019) introduced a new family of continuous distributions called the odd inverse Pareto-G (OIP-G) family which extends the exponentiated-G (Exp-G) family due to Gupta et al. (1998) and the Marshall-Olkin-G (Mo-G) family due to Marshall and Olkin (1997). The new family is constructed based on the inverse Pareto distribution and the T-X family (Alzaatreh et al., 2013). The cumulative distribution function (cdf) and probability density function (pdf) of OIP-G family are defined by :

$$F(x; \alpha, \beta, \varphi) = \frac{G(x; \varphi)^\alpha}{[1 - G(x; \varphi)]^\alpha} \left[\beta + \frac{G(x; \varphi)}{1 - G(x; \varphi)} \right]^{-\alpha}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0, \quad (1)$$

and

$$f(x; \alpha, \beta, \varphi) = \alpha \beta g(x; \varphi) G(x; \varphi)^{\alpha-1} [\beta + (1 - \beta) G(x; \varphi)]^{-\alpha-1}, \quad (2)$$

where α and β are two additional shape parameters, $G(x; \phi)$ is the baseline cdf with parameter vector ϕ and $g(x; \phi) = dG(x; \phi)/dx$.

In this paper, we define a new lifetime distribution called the OIP-W distribution. We hope that it will provide consistently better fits than other generated distributions under the same underlying distribution. This paper is organized as follows.

In Section 2, we study the OIP-W distribution. Statistical properties are calculated in Section 3 and 4. Maximum likelihood estimation of the parameters is determined in Section 5. An application of OIP-W distribution for a real data set is performed in Section 6. Finally, conclusion is appeared in Section 7.

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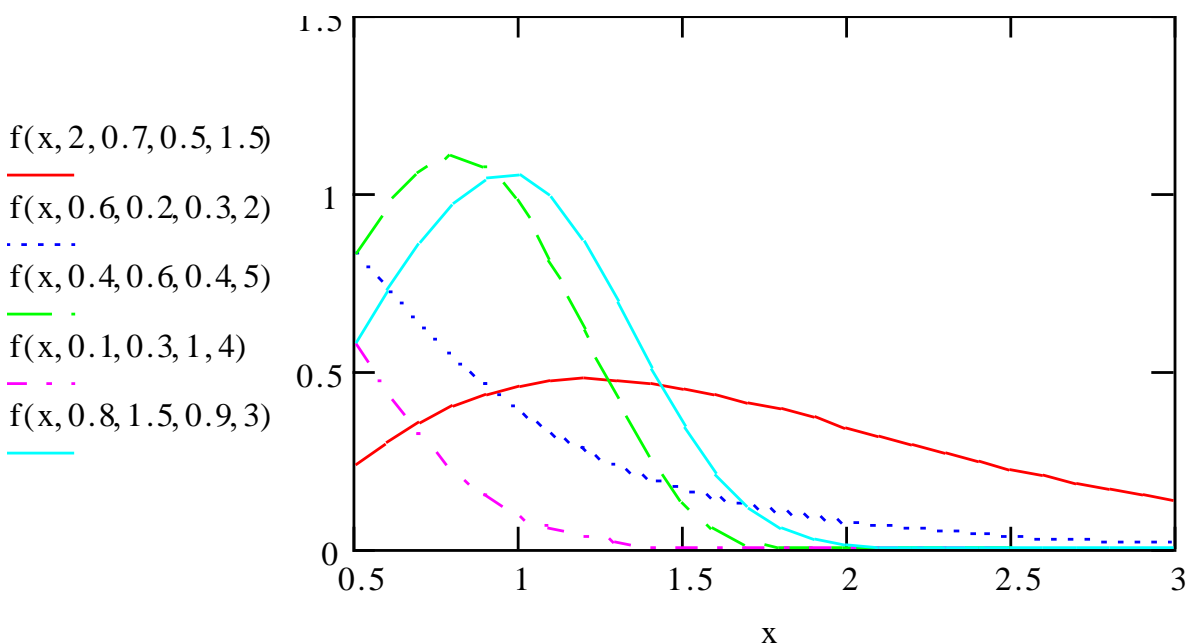


Figure 1: The pdfs of OIP-W distribution with different parameters.

