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The Kumaraswamy Transmuted-G Family of Distributions: Properties and Applications

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Abstract: We introduce a new class of continuous distributions called the *Kumaraswamy transmuted-G* family which extends the transmuted class defined by Shaw and Buckley (2007). Some special models of the new family are provided. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Rényi and Shannon entropies, order statistics and probability weighted moments are derived. The maximum likelihood is used for estimating the model parameters. The flexibility of the generated family is illustrated by means of two applications to real data sets.

Key words: Generating Function, Kumaraswamy-G Family, Maximum Likelihood, Order Statistic, Probability Weighted Moment, Transmuted Family.

1. Introduction

The interest in developing more flexible statistical distributions remains strong nowadays. Many generalized distributions have been developed over the past decades for modeling data in several areas such as biological studies, environmental sciences, economics, engineering, finance and medical sciences. Recently, there has been an increased interest in defining new generated families of univariate continuous distributions by introducing additional shape parameters to the baseline model. One example is the beta-generated family proposed by Eugene et al. (2002). Another example is the generalized transmuted-G family definned by Nofal et al. (2015). For more details about the recent development of generalized statistical distributions, we refer the reader to Alzaatreh et al. (2013) and Lee et al. (2013). However, in many applied areas, there is a clear need for extending forms of the classical models.

The generated distributions have attracted several statisticians to develop new models because the computational and analytical facilities available in most symbolic computation software platforms. Several mathematical properties of the extended distributions may be easily explored using mixture forms of exponentiated-G (exp-G for short) distributions.

Consider a baseline cumulative distribution function (cdf) $H(x;\phi)$and probability density function (pdf) $h(x;\phi)$ with a parameter vector ϕ, where $\phi=(\phi_k)=(\phi_1,\phi_2,\dots)$. Then, the cdf and pdf of the *transmuted class* (TC) of distributions are defined by

$$G(x;\lambda,\phi)=H(x;\phi)\left[1+\lambda-\lambda H(x;\phi)\right] \quad (1)$$

and

$$g\left(x;\lambda,\phi\right)=h\left(x;\phi\right)\left[1+\lambda-2\lambda H\left(x;\phi\right)\right], \qquad (2)$$

respectively.

Note that the TC is a mixture of the baseline and exp-G distributions, the last one with power parameter equal to two. For $\lambda = 0$, equation (2) gives the baseline distribution. Further details can be found in Shaw and Buckley (2007).

In this paper, we de…ne and study a new family of distributions by adding two extra shape parameters in equation (1) to provide more flexibility to the generated family. In fact, based on the Kumaraswamy-generalized (Kw-G) class pioneered by Cordeiro and de Castro (2011), we construct a new generator so-called the Kumaraswamy transmuted-G (Kw-TG) family and give a comprehen-sive description of some of its mathematical properties. We hope that the new model will attract wider applications in reliability, engineering and other areas of research.

For an arbitrary baseline cdf $\mathrm{G}(\mathrm{x})$, Cordeiro and de Castro (2011) defined the Kw-G generator by the cdf and pdf given by

$$F\left(x;a,b\right)=1-\left[1-G\left(x\right)^{a}\right]^{b} \qquad (3)$$

and

$$f(x;a,b)=a\,b\,g\left(x\right)\,G\left(x\right)^{a-1}\,\left[1-G\left(x\right)^{a}\right]^{b-1}, \qquad (4)$$

respectively, where $g(x) = dG(x)/dx$and *a* and *b* are two additional positive shape parameters. Clearly, for *a* = *b* = 1, we obtain the baseline distribution. The additional parameters *a* and *b* aim to govern skewness and tail weight of the generated distribution. An attractive feature of this family is that *a* and *b* can afford greater control over the weights in both tails and in the center of the distribution.

The rest of the paper is outlined as follows. In Section 2, we de…ne the Kw-TG family and provide some special models. In Section 3, we derive a very useful representation for the Kw-TG density function. We obtain in Section 4 some general mathematical properties of the proposed family including ordinary and incomplete moments, mean deviations, moment generating function (mgf), Rényi, Shannon and q-entropies, order statistics and their moments and probability weighted moments (PWMs). Four special models of this family are presented in Section 5 corresponding to the baseline exponential, power, log-logistic and Burr X distributions. Maximum likelihood estimation of the model parameters is investigated in Section 6. In Section 7, we provide a simulation study to test the performance of the maximum likelihood method in estimating the parameters of the Kumaraswamy transmuted-exponential (Kw-TE) model and perform two applications to real data sets to illustrate the potentiality of the new family. Finally, some concluding remarks are presented in Section 8.

2. The Kw-TG family

In this section, we generalize the TC by incorporating two additional shape parameters to yield a more flexible generator. Then, the cdf of the Kw-TG family is defined by

$$F\left(x\right)=1-\left\{1-\left[\left(1+\lambda\right)H\left(x;\phi\right)-\lambda H\left(x;\phi\right)^{2}\right]^{a}\right\}^{b}. \qquad (5)$$

The pdf corresponding of (5) is given by

$$\begin{aligned} f(x) &= a\,b\,h(x;\phi)\{1+\lambda-2\lambda H(x;\phi)\} \\ &\quad \{H(x;\phi)[1+\lambda-\lambda H(x;\phi)]\}^{a-1} \\ &\quad \left\{1-\left[(1+\lambda)H(x;\phi)-\lambda H(x;\phi)^2\right]^a\right\}^{b-1}. \end{aligned} \tag{6}$$

Henceforth, a random variable X having the density function (6) is denoted by $X \sim \text{Kw}-\text{TG}(\lambda, a, b, \phi)$

The hazard rate function (hrf) of X, say $\tau(x)$, is given by

$$\begin{aligned} \tau(x) &= a\,b\,h(x;\phi)\ \{1+\lambda-2\lambda H(x;\phi)\} \\ &\quad \{H(x;\phi)[1+\lambda-\lambda H(x;\phi)]\}^{a-1} \\ &\quad \left\{1-\left[(1+\lambda)H(x;\phi)-\lambda H(x;\phi)^2\right]^a\right\}^{-1}. \end{aligned}$$

The quantile function (qf) of X, say $Q(u) = F^{-1}(u)$, can be obtained by inverting (5) numerically and it is given by

$$Q(u) = G^{-1}\left\{\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda B}}{2\lambda}\right\} \text{ for } \lambda \neq 0,$$

where $B = [1 - (1-u)^{1/b}]^{1/a}$, otherwise $Q(u) = Q_G(u;\phi)$. Some special cases of the new family are listed in Table 1.

A physical interpretation of the Kw-TG family cdf is possible whenever a and b are positive integers. Consider a device made of a series of b independent components that are connected so that each component is made of a independent sub-components in a parallel system. If the sub-component lifetimes have a common cdf, then the lifetime of the device follows the Kw-TG family of distributions in (5). So, the system fails if any of the b components fail. Also, each component fails if all of its a sub-components fail.

Moreover, suppose a system consists of b independent subsystems functioning independently at a given time and that each sub-system consists of a independent components that are connected in parallel. Further, suppose that each component consists of two units. The overall system will follow the Kw-TG model with $\lambda = 1$ if the two units are connected in series, whereas the overall system will follow the Kw-TG distribution with $\lambda = -1$ if the two units are connected in parallel.

Table 1: Sub-models of the Kw-TG family

a	b	λ	Reduced Model	Authors
a	b	0	Kw-G Family	Cordeiro and de Castro (2011)
1	b	0	G-G Family	New
a	1	0	exp-G Family	Gupta et al. (1998)
1	b	λ	GT-G Family	New
a	1	λ	ET-G Family	New
1	1	λ	T-G Family	Shaw and Buckley (2007)
1	1	0	$H(x;\phi)$	–

3. Mixture representation

In this section, we provide a useful representation for the Kw-TG pdf. Consider the power series

$$(1-z)^{b-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \, \Gamma(b)}{k! \, \Gamma(b-k)} z^k, \tag{7}$$

which holds for $|z| < 1$ and b > 0 real non-integer.

