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An Extension of Poisson Distribution and its Applications in Human Reproduction

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Abstract: An extension of truncated Poisson distribution having two parameters for a group of two types of population is derived and named as Bounded Poisson (BP) distribution. To estimate the parameters, method of moment has been employed. To check the suitability and applicability of the model it has been applied on real data set on human fertility derived from the third round of National Family Health Survey conducted in 2005-06 in Uttar Pradesh, India. Proposed model provides a good fitting to the data under consideration.

Keywords: fertility, poisson distribution, fertility control behaviour

1. Introduction

Human birth process is a complex phenomenon which is regulated by not only biological factors but also by socio-economic, demographic and cultural factors too. In ancient time there were no ways to control (limit as well as space) fertility instead of abstaining from sex and few traditional methods like withdrawal method, rhythm method (also known as calendar method/ safe period method). These methods are not much effective and chance of conception is always high. But in modern society there are plenty of contraceptive methods available and these are highly effective as compared to traditional contraceptive methods. Thus in present time a population is consists of two kinds of women, first those women who do not use any kind of contraception, and second those who use some contraceptive method either to limiting the number of children or spacing between two children. Therefore in certain duration fertility of non-contraceptive users will be higher than fertility of contraceptive users. Thus the fertility of a population at a particular time is kind of mixture of fertility behaviour. Since the number of children ever born is count type data thus the poisson distribution is an appropriate mathematical model for studying.

Cohen in 1954 suggests that poisson distribution may be used to model data like haemocytometer counts of blood cells per square, the number of noxious weed seed per unit of field and the number of defects of per unit of a manufactured product, number of births/deaths in certain duration. Cohen (1960) introduced positive poisson distribution to describe a chance mechanism whose observational apparatus (i.e. diagnosis) becomes active only when at least one event occurs. Further (Singh, 1978) obtained a numerical example to illustrate the statistical application of the positive poisson distribution in such situations. Singh (1968) also modeled the number of births in a specified period of time using inflated version of poisson distribution. Shanmugam (1985) proposed a modified version of positive Poisson distribution considered by Cohen (1960) which is termed as the intervened poisson distribution (IPD) along with its medical applications to know the effect of intervention factor. It is widely used in medical field to explain the occurrence rare events. Later, Huang and Fung (1989) suggested intervened truncated poisson and Shanmugam (1992) again studied inferential aspect of the intervened poisson distribution. Dhanavanthan (1998; 2000) propose compound intervened poisson distribution and estimate the parameters. Scollnik (2006) proposed a generalized version of intervened poisson distribution for suitable in case of over dispersion and under dispersion. It has been observed that IPD uses in several areas of research such as reliability analysis, queuing problems, epidemiological studies etc. Kumar and Shibu (2011) modified the IPD in order to incorporate the situation of further intervention and Kumar and Shibu (2012) obtained alternative form of the truncated IPD. Further, Kumar and Shibu (2014) propose mixture of intervened poisson distribution. Also Kumar and Sreejakumari (2012) develop intervened negative binomial distribution and studied its property.

Poisson distribution is extensively used in demography for explaining phenomenon such as number of conception/births in certain duration (Feller, 1957; Brass, 1958; Dandekar, 1955; Singh and Bhattacharya, 1970). It is worthwhile to mention here the birth process affected by several intermediate variables like use of contraception, separation of couples for some time beings, age, education, media exposure and many more. Keeping the above concept, in the present paper we derive a distribution which is two parameter extension of a truncated Poisson distribution and termed it as Bounded Poisson distribution (BPD). To estimate the parameters, moment estimation procedure has been used. Since this procedure is easy in the mathematical calculation and many times need not the use of computer even. In this study our aim is to know the effect of preventing factor in order to reduce the fertility of the couple in a given period.

2. Model

Let Y be a random variable which denotes the number of births to a women. Here we are not considering Y =0 i.e. childless women because our aim is to see the effect of contraceptive use on fertility and we assume that before first birth there is very less chance to use contraceptive only about 5 percent female in Uttar Pradesh use some sort of contraceptive deliberately before first birth. In order to know the effect of preventing factors of reduction in fertility, an attempt to has been done to derive a probability model in this paper. Thus Y is a random variable which denotes number of births follows zero truncated poisson distribution as given below:

$$P(Y=y)=\frac{(e^{\theta}-1)^{-1}\theta^{y}}{y!}\ ;\ y=1,2,3...$$

where θ>0 is called incidence parameter. Let us suppose that few women are using contraceptives or any preventive techniques for limiting/spacing the fertility therefore incidence parameter θ reduces by ϕ amount resulting in incidence parameter (θ–ϕ) where 0≤ ϕ≤ θ. Thus Z is the number of births among contraceptive users and it distributed as poisson distribution given by