2 The New Distribution

The pdf and cdf of weibull (W) distribution with scale parameter $\theta > 0$ and shape parameter $\gamma > 0$ are given by $g(x; \theta, \gamma) = \theta\gamma x^{\gamma-1}e^{-(\theta x^\gamma)}$ and $G(x; \theta, \gamma) = 1 - e^{-(\theta x^\gamma)}$, respectively. Then, the OIP-W cdf with a set of parameters $\zeta = (\alpha, \beta, \theta, \gamma)$ follows from (1) given by:

$$F(x; \alpha, \beta, \theta, \gamma) = \left[1 - e^{-\theta x^\gamma}\right]^\alpha \left[\beta + (1 - \beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha}, \quad x > 0, \alpha > 0, \beta > 0, \theta > 0, \gamma > 0, \tag{3}$$

The corresponding OIP-G pdf follows from (2) is given by

$$f(x; \alpha, \beta, \theta, \gamma) = \alpha\beta\theta\gamma x^{\gamma-1}e^{-\theta x^\gamma} \left(1 - e^{-\theta x^\gamma}\right)^{\alpha-1} \left[\beta + (1 - \beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha-1}. \tag{4}$$

Also, the survival function and hazard function are given respectively as follows:

$$S(x; \alpha, \beta, \theta, \gamma) = 1 - \left(\left[1 - e^{-\theta x^\gamma}\right]^\alpha \left[\beta + (1 - \beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha}\right),$$

and

$$h(x; \alpha, \beta, \theta, \gamma) = \frac{\left[\alpha\beta\theta\gamma x^{\gamma-1}e^{-\theta x^\gamma} \left(1 - e^{-\theta x^\gamma}\right)^{\alpha-1} \left[\beta + (1 - \beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha-1}\right]}{1 - \left(\left[1 - e^{-\theta x^\gamma}\right]^\alpha \left[\beta + (1 - \beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha}\right)}.$$

Figure 1, 2 and 3 illustrate some possible shapes of the pdf, cdf and hazard function of OIP-W distribution for some values of the parameters α, β, θ and γ .

Note that the OIP-W distribution is very flexible model that approaches to different distributions when its parameters are changed. The OIP-W distribution contains following well known

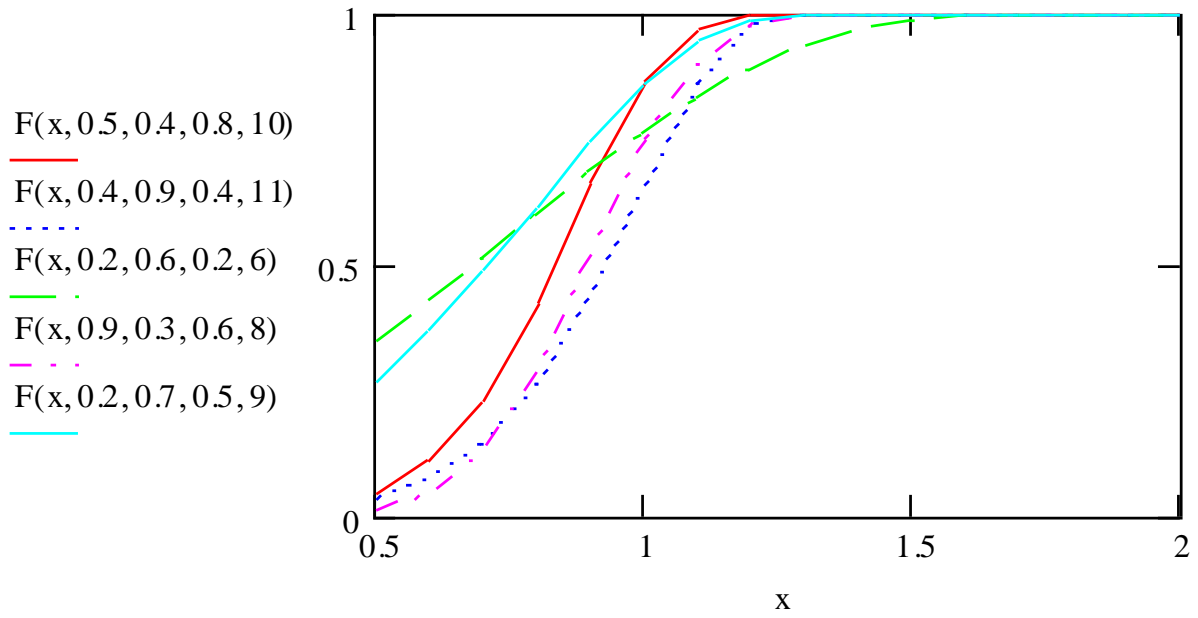


Figure 2: The cdfs of OIP-W distribution with different parameters.