After applying the power series (7) to equation (6), we obtain

$$f(x) = \underbrace{h(x)\left[1+\lambda-2\lambda H(x)\right]}_{g(x)} \sum_{k=0}^{\infty} \frac{(-1)^k \, ab\Gamma(b)}{k!\Gamma(b-k)} \times \underbrace{\left[(1+\lambda) H(x) - \lambda H(x)^2\right]^{(k+1)a-1}}_{G(x)^{(k+1)a-1}}.$$

Further, we can write the last equation as

$$f(x) = \sum_{k=0}^{\infty} \upsilon_k \, g(x) \, G(x)^{(k+1)a-1}, \tag{8}$$

where $\upsilon_k = (-1)^k ab\Gamma(b)/[k!\,\Gamma(b-k)]$

Finally, the pdf (8) can be expressed as a mixture of exp-G densities

$$f(x) = \sum_{k=0}^{\infty} \omega_k \, \pi_{(k+1)a}(x), \tag{9}$$

where $\omega_k = \upsilon_k/(k+1)a$ and $\pi_\gamma(x) = \gamma g(x) G(x)^{\gamma-1}$is the exp-G pdf with power parameter $\gamma > 0$.

Thus, several mathematical properties of the Kw-TG family can be deter-mined from those properties of the exp-G family. For example, the ordinary and incomplete moments and mgf of *X* can be obtained directly from those of the exp-G class. Equation (9) is the main result of this section.

The cdf of the Kw-TG family can also be expressed as a mixture of exp-G densities. By integrating (9), we obtain the same mixture representation

$$F(x) = \sum_{k=0}^{\infty} \omega_k \, \Pi_{(k+1)a}(x),$$

where $\prod_{(k+1)a}(x)$ the cdf of the exp-G family with power parameter $(k+1)\,a$.

4. Mathematical properties

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab because of their ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to evaluate statistical measures can be more efficient than computing them directly by numerical integration. We have noted that the infinity limit in these sums can be substituted by a large positive integer such as 50 for most practical purposes.

4.1 Moments

Henceforth, $Y_{(k+1)a}$ denotes the exp-G distribution with power parameter $(k+1)\,a$. The rth moment of X, say μ'_r, follows from (9) as

$$\mu'_r = E\left(X^r\right) = \sum_{k=0}^{\infty} \omega_k \; E\left(Y^r_{(k+1)a}\right).$$

The nth central moment of X, say μ_n, is given by

$$\begin{aligned}\mu_n &= E\left(X-\mu'_1\right)^n = \sum_{r=0}^{n}\binom{n}{r}\left(-\mu'_1\right)^{n-r} E\left(X^r\right)\\ &= \sum_{r=0}^{n}\sum_{k=0}^{\infty}(-1)^{n-r}\,\omega_k\binom{n}{r}\mu_r'^{(n-r)}\,E\left(Y^r_{(k+1)a}\right).\end{aligned}$$

The cumulants (kn) of X follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1}\binom{n-1}{r-1}\kappa_r\,\mu'_{n-r},$$

where $k_1 = \mu'_1$, $k_2 = \mu'_2 - \mu_1'^2$, $k_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu_1'^3$, etc. The measures of skewness and kurtosis can be calculated from the ordinary moments using well-known relationships

4.2 Generating function

Here, we provide two formulae for the mgf $M_X(t) = E(e^{tX})$ of X. Clearly, the first one can be derived from equation (9) as

$$M_X\left(t\right) = \sum_{k=0}^{\infty}\omega_k\, M_{(k+1)a}\left(t\right),$$

where $M_{(k+1)a}$ (t) is the mgf of $Y_{(k+1)a}$. Hence, $M_X(t)$ can be determined from the exp-G generating function.

A second formula for $M_X(t)$ follows from (9) as

$$M_X\left(t\right) = \sum_{k=0}^{\infty}\omega_k\,\tau\left(t,k\right),$$

where $\tau(t,k) = \int_0^1 exp\,[tQ_H(u)]u^{(k+1)a-1}du$ and Q_H (u) is the qf corresponding to $H(x;\phi$, i.e., $QH(u) = H^{-1}(u;\phi)$.

4.3 Incomplete moments

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The sth incomplete moment, $\varphi_s(t)$, of X can be expressed from (9) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx = \sum_{k=0}^{\infty} \omega_k \int_{-\infty}^{t} x^s \pi_{(k+1)a}(x)\, dx. \qquad (10)$$

The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median $[\delta_2 = E(|X - M|)]$ of X are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = \text{Median}(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily evaluated from (5) and $\varphi_1(t)$ is the first incomplete moment given by (10) with $s = 1$.

Now, we provide two ways to determine δ_1 and δ_2. First, a general equation for $\varphi_1(t)$ can be derived from (10) as

$$\varphi_1(t) = \sum_{k=0}^{\infty} \omega_k\, J_{(k+1)a}(t),$$

where $J_{(k+1)a}(t) = \int_{-\infty}^{t} x\pi_{(k+1)a}(x)dx$ is the first incomplete moment of the exp-G distribution.

A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k=0}^{\infty} \omega_k\, \upsilon_k(t),$$

where $\upsilon_k(t) = (k+1)a \int_0^{H(t)} Q_H(u) u^{(K+1)a-1} du$ can be computed numerically.

These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q)/(\pi\mu_1')$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π.

4.4 Entropies

The Rényi entropy of a random variable *X* represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\theta(X) = \frac{1}{1-\theta} \log\left(\int_{-\infty}^{\infty} f(x)^\theta dx\right),\ \theta > 0 \text{ and } \theta \neq 1.$$

Using the pdf (6), we can write

$$I_\theta(X) = \frac{1}{1-\theta} \log\left(\int_{-\infty}^{\infty} f(x)^\theta dx\right),\ \theta > 0 \text{ and } \theta \neq 1.$$

Using the pdf (6), we can write

$$\begin{aligned} f(x)^\theta \;=\; & (ab)^\theta h(x)^\theta \{1+\lambda-2\lambda H(x)\}^\theta \\ & \times \{H(x)[1+\lambda-\lambda H(x)]\}^{\theta(a-1)} \\ & \times \left\{1-\left[(1+\lambda)H(x)-\lambda H(x)^2\right]^a\right\}^{\theta(b-1)}. \end{aligned}$$

Applying the power series (7) to the last term, we obtain

$$f(x)^{\theta} = (ab)^{\theta}\sum_{k=0}^{\infty}(-1)^{j}\binom{\theta(b-1)}{k}g(x)^{\theta}G(x)^{ak+\theta(a-1)}$$
$$= \sum_{k=0}^{\infty}m_k\, g(x)^{\theta}G(x)^{ak+\theta(a-1)}.$$

Then, the Rényi entropy of the Kw-TG family is given by

$$I_{\theta}(X) = \frac{1}{1-\theta}\log\left\{\sum_{k=0}^{\infty}m_k\int_{-\infty}^{\infty}g(x)^{\theta}G(x)^{ak+\theta(a-1)}\,dx\right\},$$

where

$$m_k = (-1)^k(ab)^{\theta}\binom{\theta(b-1)}{k}.$$

The θ-entropy, say $H_{\theta}(X)$, can be obtained, for $\theta > 0$, $\theta \neq 1$, as

$$H_{\theta}(X) = \frac{1}{\theta-1}\log\left\{1-\sum_{k=0}^{\infty}m_k\int_{-\infty}^{\infty}g(x)^{\theta}G(x)^{ak+\theta(a-1)}\,dx\right\}.$$

The Shannon entropy of a random variable X, say SI, is defined by

$$SI = E\{-[\log f(\boldsymbol{X})]\},$$

which follows by taking the limit of $I_{\theta}(X)$ as θ tends to 1.

4.5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1,\ldots\ldots,X_n$ be a random sample from the Kw-TG family. The pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}F^{j+i-1}(x), \qquad (11)$$

where B(· ; ·) is the beta function. Based on equation (5), we have

$$F^{j+i-1}(x) = \sum_{l=0}^{\infty}(-1)^{l}\binom{j+i-1}{l}\left\{1-\left[(1+\lambda)\,H(x)-\lambda H(x)^{2}\right]^{a}\right\}^{l\,b}. \qquad (12)$$

Using (6) and (12) and after a power series expansion, we can write

$$f(x)\,F^{j+i-1}(x) = \sum_{k=0}^{\infty}d_{k,j}\,\pi_{(k+1)a}(x), \qquad (13)$$

where

$$d_{k,j} = \sum_{l=0}^{\infty}\frac{(-1)^{l+k}\,a\,b}{(k+1)\,a}\binom{j+i-1}{l}\binom{b(l+1)-1}{k}.$$

Substituting (13) in equation (11), the pdf of Xi:n can be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j}\binom{n-i}{j}}{\mathrm{B}(i,n-i+1)}d_{k,j}\,\pi_{(k+1)a}(x),$$

where π(k+1)a (x) is the exp-G density with power parameter (k + 1) a.