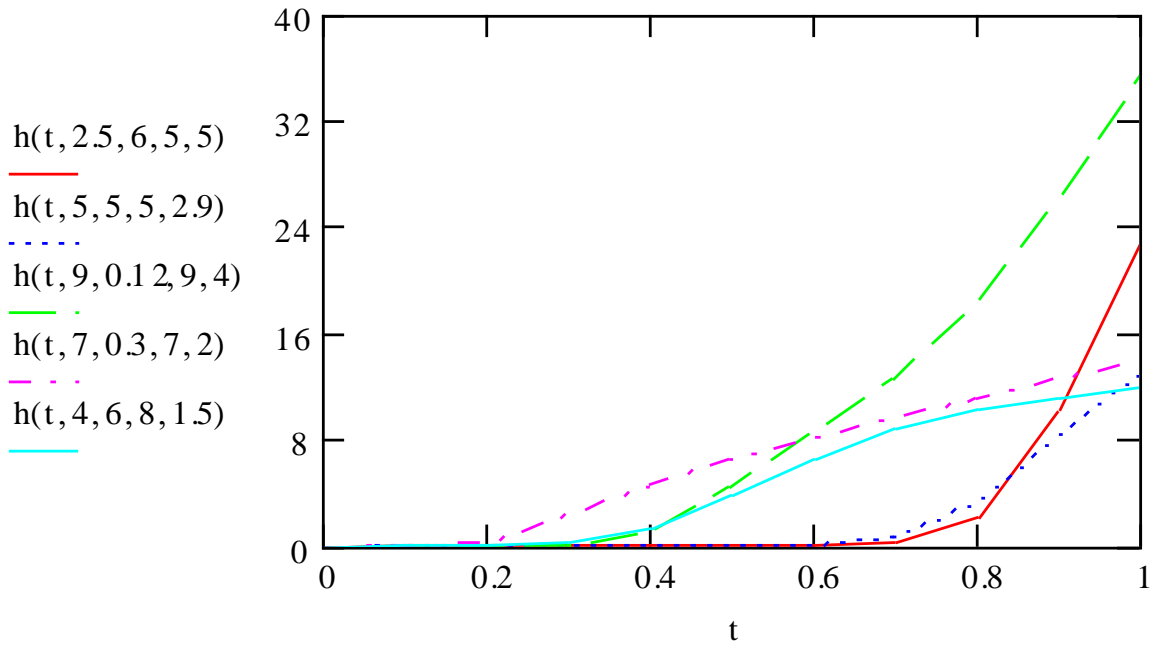


Figure 3: The hazard functions of OIP-W distribution with different parameters.

distributions as its special cases. In particular, when $\beta = 1$ the OIP-W distribution reduces to the exponentiated-weibull (EXP-W) distribution (Mudholkar and Srivastava, 1993). When $\alpha = 1$, The OIP-W distribution reduces to Marshall-Olkin-Weibull (MO-W) distribution. When $\beta = \gamma = 1$, then the resulting distribution is the exponentiated-exponential distribution. When $\alpha = \gamma = 1$, we get the Marshall-Olkin-exponential distribution. Clearly, when $\alpha = \beta = \gamma = 1$, we obtain the weibull distribution. Finally, when $\alpha = \beta = \gamma = 1$, then the resulting distribution is the exponential distribution.

3 Statistical Properties

In this section, some statistical properties for the OIP-W distribution are obtained.

3.1 Useful Expansions

In this subsection, some useful expansions of pdf and cdf for OIP-W distribution are covered.

Firstly, we obtain an expansion for pdf of OIP-W distribution as a linear combination of weibull distribution. Using the power series

$$(1 + dz)^{-t} = \sum_{k=0}^{\infty} \binom{-t}{k} d^k z^k,$$

for the last term and inserting the resulting expansion in (4) gives:

$$f(x; \alpha, \beta, \theta, \gamma) = \alpha \theta \gamma x^{\gamma-1} e^{-\theta x^\gamma} \sum_{k=0}^{\infty} \binom{-\alpha-1}{k} \beta^{-\alpha-k} (1-\beta)^k \left(1 - e^{-\theta x^\gamma}\right)^{\alpha+k-1}. \tag{5}$$

For $|z| < 1$, the power series holds

$$(1 - z)^{-t} = \sum_{j=0}^{\infty} (-1)^j \binom{-t}{j} z^j. \tag{6}$$

Applying (6) to last term of (5), gives

$$f(x; \alpha, \beta, \theta, \gamma) = \sum_{k,j=0}^{\infty} \psi_{k,j} \theta \gamma x^{\gamma-1} e^{-\theta(j+1)x^\gamma}, \tag{7}$$

where

$$\psi_{k,j} = \alpha \sum_{k,j=0}^{\infty} (-1)^j \binom{-\alpha-1}{k} \binom{\alpha+k-1}{j} \beta^{-\alpha-k} (1-\beta)^k.$$

Secondly, an expansion of $[F(x)]^h$ is obtained as following, Now

$$[F(x)]^h = \left[1 - e^{-\theta x^\gamma}\right]^{\alpha h} \left[\beta + (1-\beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha h},$$

We can rewrite the last term as

$$\left[\beta + (1-\beta) \left(1 - e^{-\theta x^\gamma}\right)\right]^{-\alpha h} = \sum_{t=0}^{\infty} \binom{-\alpha h}{t} \beta^{-\alpha h-t} (1-\beta)^t \left(1 - e^{-\theta x^\gamma}\right)^t.$$

Then

$$[F(x)]^h = \sum_{t=0}^{\infty} \binom{-\alpha h}{t} \beta^{-\alpha h - t} (1 - \beta)^t (1 - e^{-\theta x^\gamma})^{\alpha h + t}.$$

Using binomial expansion, we get

$$[F(x)]^h = \sum_{t,l=0}^{\infty} \eta_{t,l} e^{-\theta l x^\gamma}, \quad (8)$$

where

$$\eta_{t,l} = \sum_{t,l=0}^{\infty} (-1)^l \binom{-\alpha h}{t} \binom{\alpha h + t}{l} \beta^{-\alpha h - t} (1 - \beta)^t.$$

3.2 Quantile and Median

The quantile function x_q of OIP-W distribution is

$$x_q = \left[\frac{-1}{\theta} \ln \left(1 - \left[\frac{\beta q^{\frac{1}{\alpha}}}{1 - (1 - \beta) q^{\frac{1}{\alpha}}} \right] \right) \right]^{\frac{1}{\gamma}}, \quad 0 \leq q \leq 1. \quad (9)$$

By putting $q = 0.5$ in formula (9) we can get the median of odd inverse Pareto-Weibull distribution.

3.3 Ordinary Moments

If X has the pdf (7), the r th moment of the OIP-W distribution can be calculated through the following relation.

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f(x) dx \quad (10)$$

Substituting (7) into (10), we obtain

$$\mu'_r = \sum_{k,j=0}^{\infty} \psi_{k,j} \theta^\gamma \int_0^{\infty} x^{r+\gamma-1} e^{-\theta(j+1)x^\gamma} dx.$$

Let $y = \theta(j+1)x^\gamma$, then the moments becomes

$$\mu'_r = \sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{\frac{r}{\gamma}} (j+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{r}{\gamma} + 1\right).$$

The moment generating function of the OIP-W distribution is obtained as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{k,j,r=0}^{\infty} \frac{t^r}{r!} \frac{\psi_{k,j}}{\theta^{\frac{r}{\gamma}} (j+1)^{\frac{r}{\gamma}+1}} \Gamma\left(\frac{r}{\gamma} + 1\right).$$

3.4 Incomplete and Conditional Moments

If X has the pdf (7), the s th incomplete moment of the OIP-W distribution can be calculated as follows:

$$m_s(z) = \int_0^z x^s f(x) dx = \sum_{k,j=0}^{\infty} \psi_{k,j} \theta^\gamma \int_0^z x^{s+\gamma-1} e^{-\theta(j+1)x^\gamma} dx,$$