Then, the density function of the Kw-TG order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of Xi:n follow from those properties of Y(k+1)a. For example, the moments of Xi:n can be expressed as

$$E\left(X_{i:n}^{q}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j}\binom{n-i}{j}}{\mathrm{B}\left(i,n-i+1\right)}d_{k,j}\,E\left(Y_{(k+1)a}\right). \tag{14}$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in equation (14), we can derive explicit expressions for the L-moments of X as in…nite weighted linear combinations of the means of suitable Kw-TG order statistics.

We have

$$\lambda_{r}=\frac{1}{r}\sum_{d=0}^{r-1}(-1)^{d}\binom{r-1}{d}E\left(X_{r-d:r}\right),\ r\geq 1.$$

The first four L-moments are given by: $\lambda_1 = E(X_{1:1}), \lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}), \lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$.

One simply can obtain the λ's for X from (14) with $q = 1$

4.6 Probability weighted moments

The *(s, r)*th PWM of *X* following the Kw-TG distribution, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r}=E\left\{X^{s}F(X)^{r}\right\}=\int_{-\infty}^{\infty}x^{s}f\left(x\right)F(X)^{r}\,dx.$$

From equation (13), we can write

$$\rho_{s,r}=\sum_{k=0}^{\infty}b_{k,r}\int_{-\infty}^{\infty}x^{s}\pi_{(k+1)a}\left(x\right)=\sum_{k=0}^{\infty}b_{k,r}\;E\left(Y_{(k+1)a}\right),$$

where

$$b_{k,r}=\sum_{l=0}^{\infty}\frac{(-1)^{l+k}\,a\,b}{(k+1)\,a}\binom{r}{l}\binom{b\,(l+1)\;-1}{k}.$$

5. Special models

In this section, we provide four special models of the Kw-TG family, namely, Kw-T exponential, Kw-T power, Kw-T log logistic and Kw-T Burr X distributions. These sub-models generalize important existing distributions in the literature. Section 4 is used to obtain some properties of the Kw-TE distribution.

5.1 The Kw-TE distribution

The exponential distribution with scale parameter > 0 has pdf and cdf given by h (x) = $\alpha e^{-\alpha x}$ (for $x > 0$) and H (x) = 1 - $e^{-\alpha x}$, respectively. Then, the Kw-TE density function reduces to

$$\begin{aligned} f(x) &= \alpha a b \mathrm{e}^{-\alpha x}\left\{1-\lambda+2\lambda \mathrm{e}^{-\alpha x}\right\}\left\{\left(1-\mathrm{e}^{-\alpha x}\right)\left[1+\lambda \mathrm{e}^{-\alpha x}\right]\right\}^{a-1} \\ &\quad \times\left[1-\left\{\left(1-\mathrm{e}^{-\alpha x}\right)\left[1+\lambda \mathrm{e}^{-\alpha x}\right]\right\}^{a}\right]^{b-1}. \end{aligned}$$

The Kw-TE distribution reduces to the transmuted exponential (TE) distribution when a = b = 1. Also, when = 0, it reduces to the Kw-E distribution. Figure 1 displays some possible shapes of the density and hazard rate functions of this distribution. Figure 1 reveals that the pdf of the Kw-TE can be reversed J-shape or right skewed. The hrf can be decreasing (DFR), increasing (IFR) or constant (CFR) failure rate.

Next, some properties of the Kw-TE are obtained by using the general properties discussed in Section 4.

(1) Moments: From Section 4.1, the *r*th moment of Kw-TE distribution can be written as

$$\begin{aligned} \mu_r' &= E\left(X^r\right)=\sum_{k=0}^{\infty}\frac{(-1)^r}{\lambda^r}(k+1)\,a\omega_k \\ &\quad \times\frac{\partial^r}{\partial p^r}\mathbf{B}\left((k+1),p+1-(k+1)\,a\right)|_{p=(k+1)a}. \end{aligned}$$

The *n*th central moment of the Kw-TE model is given by

$$\begin{aligned} \mu_n &= \sum_{r=0}^{n}\sum_{k=0}^{\infty}\frac{(-1)^n}{\lambda^r}\binom{n}{r}(k+1)\,a\mu_r'^{(n-r)}\omega_k \\ &\quad \times\frac{\partial^r}{\partial p^r}\mathbf{B}\left((k+1),p+1-(k+1)\,a\right)|_{p=(k+1)a}. \end{aligned}$$

(2) Moment generating function: From Section 4.2, the mgf for Kw-TE can be expressed as

$$M_X(t)=\sum_{k,r,h=0}^{\infty}\frac{t^r(-1)^h(k+1)\,a\Gamma(k+1)\,\omega_k}{h!r!\lambda^r\,\Gamma\left((k+1)\,a-h\right)(h+1)^{r+1}}\Gamma(r+1),\;\; r>-1.$$

A second formula for MX (t) of the Kw-TE model (for $t < \lambda$) is

$$M_X(t)=\sum_{k=0}^{\infty}\omega_k\,(k+1)\,a\mathbf{B}\left(1-\frac{t}{\lambda},(k+1)\,a\right).$$

(3) Incomplete moments: From Section 4.3, the sth incomplete moment of the Kw-TE model is given by

$$\varphi_s(t)=\sum_{k,h=0}^{\infty}\frac{(-1)^h(k+1)\,a\Gamma\left((k+1)\,a\right)\omega_k}{i!\lambda^s\,(h+1)^{s+1}\,\Gamma\left((k+1)\,a-h\right)}\gamma\left(s+1,\left(\frac{\lambda}{t}\right)^{\beta}\right),$$

where $\gamma(a, z) = \int_0^z y^{\alpha-1}e^{-y}dy$ is the incomplete gamma function.

(4) Entropies: From Section 4.4, the Rényi entropy of the Kw-TE model is given by

$$I_\theta(X) = \frac{1}{1-\theta}\log\left(\sum\nolimits_{k,i,j,l=0}^{\infty} m_{k,i,j,l}\right),$$

where

$$\begin{aligned} m_{k,i,j,l} &= \frac{2^i \lambda^{i+j+2\theta-1}(ab)^\theta(-1)^{k+i+j+l}}{(\theta+l)(1+\lambda)^{i+j-a(k+\theta)}}\binom{\theta}{i}\binom{(b-1)\theta}{k} \\ &\quad\times\binom{ak+(a-1)\theta}{j}\binom{ak+\theta(a-1)+i+j}{l}. \end{aligned}$$

The Shannon entropy of the Kw-TE distribution follows by taking the limit of $I_\theta(X)$ when θ goes to 1.

(5) Order statistics: From Section 4.5, the moments of order statistics for the Kw-TE distribution can be written as

$$\begin{aligned} E\left(X_{i:n}^q\right) &= \sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{j+q}(k+1)a}{\lambda^q \mathrm{B}(i,n-i+1)} d_{k,j}\binom{n-i}{j} \\ &\quad\times\frac{\partial^q}{\partial p^q}\mathbf{B}\left((k+1),p+1-(k+1)a\right)|_{p=(k+1)a}. \end{aligned}$$

(6) Probability weighted moments: From Section 4.6 we have

$$\rho_{s,r} = \sum_{k=0}^{\infty}\frac{(-1)^s}{\lambda^s}(k+1)ab_{k,r}\frac{\partial^s}{\partial p^s}\mathbf{B}\left((k+1),p+1-(k+1)a\right)|_{p=(k+1)a}.$$

5.2 The Kw-T power (Kw-TPo) distribution

The power (Po) distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ has pdf and cdf given by $h(x) = \alpha\beta^\alpha x^{\alpha-1}$ (for $0 < x < \beta^{-1}$) and $H(x) = (\beta x)^\alpha$, respectively. Then, the Kw-TPo density is given by

$$\begin{aligned} f(x) &= a\,b\,\alpha\,\beta^\alpha x^{\alpha-1}\left\{1+\lambda-2\lambda(\beta x)^\alpha\right\}\left\{(\beta x)^\alpha\left[1+\lambda-\lambda(\beta x)^\alpha\right]\right\}^{a-1} \\ &\quad\times\left\{1-\left[(\beta x)^\alpha\left[1+\lambda-\lambda(\beta x)^\alpha\right]\right]^a\right\}^{b-1}. \end{aligned}$$

This distribution reduces to the transmuted power (TPo) distribution if a = b = 1. For $\lambda = 0$, we obtain the Kw-Po distribution. For $\alpha = 1$, it follows as a special case the uniform distribution defined on the interval $(0, 1/\beta)$. Figure 2 displays plots of the density and hazard rate functions for the Kw-TPo distribution for selected parameter values. These plots reveal that the pdf of the Kw-TPo can be reversed J-shape, J-shape, concave up or concave down. The hrf can be decreasing (DFR), increasing (IFR) or bathtub (BT) failure rate shapes.