Using lower incomplete gamma function, we get

$$m_s(z) = \sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{\frac{s}{\gamma}(j+1)^{\frac{s}{\gamma}+1}} v\left(\frac{s}{\gamma} + 1, \theta(j+1)z^\gamma\right)},$$

where $v(s, z) = \int_0^z x^{s-1} e^{-x} dx$ is the lower incomplete gamma function. Further, the conditional moments of the OIP-W distribution can be calculated from the following relation:

$$\Delta_s(z) = \int_z^\infty x^s f(x) dx = \sum_{k,j=0}^{\infty} \psi_{k,j} \theta^\gamma \int_z^\infty x^{s+\gamma-1} e^{-\theta(j+1)x^\gamma} dx.$$

Using upper incomplete gamma function, we get

$$\Delta_s(z) = \sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{\frac{s}{\gamma}(j+1)^{\frac{s}{\gamma}+1}} \Gamma\left(\frac{s}{\gamma} + 1, \theta(j+1)z^\gamma\right)},$$

where $\Gamma(s, z) = \int_z^\infty x^{s-1} e^{-x} dx$, is the upper incomplete gamma function.

3.5 Probability Weighted Moment

The probability weighted moment (PWM) can be obtained using the following relation:

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x) (F(x))^s dx. \tag{11}$$

The PWM of the OIP-W distribution is obtained by substituting (7) and (8) into (11), replacing h with s , leads to

$$\begin{aligned} \tau_{r,s} &= \int_0^\infty \sum_{k,j,t,l=0}^{\infty} \psi_{k,j} \eta_{t,l} \theta^\gamma x^{r+\gamma-1} e^{-\theta(j+l+1)x^\gamma} dx, \\ &= \sum_{k,j,t,l=0}^{\infty} \psi_{k,j} \eta_{t,l} \frac{1}{\theta^{r/\gamma(j+l+1)^{(r/\gamma)+1}} \Gamma\left(\frac{r}{\gamma} + 1\right)}. \end{aligned}$$

3.6 Inequality Measures

The first incomplete moment can be used to construct inequality measures called Lorenz and Bonferroni curves, which are most widely used in income and wealth distributions. The Lorenz and Bonferroni curves of the OIP-W distribution are obtained as follows:

$$L_F(x) = \frac{\int_0^z x f(x) dx}{E(X)} = \frac{\sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{1/\gamma(j+1)^{(1/\gamma)+1}} v\left(\frac{1}{\gamma} + 1, \theta(j+1)z^\gamma\right)}}{\sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{1/\gamma(j+1)^{(1/\gamma)+1}} \Gamma\left(\frac{1}{\gamma} + 1\right)}},$$

and

$$B_F(x) = \frac{\int_0^z xf(x)dx}{E(X)F(x)} = \frac{\sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{1/\gamma}(j+1)^{(1/\gamma)+1}} v\left(\frac{1}{\gamma} + 1, \theta(j+1)z^\gamma\right)}{\sum_{k,j=0}^{\infty} \frac{\psi_{k,j}}{\theta^{1/\gamma}(j+1)^{(1/\gamma)+1}} \Gamma\left(\frac{1}{\gamma} + 1\right) \left\{ \frac{(1-e^{-\theta x^\gamma})}{\beta+(1-\beta)(1-e^{-\theta x^\gamma})} \right\}^\alpha}.$$

3.7 Rènyi and q-Entropies

The Rènyi entropy of a random variable (r.v.) X represents a measure of variation of the uncertainty. The Rènyi entropy is defined by:

$$I_R(X) = \frac{1}{1-\lambda} \log \left\{ \int_{-\infty}^{\infty} f^\lambda(x) dx \right\}, \quad \lambda > 0, \quad \lambda \neq 1.$$

Therefore, the Rènyi entropy of a r.v. X which follows the OIP-W is given by:

$$I_R(X) = \frac{1}{1-\lambda} \log \left[\frac{\gamma^{\lambda-1} \theta^{\frac{\lambda}{\gamma} + \gamma - 2}}{(\lambda(j+1))^{\frac{\lambda}{\gamma}(\gamma-1) - \gamma + 2}} \sum_{k,j=0}^{\infty} \psi_{k,j} \Gamma\left(\frac{\lambda}{\gamma}(\gamma-1) - \gamma + 2\right) \right].$$