5.3 The Kw-T log logistic (Kw-TLL) distribution

The log-logistic (LL) distribution with positive parameters α and β has pdf and cdf given by $h(x) = \beta\alpha^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}$ (for x > 0) and $H(x) = 1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}$, respectively. Then, the pdf of the Kw-TLL distribution is given by

$$\begin{aligned} f(x) \;=\; & a\,b\,\beta\,\alpha^{-\beta}\,x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a-1} \\ & \times\left\{1-\lambda\left[1-2\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right]\right\}\left\{1+\lambda\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a-1} \\ & \times\left\{1-\left[1+\lambda\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right]^{a}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a}\right\}^{b-1}. \end{aligned}$$

The Kw-TLL model reduces to the transmuted log-logistic (TLL) distribution when a = b = 1. For $\lambda = 0$, we obtain the Kw-LL model. Plots of the density and hazard rate functions of the Kw-TLL distribution are displayed in Figure 3 for some parameter values. These plots show that the pdf of the Kw-TLL model can be reversed J-shape or concave down. The hrf can be decreasing (DFR), increasing (IFR) or upside down bathtub (UBT) failure rate shapes.

5.4 The Kw-T Burr X (Kw-TBrX) distribution

The Burr X (also known as generalized Raleigh) distribution with positive parameters α and β has pdf and cdf given by $h(x) = 2\alpha\beta^2 x e^{-(\beta x)^2}\{1-e^{-(\beta x)^2}\}^{\alpha-1}$ (for $x > 0$) and $H(x) = \left[1-e^{-(\beta x)^2}\right]^{\alpha}$, respectively. Then, the pdf of the Kw-TBrX distribution reduces to

$$\begin{aligned} f(x) \;=\; & 2\,a\,b\,\alpha\,\beta^2\,x e^{-(\beta x)^2}\left\{1+\lambda-2\lambda\left[1-e^{-(\beta x)^2}\right]^{\alpha}\right\} \\ & \times\left[1-e^{-(\beta x)^2}\right]^{\alpha a-1}\left\{1+\lambda-\lambda\left[1-e^{-(\beta x)^2}\right]^{\alpha}\right\}^{a-1} \\ & \times\left\{1-\left(1+\lambda-\lambda\left[1-e^{-(\beta x)^2}\right]^{\alpha}\right)^{a}\left[1-e^{-(\beta x)^2}\right]^{\alpha a}\right\}^{b-1}. \end{aligned}$$

The Kw-TBrX distribution includes the transmuted Burr X (TBrX) distribution when a = b = 1. For $\lambda = 0$, we obtain the Kw-BrX distribution. The plots in Figure 4 show some possible shapes of the density and hazard rate functions of the Kw-TBrX distribution. Figure 4 shows that the pdf of the Kw-TBrX model is very ‡exible. It can be reversed J-shape, left skewed or right skewed. The hrf can be decreasing (DFR), increasing (IFR) or bathtub (BT) failure rate shapes.

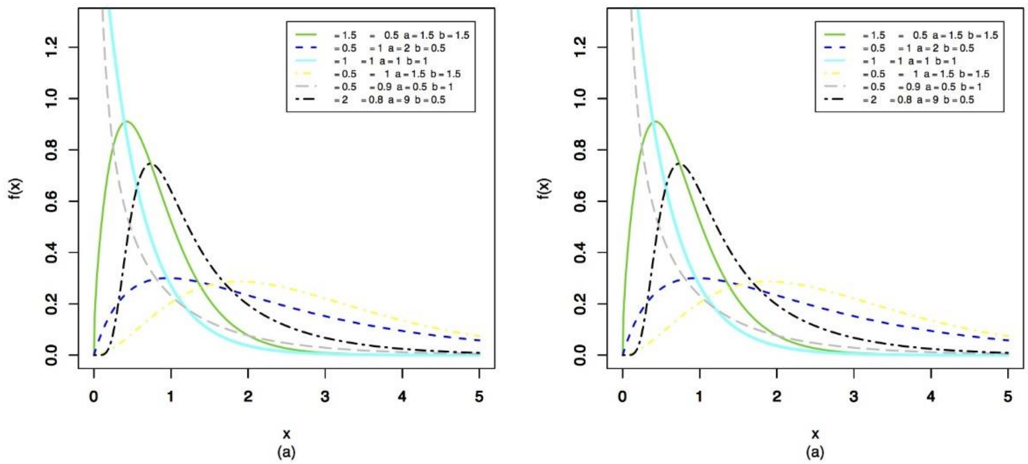

Figure 1: (a) The Kw-TE density plots. (b) The Kw-TE hrf plots

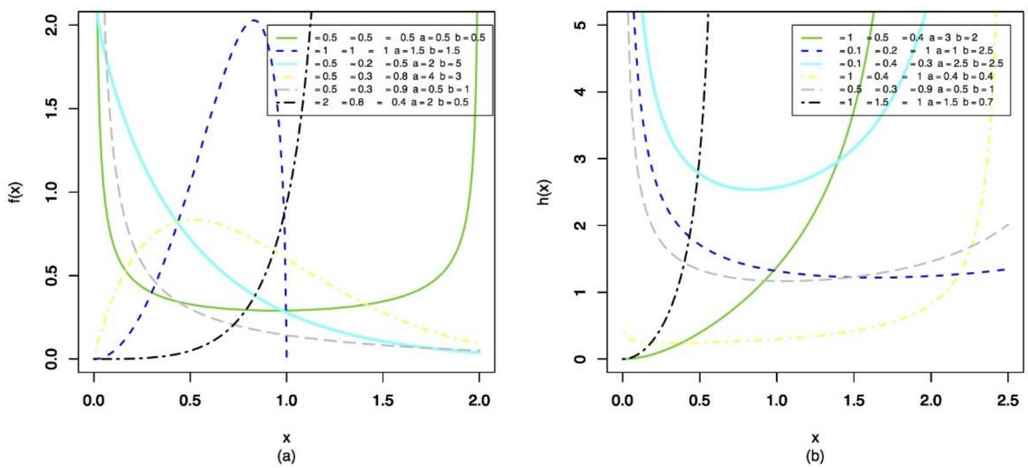

Figure 2: (a) Plots of the Kw-TPo pdf. (b) Plots of the Kw-TPo hrf.

6. Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the litera-ture but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing con…dence intervals and also in test statistics. The nor-mal approximation for these estimators in large sample theory is easily handled either analytically or numerically.