The q-entropy say, $H_q(X)$, is defined by:

$$H_q(X) = \frac{1}{q-1} \log \left\{ 1 - \int_{-\infty}^{\infty} f^q(x) dx \right\}, \quad q > 0, \quad q \neq 1.$$

For the OIP-W distribution, the q-entropy is defined as follows:

$$H_q(X) = \frac{1}{q-1} \log \left\{ 1 - \left[\frac{\gamma^{q-1} \theta^{\frac{q}{\gamma} + \gamma - 2}}{(q(j+1))^{\frac{q}{\gamma}(\gamma-1) - \gamma + 2}} \sum_{k,j=0}^{\infty} \psi_{k,j} \Gamma\left(\frac{q}{\gamma}(\gamma-1) - \gamma + 2\right) \right] \right\}$$

4 Order Statistic

Let X_1, X_2, \dots, X_n be a simple random sample of size n from OIP-W distribution and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding order statistics. The pdf of $X_{(r)}$ is given by

$$f_{(r)}(x) = \frac{1}{B(r, n-r+1)} f(x) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x)]^{i+r-1}. \quad (12)$$

where $F(x)$ and $f(x)$ are the cdf and pdf of the OIP-W distribution given by (3) and (4) respectively. $B(.,.)$ is the beta function. Based on equations (7) and (8), we can write

$$f(x)F(x)^{i+r-1} = \theta \gamma \sum_{k,j,t,l=0}^{\infty} \frac{(-1)^l (1-\beta)^t}{\beta^{\alpha(i+r-1)+t}} \binom{-\alpha(i+r-1)}{t} \binom{\alpha(i+r-1)+t}{l} \times \psi_{k,j} x^{\gamma-1} e^{-\theta(j+l+1)x^\gamma}. \quad (13)$$

Inserting (13) into (12), the pdf of $X_{(r)}$ could be written as

$$f_{(r)}(x) = \sum_{k,j,t,l=0}^{\infty} \omega_{k,j,t,l} x^{\gamma-1} e^{-\theta(j+l+1)x^\gamma},$$

where

$$\omega_{k,j,t,l} = \sum_{i=0}^{n-r} \frac{(-1)^{i+l} \theta \gamma (1-\beta)^t}{\beta^{\alpha(i+r-1)+t} B(r, n-r+1)} \binom{n-r}{i} \binom{-\alpha(i+r-1)}{t} \binom{\alpha(i+r-1)+t}{l} \psi_{k,j}.$$

5 Maximum Likelihood Estimation

In this section, we discuss the estimation of the unknown parameters of the OIP-W distribution from complete sample by using the method of maximum likelihood. Let X_1, X_2, \dots, X_n be a random sample of size n from OIP-W (ζ), where $\zeta = (\alpha, \beta, \theta, \gamma)$. The log-likelihood function L is given by:

$$L = n \ln \alpha + n \ln \beta + n \ln \theta + n \ln \gamma + (\gamma - 1) \sum_{i=1}^n \ln (x_i) - \theta \sum_{i=1}^n (x_i^\gamma) + (\alpha - 1) \sum_{i=1}^n \ln (1 - e^{-\theta x_i^\gamma}) - (\alpha + 1) \sum_{i=1}^n \ln [\beta + (1 - \beta)e^{-\theta x_i^\gamma}].$$

The maximum likelihood estimates (MLEs) of the parameters are obtained by Differentiating the log-likelihood function L with respect to the parameters α, β, θ and γ and setting the results to zero.

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln \phi (x_i, \theta, \gamma) - \sum_{i=1}^n \ln [\beta + (1 - \beta)\phi (x_i, \theta, \gamma)] = 0, \tag{14}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} - (\alpha + 1) \sum_{i=1}^n \left[\frac{e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi (x_i, \theta, \gamma)]} \right] = 0, \tag{15}$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i^\gamma + (\alpha - 1) \sum_{i=1}^n \left[\frac{x_i^\gamma e^{-\theta x_i^\gamma}}{\phi (x_i, \theta, \gamma)} \right] - (\alpha + 1) \sum_{i=1}^n \left[\frac{(1 - \beta)x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi (x_i, \theta, \gamma)]} \right] = 0, \tag{16}$$

$$\frac{\partial L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \ln (x_i) - \theta \sum_{i=1}^n (x_i^\gamma \ln (x_i)) + (\alpha - 1) \sum_{i=1}^n \left[\frac{\theta x_i^\gamma \ln (x_i) e^{-\theta x_i^\gamma}}{\phi (x_i, \theta, \gamma)} \right] - (\alpha + 1) \times \sum_{i=1}^n \left[\frac{(1 - \beta)\theta x_i^\gamma \ln (x_i) e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi (x_i, \theta, \gamma)]} \right] = 0, \tag{17}$$

where $\phi (x_i, \theta, \gamma) = (1 - e^{-\theta x_i^\gamma})$. From (14), we obtain the MLE of α in a closed form as

$$\hat{\alpha} = \frac{n}{\left[\sum_{i=1}^n \ln [\hat{\beta} + (1 - \hat{\beta})\phi (x_i, \hat{\theta}, \hat{\gamma})] - \sum_{i=1}^n \ln (\phi (x_i, \hat{\theta}, \hat{\gamma})) \right]}. \tag{18}$$

Substituting from (18) into (15), (16), and (17), we get the MLEs of (β, θ, γ) by solving the resulting system of non-linear equations. These equations cannot be solved analytically. We apply statistical software such as Mathcad to solve them numerically.

The elements of the observed information matrix are:

$$\begin{aligned} \frac{\partial^2 L}{\partial \alpha^2} &= \frac{-n}{\alpha^2}, \\ \frac{\partial^2 L}{\partial \alpha \partial \beta} &= - \sum_{i=1}^n \left[\frac{e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi (x_i, \theta, \gamma)]} \right], \\ \frac{\partial^2 L}{\partial \alpha \partial \theta} &= \sum_{i=1}^n \left[\frac{x_i^\gamma e^{-\theta x_i^\gamma}}{\phi (x_i, \theta, \gamma)} \right] - \sum_{i=1}^n \left[\frac{(1 - \beta)x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi (x_i, \theta, \gamma)]} \right], \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial \beta^2} &= \frac{-n}{\beta^2} + (\alpha + 1) \sum_{i=1}^n \left[\frac{e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right]^2, \\
\frac{\partial^2 L}{\partial \beta \partial \theta} &= (\alpha + 1) \sum_{i=1}^n \left[\frac{x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right] \left[1 + \frac{(1 - \beta)e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right], \\
\frac{\partial^2 L}{\partial \beta \partial \gamma} &= (\alpha + 1) \sum_{i=1}^n \left[\frac{\theta x_i^\gamma \ln(x_i) e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right] \left[1 - \frac{(1 - \beta)e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right], \\
\frac{\partial^2 L}{\partial \theta^2} &= \frac{-n}{\theta^2} - (\alpha - 1) \sum_{i=1}^n \left[\frac{x_i^{2\gamma} e^{-\theta x_i^\gamma}}{\phi(x_i, \theta, \gamma)} \left[1 + \frac{e^{-\theta x_i^\gamma}}{\phi(x_i, \theta, \gamma)} \right] \right] + \\
&\quad (\alpha + 1) \sum_{i=1}^n \left[\frac{(1 - \beta)x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right] \left[x_i^\gamma + \frac{(1 - \beta)x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right], \\
\frac{\partial^2 L}{\partial \theta \partial \gamma} &= (\alpha - 1) \sum_{i=1}^n \left[\frac{x_i^\gamma \ln(x_i^\gamma) e^{-\theta x_i^\gamma}}{\phi(x_i, \theta, \gamma)} \left[1 - \theta x_i^\gamma - \frac{\theta x_i^\gamma e^{-\theta x_i^\gamma}}{\phi(x_i, \theta, \gamma)} \right] \right] - \sum_{i=1}^n x_i^\gamma \ln(x_i^\gamma) - \\
&\quad (\alpha + 1) \sum_{i=1}^n \left[\frac{(1 - \beta)x_i^\gamma \ln(x_i^\gamma) e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right] \left[1 - \theta x_i^\gamma - \frac{(1 - \beta)\theta x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right], \\
\frac{\partial^2 L}{\partial \gamma^2} &= \frac{-n}{\gamma^2} - \theta \sum_{i=1}^n x_i^\gamma [\ln(x_i)]^2 + (\alpha - 1) \sum_{i=1}^n \left[\frac{\theta x_i^\gamma [\ln(x_i)]^2 e^{-\theta x_i^\gamma}}{\phi(x_i, \theta, \gamma)} \left[1 - \theta x_i^\gamma - \frac{\theta x_i^\gamma e^{-\theta x_i^\gamma}}{\phi(x_i, \theta, \gamma)} \right] \right] - \\
&\quad (\alpha + 1) \sum_{i=1}^n \left[\frac{(1 - \beta)\theta x_i^\gamma [\ln(x_i)]^2 e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right] \left[1 - \theta x_i^\gamma - \frac{(1 - \beta)\theta x_i^\gamma e^{-\theta x_i^\gamma}}{[\beta + (1 - \beta)\phi(x_i, \theta, \gamma)]} \right].
\end{aligned}$$