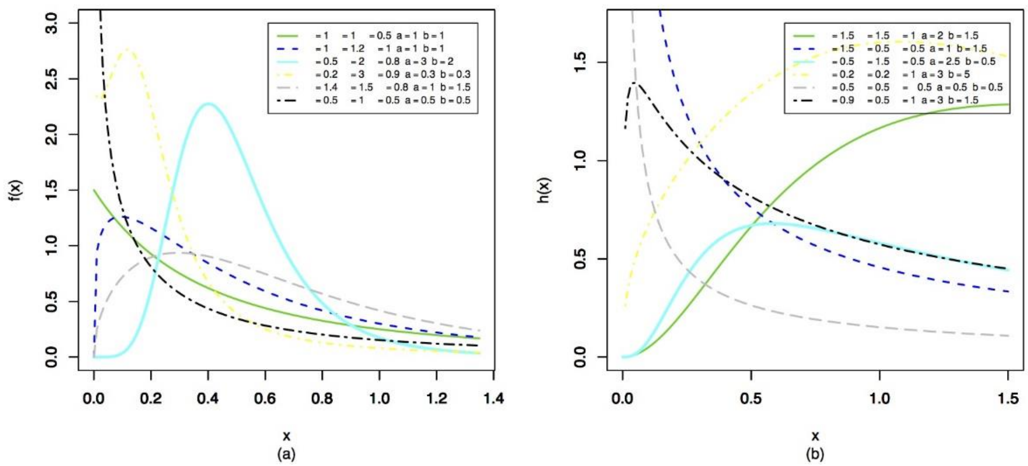

Figure 3: (a) The Kw-TLL density plots. (b) The Kw-TLL hrf plots

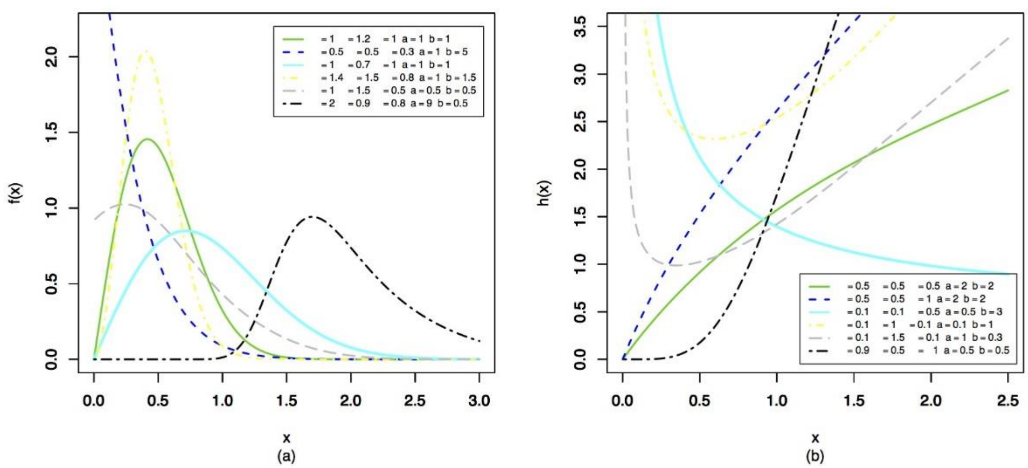

Figure 4: (a) The Kw-TBrX density plots. (b) The Kw- TBrX hrf plots

So, we determine the MLEs of the parameters of the new family of distri-butions from complete samples only. Let $x_1,\ldots,x_n$ be a random sample from the Kw-TG family with parameters λ,a,b and ϕ . Let $\theta = \left(a, b, \lambda, \phi^{\mathrm{T}}\right)^{\mathrm{T}}$ be the (p×1) parameter vector. Then, the log-likelihood function for θ, say $l = l\,(\theta)$, is given by

$$\begin{aligned}\ell \quad = \quad & n \log a + n \log b + \sum_{i=1}^{n} \log h\left(x_i; \boldsymbol{\phi}\right) + \sum_{i=1}^{n} \log p_i \\ & +(a-1) \sum_{i=1}^{n} \log q_i + (b-1) \sum_{i=1}^{n} \log\left(1-q_i\right), \qquad (15)\end{aligned}$$

where $p_i = 1 + \lambda - 2\lambda\, H\left(x_i; \phi\right)$, $z_i = 1 + \lambda - \lambda H\left(x_i; \phi\right)$ and $q_i = z_i\, H\left(x_i; \phi\right)$.

Equation (15) can be maximized either directly by using the R (optim func-tion), SAS (PROC NLMIXED) or Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by di¤erentiating (15).

The score vector components, say $U(\theta) = \frac{\partial t}{\partial \theta} = (\frac{\partial t}{\partial \alpha}, \frac{\partial t}{\partial b}, \frac{\partial t}{\partial \lambda}, \frac{\partial t}{\partial \phi_k})^T = (U_a, U_b, U_\lambda, U_{\phi_k})^T$ are given by

$$U_a = \frac{n}{a} + \sum_{i=1}^{n} \log q_i, \quad U_b = \frac{n}{b} + \sum_{i=1}^{n} \log(1-q_i),$$

$$\begin{aligned} U_\lambda &= \sum_{i=1}^{n} \frac{[1-2H(x_i;\phi)]}{p_i} + (a-1)\sum_{i=1}^{n} \frac{\left\{H(x_i;\phi) - H(x_i;\phi)^2\right\}}{q_i} \\ &\quad -(b-1)\sum_{i=1}^{n} \frac{\left\{H(x_i;\phi) - H(x_i;\phi)^2\right\}}{1-q_i} \end{aligned}$$

and

$$\begin{aligned} U_{\phi_k} &= \sum_{i=1}^{n} \frac{h'(x_i;\phi)}{h(x_i;\phi)} - 2\lambda \sum_{i=1}^{n} \frac{H'(x_i;\phi)}{p_i} \\ &\quad +(a-1)\sum_{i=1}^{n} \frac{H'(x_i;\phi)\{z_i - \lambda H(x_i;\phi)\}}{q_i} \\ &\quad -(b-1)\sum_{i=1}^{n} \frac{H'(x_i;\phi)\{z_i - \lambda H(x_i;\phi)\}}{(1-q_i)}, \end{aligned}$$

where $h'(x_i;\phi) = \partial h(x_i;\phi)/\partial h(x_i;\phi)$ and $H'(x_i;\phi) = \partial H(x_i;\phi)/\partial \phi_k$

Setting the nonlinear system of equations $U_a = U_b = U_\lambda = U_{\phi k} = 0$ and solving them simultaneously yields the MLE $\hat{\theta} = (\hat{a}, \hat{b}, \hat{\lambda}, \hat{\phi}^T)$ of $\theta = (a, b, \lambda, \phi^T)^T$.These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms. For interval estimation of the model parameters, we require the observed information matrix

$$J(\boldsymbol{\theta}) = -\begin{pmatrix} U_{aa} & U_{ab} & U_{a\lambda} & | & U_{a\phi}^T \\ U_{ba} & U_{bb} & U_{b\lambda} & | & J_{b\phi}^T \\ J_{\lambda a} & U_{\lambda b} & U_{\lambda\lambda} & | & U_{\lambda\phi}^T \\ -- & -- & -- & -- & -- \\ U_{a\phi} & J_{b\phi} & U_{\lambda\phi} & | & U_{\varphi\phi} \end{pmatrix},$$

whose elements are given in appendix A.

Under standard regularity conditions when $n \to \infty$, the distribution of $\hat{\theta}$ can be approximated by a multivariate normal $N_p(0; J(\hat{\theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\hat{\theta})$ is the total observed information matrix evaluated at $\hat{\theta}$. The method of the resampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Interval estimates may also be obtained using the bootstrap percentile method. Likelihood ratio tests can be performed for the proposed family of distributions in the usual way.

7. Applications

In this section, we illustrate the applicability of the Kw-TG family to real data sets. We focus on the Kw-TE distribution presented in Section 5. The method of maximum likelihood is used to estimate the model parameters. This section is divided into two subsections. The …rst subsection is devoted to study the performance of the MLEs for estimating the Kw-TE parameters using Monte Carlo simulation for di¤erent parameter values and various sample sizes. In the second subsection, two data sets are used to prove empirically the applicability of the Kw-TE distribution.

7.1 Monte Carlo simulation

A simulation study is conducted in order to test the performance of the MLEs for estimating the Kw-TE parameters. We consider three di¤erent sets of parameters: I: a = 3, b = 2, λ = 0.5, α = 2, II: a = 2, b = 3, λ = 0.8, α = 3 and III: a = 1, b = 5, λ = -0.7, α = 2. For each parameter combination, we simulate data from the Kw-TE model with di¤erent sample sizes, n = 50, n = 100, n = 150, n = 200, and calculate the MLEs by maximizing the log-likelihood equation in (15), where $h(x) = \alpha e^{-\alpha x}$. The process is repeated 1; 000 times and for each set of parameters and each sample size, the average bias (estimate-actual) and the standard deviation are evaluated. The results are presented in Table 2. From the results in Table 2, the biases and standard deviations decrease as the sample size increases. Furthermore, $\hat{a}$ and $\hat{\alpha}$ are overestimated and $\hat{b}$ is underestimated for the three sets of the parameters.

Whilst $\hat{\lambda}$ is underestimated for sets I and II and overestimated for group III. In general, Table 2 indicates that the MLE method performs quite good for estimating the Kw-TE parameters.