Approximate two-sided $100(1-v)\%$ confidence intervals of the unknown parameters $(\alpha, \beta, \theta, \gamma)$ are $\hat{\alpha} \pm z_{\frac{v}{2}} \sqrt{\text{Var}(\hat{\alpha})}$, $\hat{\beta} \pm z_{\frac{v}{2}} \sqrt{\text{Var}(\hat{\beta})}$, $\hat{\theta} \pm z_{\frac{v}{2}} \sqrt{\text{Var}(\hat{\theta})}$, and $\hat{\gamma} \pm z_{\frac{v}{2}} \sqrt{\text{Var}(\hat{\gamma})}$, respectively, where $z_{\frac{v}{2}}$ is the upper $(\frac{v}{2})$ th percentile of the standard normal distribution.

6 Applications

In this section, we perform an application to a real data set to evaluate the flexibility of the OIP-W model. In order to compare the OIP-W distribution with other fitted distributions, including Odd log-logistic Weibull (OLLW), Kumaraswamy Weibull (KwW), Beta-Weibull (BW), Odd log-logistic Marshall Olkin Weibull (OLLMOW), and McDonald Weibull (McW). The data set is taken from [Van Montfort \(1970\)](#) which represents the Maximum Annual Flood Discharges of the North Saskatchewan in units of 1000 cubic feet per second, of the North Saskatchewan River at Edmonton, over a period of 48 years. The data are in [Table 1](#). The MLEs that are computed using adequacy model is given in [Table 2](#). The goodness of fit measures including the values of log-likelihood function, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and Corrected Akaike Information Criteria (CAIC) are given in [Table 3](#) for the five models in order to verify which distribution fits better to these the data. Generally, we consider the best fit using the smaller values of these statistics. [Table 3](#) show that the OIP-W model has the lowest values of all statistical measures among all fitted distributions.

Table 1: The data.

19.885	20.940	21.820	23.700	24.888	25.460	25.760	26.720	27.500
28.100	28.600	30.200	30.380	31.500	32.600	32.680	34.400	35.347
35.700	38.100	39.020	39.200	40.000	40.400	40.400	42.250	44.020
44.730	44.900	46.300	50.330	51.442	57.220	58.700	58.800	61.200
61.740	65.440	65.597	66.000	74.100	75.800	84.100	106.600	109.700
121.970	121.970	185.560						

Table 2: MLE values from fitting the data set.

Model	Estimates					
OLLW	0.107	0.494	4.878	-	-	-
KwW	0.049	1.175	8.908	0.306	-	-
BW	0.074	1.061	8.887	0.373	-	-
OLLMOW	0.138	0.423	5.580	0.874	-	-
McW	0.134	1.041	5.404	0.216	4.820	-
OIPW	1.497	7.410	1.617	0.371	-	-

Table 3: Goodness of fit measures for the data set.

Model	Measures			
	-2L	AIC	BIC	CAIC
OLLW	-438.300	444.300	449.914	443.846
KwW	-435.443	443.443	450.928	444.373
BW	-435.190	443.190	450.673	444.121
OLLMOW	-438.060	446.060	453.544	446.989
McW	-432.272	442.372	451.728	443.800
OIPW	-431.607	439.607	447.151	440.596

7 Conclusion

In this paper, a new four-parameter model, which is called OIP-W distribution, is introduced. Explicit expressions for some statistical properties of the distribution are derived and discussed. Maximum likelihood estimation is derived and the observed fisher information matrix is obtained. The OIP-W distribution is employed to fit a real data set, and it performs better than some other competitive distributions.

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