7.2 Real data

In this section, the Kw-TE model is fitted to two data sets and compared with other existing distributions. In order to compare the …ts of the distri-butions, we consider various measures of goodness-of-fit including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), consistent Akaike information criterion (CAIC), maximized log-likelihood under the model ($-2\hat{\ell}$), Anderson-Darling (A*) and Cramér-Von Mises (W*) statistics. The measures of goodness-of-fit measures are given by

Table 2: Bias and standard deviation for yhe parameter estimates

n	Actual values				Bias (Standard deviation)			
	a	b	λ	α	$\widehat{a}$	$\widehat{b}$	$\widehat{\lambda}$	$\widehat{\alpha}$
50	3	2	0.5	2	0.7149	-0.3479	-0.1002	0.8820
					(1.2819)	(1.2811)	(0.4187)	(1.1529)
100	3	2	0.5	2	0.4484	-0.3413	-0.0952	0.7944
					(0.8483)	(1.0106)	(0.3439)	(1.0666)
150	3	2	0.5	2	0.2437	-0.1812	-0.0305	0.6731
					(0.7962)	(0.9632)	(0.3413)	(1.0085)
200	3	2	0.5	2	0.1141	-0.1090	-0.0155	0.4657
					(0.5740)	(0.7185)	(0.3137)	(0.6734)
50	2	3	0.8	3	0.0676	-0.1147	-0.2365	1.1136
					(0.3050)	(1.4579)	(0.3230)	(1.3991)
100	2	3	0.8	3	0.0722	-0.1105	-0.0831	0.7971
					(0.2534)	(1.3926)	(0.3128)	(1.3679)
150	2	3	0.8	3	0.0585	-0.0889	-0.0430	0.4021
					(0.2513)	(1.2794)	(0.2564)	(1.3183)
200	2	3	0.8	3	0.0524	-0.0611	-0.0014	0.1680
					(0.2149)	(1.1083)	(0.2547)	(0.9225)
50	1	5	-0.7	2	0.1377	-1.6994	0.2057	1.2947
					(0.3065)	(2.1832)	(0.4187)	(1.4972)
100	1	5	-0.7	2	0.1323	-1.6320	0.1659	1.0894
					(0.3052)	(1.6012)	(0.3789)	(1.2530)
150	1	5	-0.7	2	0.0518	-0.9354	0.0871	0.8195
					(0.2087)	(1.1508)	(0.2750)	(1.0414)
200	1	5	-0.7	2	0.0454	-0.5060	0.0656	0.3960
					(0.1614)	(0.7658)	(0.1953)	(0.3134)

$$\mathrm{AIC} = -2\hat{\ell} + 2\mathrm{p}, \mathrm{BIC} = -2\hat{\ell} + 2\mathrm{p}\log(\mathrm{n})$$

$$\mathrm{HQIC} = -2\hat{\ell} + 2\mathrm{p}\log(\log(\mathrm{n})), \mathrm{CAIC} = -2\hat{\ell} + 2\mathrm{pn}/(\mathrm{n} - \mathrm{p} - 1)$$

$$A^{*} = \left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right]\right\}$$

and

$$W^{*} = \left(\frac{1}{2n} + 1\right)\left\{\sum_{j=1}^{n}\left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\},$$

respectively, where $z_i = F(y_j)$, p is the number of parameters, n is the sample size and the values $\acute{y}_j$s are the ordered observations. The smaller these statistics are, the better the fit is. Upper tail percentiles of the asymptotic distributions of these goodness-of-fit statistics were tabulated in Nichols and Padgett (2006).

Data set I: Failure times of 84 aircraft windshield

The first data set was studied by Murthy et al. (2004), which repre-sents failure times for a particular windshield device. The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the non-structural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the windshield. For further details, see, for example, Murthy et al. (2004). The data consist of 84 observations. These data were previously studied by Cordeiro et al. (2014) to compare the …ts of the McDonald Weibull (McW), beta Weibull (BW), Kumaraswamy Weibull (Kw-W) and Weibull distributions. Here, we shall compare the …ts of the Kw-TE model with other models: the McW (Cordeiro et al., 2014), gamma Weibull (GW) (Provost et al., 2011), BW (Lee et al., 2007), modi…ed beta Weibull (MBW) (Khan, 2015), transmuted modi…ed Weibull (TMW) (Khan and King, 2013) and transmuted exponenti-ated generalized Weibull (TExGW) (Yousof et al., 2015) distributions, whose pdf's (for x > 0) are given by

$$\text{McW: } f(x)=\frac{\beta c\alpha^{\beta}}{B(a/c,b)}\,x^{\beta-1}\,\mathrm{e}^{-(\alpha x)^{\beta}}\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}$$
$$\times\left\{1-\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{c}\right\}^{b-1};$$
$$\text{GW: } f(x)=\frac{\beta\alpha^{\gamma/\beta+1}}{\Gamma(1+\gamma/\beta)}x^{\beta+\gamma-1}\,\mathrm{e}^{-\alpha x^{\beta}};$$
$$\text{BW: } f(x)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}\mathrm{e}^{-b(\alpha x)^{\beta}}\left[1-\mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1};$$
$$\text{MBW: } f(x)=\frac{\beta\gamma^{a}\alpha^{-\beta}}{B(a,b)}x^{\beta-1}\mathrm{e}^{-b\left(\frac{x}{\alpha}\right)^{\beta}}\left[1-\mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a-1}$$
$$\times\left\{1-(1-\gamma)\left[1-\mathrm{e}^{-b\left(\frac{x}{\alpha}\right)^{\beta}}\right]\right\}^{-a-b};$$
$$\text{TMW: } f(x)=\left(\alpha+\gamma\beta x^{\beta-1}\right)\mathrm{e}^{-\alpha x-\gamma x^{\beta}}\left(1-\lambda+2\lambda\mathrm{e}^{-\alpha x-\gamma x^{\beta}}\right);$$
$$\text{TExGW: } f(x)=ab\beta\alpha^{\beta}x^{\beta-1}\mathrm{e}^{-a(\alpha x)^{\beta}}\left[1-\mathrm{e}^{-a(\alpha x)^{\beta}}\right]^{b-1}$$
$$\times\left\{1+\lambda-2\lambda\left[1-\mathrm{e}^{-a(\alpha x)^{\beta}}\right]^{b}\right\}.$$

The parameters of the above densities are all positive real numbers except for the TMW and TExGW distributions for which $|\lambda|\leq 1$.

Data set II: Breaking stress of carbon fibres

The second real data set consists of 100 observations from Nichols and Padgett (2006) on breaking stress of carbon fibres (in Gba). Here, we use these data to compare the Kw-TE model with other models, namely: Weibull Fréchet (WFr) (Afify et al., 2016), TLL, exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al. 2015), transmuted Marshall-Olkin Fréchet (TMOFr) (Afify et al., 2015), transmuted Fréchet (TFr) (Mahmoud and Mandouh, 2013) and Marshall-Olkin Fréchet (MOFr) (Krishna et al, 2013) distributions with pdf's (for $x > 0$) given by

WFr: $f(x)=ab\beta\alpha^{\beta}x^{-\beta-1}e^{-b\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-b-1}e^{-a\left[e^{-\left(\frac{\alpha}{x}\right)^{\beta}}-1\right]^{-b}}$;

ETGR: $f(x)=2\alpha\delta\beta^{2}xe^{-(\beta x)^{2}}\left\{1+\lambda-2\lambda\left[1-e^{-(\beta x)^{2}}\right]^{\alpha}\right\}$

$\times\left[1-e^{-(\beta x)^{2}}\right]^{\alpha\delta-1}\left\{1+\lambda-\lambda\left[1-e^{-(\beta x)^{2}}\right]^{\alpha}\right\}^{\delta-1}$;

TMOFr: $f(x)=\frac{\alpha\beta\gamma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\gamma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\gamma}{x}\right)^{\beta}}\right]^{2}}\left[1+\lambda-\frac{2\lambda e^{-\left(\frac{\gamma}{x}\right)^{\beta}}}{\alpha+(1-\alpha)e^{-\left(\frac{\gamma}{x}\right)^{\beta}}}\right]$;

TFr: $f(x)=\frac{\beta}{\gamma}\left(\frac{\gamma}{x}\right)^{\beta+1}e^{-\left(\frac{\gamma}{x}\right)^{\beta}}\left[1+\lambda-2\lambda e^{-\left(\frac{\gamma}{x}\right)^{\beta}}\right]$;

MOFr: $f(x)=\frac{\alpha}{\gamma}\beta\left(\frac{\gamma}{x}\right)^{\beta+1}e^{-\left(\frac{\gamma}{x}\right)^{\beta}}\left[\alpha+(1-\alpha)e^{-\left(\frac{\gamma}{x}\right)^{\beta}}\right]^{-2}$.

The parameters of the above densities are all positive real numbers except for the ETGR and TFr distributions for which $|\lambda|\leq 1$.

Tables 3 and 4 list the numerical values of 2`, AIC, BIC, HQIC, CAIC, W and A for the models fitted to both data sets. The MLEs and their corre-sponding standard errors (in parentheses) of the model parameters are given in Tables 5 and 6. These figures are obtained using the MATH-CAD PROGRAM.

In Table 3, we compare the fits of the Kw-TE model with the McW, GW, BW, MBW, TMW and TExGW models. The figures in this table indicate that the Kw-TE model has the lowest values for all goodness-of-fit statistics (for failure times of Aircraft Windshield data) among the fitted models. So, the Kw-TE model could be chosen as the best model. In Table 4, we compare the fits of the Kw-TE, WFr, ETGR, TLL, TMOFr, TFr and MOFr models. It is shown that the Kw-TE model has the lowest values for all goodness-of-fit statistics (for breaking stress of carbon fibres data) among all fitted models. So, the Kw-TE model can be chosen as the best model.

Table 3: Goodness-of-fit statistics for failure times of aircraft windshield data

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	A^*	W^*
Kw-TE	257.596	265.596	275.32	269.505	266.103	0.5485	0.0578
McW	273.899	283.899	296.053	288.785	284.669	1.5906	0.1986
GW	277.721	283.721	291.013	286.653	284.021	1.9489	0.2553
BW	297.028	305.028	314.751	308.937	305.534	3.2197	0.4652
MBW	299.573	309.573	321.727	314.459	310.342	3.2656	0.4717
TMW	333.893	341.893	351.616	345.801	342.399	11.2047	0.8065
TExGW	352.594	362.594	374.748	367.48	363.363	6.2332	1.0079

Table 4: Goodness-of-fit statistics for breaking stress of carbon fibres data

Model	$-2\hat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	A^*	W^*
Kw-TE	286.44	294.44	304.861	298.657	294.861	0.3402	0.0556
WFr	286.55	294.55	304.97	298.767	294.971	0.35156	0.05481
ETGR	292.046	300.046	310.5	304.263	300.467	0.7413	0.1419
TLL	294.9	300.873	308.689	304.036	301.123	0.9643	0.1839
TMOFr	301.973	309.973	320.393	314.19	310.394	1.2677	0.2376
TFr	344.475	350.475	358.29	353.638	350.725	3.1782	0.5559
MOFr	345.328	351.328	359.143	354.491	351.578	3.3825	0.5927

Table 5: MLEs and their standard errors (in parentheses) for data set I

Model	Estimates		
Kw-TE	$\hat{\alpha}=0.0965$ (0.053)	$\hat{\lambda}=-0.8971$ (0.129)	$\hat{a}=1.6346$ (0.35)
	$\hat{b}=65.0082$ (76.536)		
McW	$\hat{\alpha}=1.9401$ (1.011)	$\hat{\beta}=0.306$ (0.045)	$\hat{a}=17.686$ (6.222)
	$\hat{b}=33.6388$ (19.994)	$\hat{c}=16.7211$ (9.622)	
MBW	$\hat{\alpha}=10.1502$ (18.697)	$\hat{\beta}=0.1632$ (0.019)	$\hat{a}=57.4167$ (14.063)
	$\hat{b}=19.3859$ (10.019)	$\hat{c}=2.0043$ (0.662)	
TExGW	$\hat{\alpha}=4.2567$ (33.401)	$\hat{\beta}=0.1532$ (0.017)	$\hat{\lambda}=0.0978$ (0.609)
	$\hat{a}=5.2313$ (9.792)	$\hat{b}=1173.3277$ (129.165)	
BW	$\hat{\alpha}=1.36$ (1.002)	$\hat{\beta}=0.2981$ (0.06)	$\hat{a}=34.1802$ (14.838)
	$\hat{b}=11.4956$ (6.73)		
TMW	$\hat{\alpha}=0.2722$ (0.014)	$\hat{\beta}=1$ (5.185×10^{-5})	$\hat{\lambda}=0.4685$ (0.165)
	$\hat{\gamma}=4.5912\times10^{-6}$ (1.927×10^{-4})		
GW	$\hat{\alpha}=2.376973$ (0.378)	$\hat{\beta}=0.848094$ (5.296×10^{-4})	$\hat{\gamma}=3.534401$ (0.665)

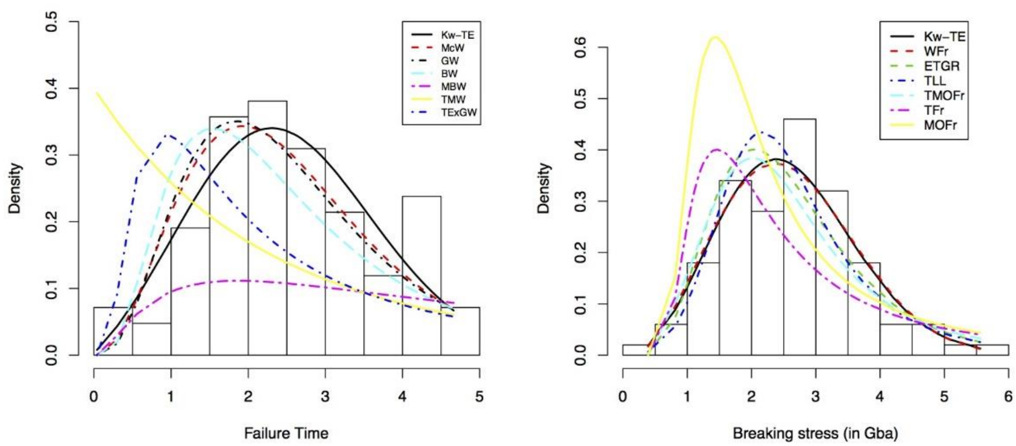

Figure 5: The fitted densities of the Kw-TE model and other models for both data sets

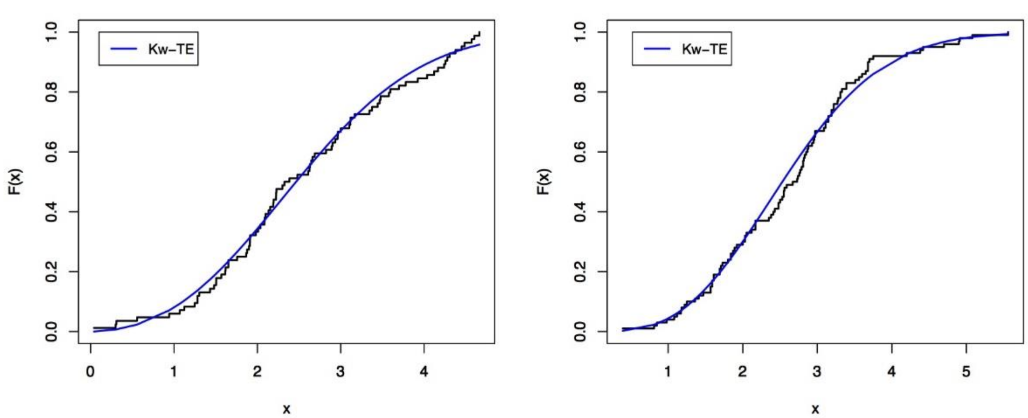

Figure 6: The fitted cdf of the Kw-TE model for data sets I (left panel) and for data sets II (right panel)

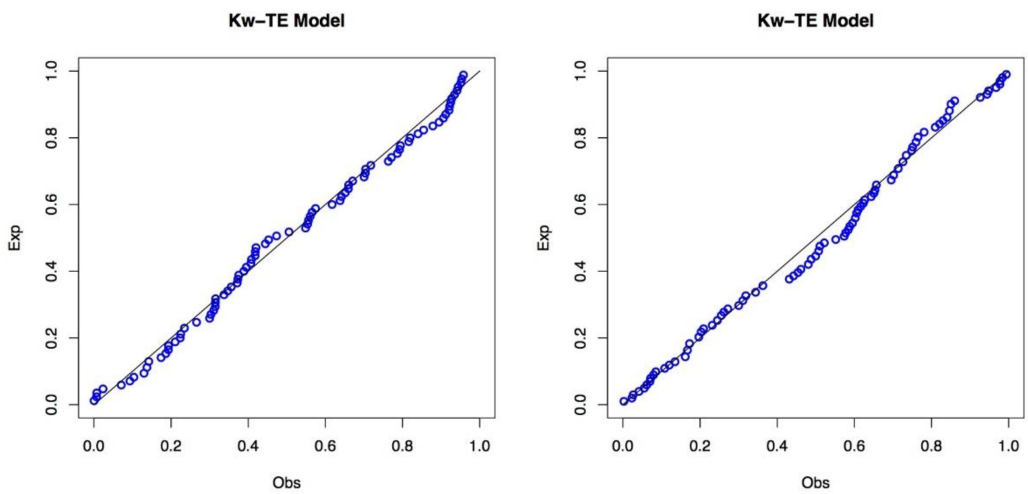

Figure 7: The QQ plots of the Kw-TE model for data sets I (left panel) and for data sets II (right panel)

Table 6: MLEs and their standard errors (in parentheses) for data set II

Model	Estimates			
Kw-TE	$\widehat{\alpha}$= 0.0955 (0.362)	$\widehat{\lambda}$= 0.5369 (7.052)	$\widehat{a}$= 3.3654 (0.916)	$\widehat{b}$= 37.2397 (29.423)
WFr	$\widehat{\alpha}$= 0.6942 (0.363)	$\widehat{\beta}$= 0.6178 (0.284)	$\widehat{a}$= 0.0947 (0.456)	$\widehat{b}$= 3.5178 (2.942)
ETGR	$\widehat{\alpha}$= 0.123 (0.088)	$\widehat{\beta}$= 0.3041 (0.034)	$\widehat{\delta}$= 41.3782 (50.268)	$\widehat{\lambda}$= 0.9318 (0.069)
TMOFr	$\widehat{\beta}$= 3.3041 (0.206)	$\widehat{\gamma}$= 0.6496 (0.068)	$\widehat{\alpha}$= 101.923 (47.625)	$\widehat{\lambda}$= 0.2936 (0.27)
TFr	$\widehat{\beta}$= 1.7435 (0.076)	$\widehat{\gamma}$= 1.9315 (0.097)	$\widehat{\lambda}$= 0.0819 (0.198)	
TLL	$\widehat{\alpha}$= 2.4602 (0.6484)	$\widehat{\beta}$= 4.011 (0.333)	$\widehat{\lambda}$= −0.0006 (1.043)	
MOFr	$\widehat{\alpha}$= 0.5988 (0.3091)	$\widehat{\beta}$= 1.5796 (0.16)	$\widehat{\gamma}$= 2.3066 (0.489)	

It is clear from Tables 3 and 4 that the Kw-TE model provide the best fits to both data sets. The histograms of the fitted distributions to data sets I and II are displayed in Figure 5. The plots support the results obtained from Tables 3 and 4. Figures 6 and 7 display the fitted cdf and the QQ plots for the Kw-TE model to the two data sets. It is evident from these plots that the Kw-TE provides good fit to the two data sets.

8. Conclusions

There is a great interest among statisticians and practitioners in the past decade to generate new extended families from classic ones. We present a new Ku-*maraswamy transmuted*-G (Kw-TG) family of distributions, which extends the transmuted family by adding two extra shape parameters. Many well-known distributions emerge as special cases of the Kw-TG family by using special para-meter values. The mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations, entropies, order statistics and probability weighted moments are pro-vided. The model parameters are estimated by the maximum likelihood method and the observed information matrix is determined. It is shown, by means of two real data sets, that special cases of the Kw-TG family can give better …ts than other models generated by well-known families.

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Appendix A:

The elements of the observed information matrix are:

$$U_{aa} = -\frac{n}{a^2},\ U_{ab} = 0,\ U_{a\lambda} = \sum_{i=1}^{n} \frac{H(x_i;\phi)\left[1 - H(x_i;\phi)\right]}{q_i},$$

$$U_{a\phi_k} = \sum_{i=1}^{n} \frac{-\lambda H'(x_i;\phi)\, H(x_i;\phi) - z_i H'(x_i;\phi)}{q_i},$$

$$U_{bb} = -\frac{n}{b^2},\ U_{b\lambda} = -\sum_{i=1}^{n} \frac{H(x_i;\phi)\left[1 - H(x_i;\phi)\right]}{1 - q_i},$$

$$U_{b\phi_k} = -\sum_{i=1}^{n} \frac{1}{1 - q_i} H'(x_i;\phi)\left\{(1+\lambda) - 2\lambda H(x_i;\phi)\right\},$$

$$\begin{aligned} U_{\lambda\lambda} \;=\;& -\sum_{i=1}^{n} \frac{\left[1 - 2H(x_i;\phi)\right]^2}{p_i^2} + (a-1)\sum_{i=1}^{n} \frac{\left[H(x_i;\phi) - H(x_i;\phi)^2\right]^2}{q_i^2} \\ & +(b-1)\sum_{i=1}^{n} \frac{\left[H(x_i;\phi) - H(x_i;\phi)^2\right]^2}{(1-q_i)^2}, \end{aligned}$$

$$\begin{aligned} U_{\lambda\phi_k} \;=\;& -\sum_{i=1}^{n} \left\{ \frac{2H'(x_i;\phi)}{p_i} + \frac{(\partial p_i/\partial\phi)\left[1 - 2H(x_i;\phi)\right]}{p_i^2} \right\} \\ & +(a-1)\sum_{i=1}^{n} \frac{\left\{H'(x_i;\phi) - 2H(x_i;\phi)\, H'(x_i;\phi)\right\}}{q_i} \\ & -(a-1)\sum_{i=1}^{n} \frac{\left\{H(x_i;\phi) - H(x_i;\phi)^2\right\} H'(x_i;\phi)\left\{z_i - \lambda H(x_i;\phi)\right\}}{q_i^2} \\ & -(b-1)\sum_{i=1}^{n} \frac{\left\{H'(x_i;\phi) - 2H(x_i;\phi)\, H'(x_i;\phi)\right\}}{(1-q_i)} \\ & +(b-1)\sum_{i=1}^{n} \frac{\left[H(x_i;\phi) - H(x_i;\phi)^2\right] H'(x_i;\phi)\left\{z_i - \lambda H(x_i;\phi)\right\}}{(1-q_i)^2} \end{aligned}$$

and

$$
\begin{aligned}
U_{\phi_k\phi_k} = & \sum_{i=1}^{n} \frac{h(x_i;\phi)\, h''(x_i;\phi) - [h'(x_i;\phi)]^2}{h(x_i;\phi)^2} \\
& -2\lambda \sum_{i=1}^{n} \frac{p_i H''(x_i;\phi) + 2\lambda [H'(x_i;\phi)]^2}{p_i^2} \\
& +(a-1)\sum_{i=1}^{n} \frac{-2\lambda H'(x_i;\phi)^2 + \{z_i - \lambda H(x_i;\phi)\} H''(x_i;\phi)}{q_i} \\
& -(a-1)\sum_{i=1}^{n} \frac{H'(x_i;\phi)\{z_i - \lambda H(x_i;\phi)\}(\partial q_i/\partial \phi_k)}{q_i^2} \\
& -(b-1)\sum_{i=1}^{n} \frac{-2\lambda H'(x_i;\phi)^2 + \{z_i - \lambda H(x_i;\phi)\} H''(x_i;\phi)}{(1-q_i)} \\
& +(b-1)\sum_{i=1}^{n} \frac{H'(x_i;\phi)\{z_i - \lambda H(x_i;\phi)\}[\partial(1-q_i)/\partial \phi_k]}{(1-q_i)^2},
\end{aligned}
$$

where $h''(x_i;\phi) = \partial^2 h(x_i;\phi)/\partial\phi_k^2$ and $H''(x_i;\phi) = \partial^2 H(x_i;\phi)/\partial\phi_k^2$

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