$$P(Z=z)=\frac{e^{(\theta-\phi)}(\theta-\phi)^{z}}{z!}\ ;\ Z=0,1,2,3...$$

The resultant fertility is the outcome of fertility among contraceptive users as well as non-contraceptive users. Thus the final distribution becomes X=Y+Z and we call it as bounded poisson distribution (BPD). The density function of BPD is given below:

$$X=Y+Z$$

$$P(X=x)=\sum_{l=0}^{x-1}P(Y=x-l).P(Z=l\,|\,Y=x-l)$$

$$P(x)=(e^{\theta}-1)^{-1}e^{-(\theta-\phi)}\sum_{l=0}^{x-1}\frac{\theta^{l}}{(x-l)!}.\frac{(\theta-\phi)^{l}}{l!}$$

$$P(x)=\frac{1}{(e^{\theta}-1)e^{(\theta-\phi)}}\left[\frac{\theta^{x}}{x!}+\frac{\theta^{x-1}}{(x-1)!}.\frac{(\theta-\phi)}{1!}+\frac{\theta^{x-2}}{(x-2)!}.\frac{(\theta-\phi)^{2}}{2!}+\frac{\theta^{x-3}}{(x-3)!}.\frac{(\theta-\phi)3}{3!}+...\frac{\theta}{1!}.\frac{(\theta-\phi)^{x-1}}{(x-1)!}\right]$$

$$P(x)=\frac{1}{(e^{\theta}-1)e^{(\theta-\phi)}}\left[\left\{\frac{\theta^{x}}{x!}+\frac{\theta^{x-1}}{(x-1)!}.\frac{(\theta-\phi)}{1!}+\frac{\theta^{x-2}}{(x-2)!}.\frac{(\theta-\phi)^{2}}{2!}+...+\frac{(\theta-\phi)^{x}}{x!}\right\}-\frac{(\theta-\phi)^{x}}{x!}\right]$$

$$P(x)=\frac{1}{(e^{\theta}-1)e^{(\theta-\phi)}}\left[\frac{\theta^{x}}{x!}\left\{1+x\frac{(\theta-\phi)}{\theta}+x(x-1)\frac{(\theta-\phi)^{2}}{\theta^{2}}+...+\frac{(\theta-\phi)^{x}}{\theta^{x}}\right\}-\frac{(\theta-\phi)^{x}}{x!}\right]$$

Thus the probability density function is given by

$$p(x)=\frac{(2\theta-\phi)^{x}-(\theta-\phi)^{x}}{(e^{\theta}-1)\,e^{(\theta-\phi)}x!}\ ;\ x=1,2,3.. \qquad (1)$$

3. Estimation of Parameters

In the present paper we have considered method of moment estimation procedure. The moments are calculated as follows

$$E(x) = \frac{\theta e^{\theta}}{(e^{\theta}-1)} + (\theta - \phi) \tag{2}$$

$$E(x^2) = \frac{1}{(e^{\theta}-1)}\left[(2\theta-\phi)^2 e^{\theta} - (\theta-\phi)^2\right] + \frac{\theta e^{\theta}}{(e^{\theta}-1)} + (\theta - \phi) \tag{3}$$

With the help of above equation we get a function of θ as

$$\Rightarrow \left[E(x) - Var(x)\right] - \frac{\theta^2 e^{\theta}}{(e^{\theta}-1)^2} = 0 \tag{4}$$

So the value of parameter θ can be obtained by solving above equation. Once value of parameter θ is estimated ϕ can be estimate using first moment given in equation 2.

4. Application of the model

We have used real dataset from third round of National Family Health Survey (NFHS-III) of India for illustration of the proposed model. Here in this study a middle age grouped (30-35 years) females of Uttar Pradesh, whose age at marriage is 16 years are considered. This group of females are more prone to use contraceptive to limit their fertility. The data on the distribution of number of births in past seven years prior to the survey have been taken with intention that adequate number of cell frequencies will be there. The proposed model is applied on the data for contraceptive users and non users separately as well as on total population and corresponding results are shown below in table 1. Total sample size, n=209 and mean and variance of the population are 1.76 and 0.82 respectively. Among these 209 women 101 women have never used any contraceptive and rest 108 has ever used any contraceptive during the study period. The distribution of number of births in seven years prior to the survey date is shown in Table 1.The mean number of children in last seven years prior to the survey is 1.73 and 1.79 for non-contraceptive users and contraceptive users respectively whereas the mean number of children is 1.76 for whole population. The variance of the population is 0.82 and 0.83 for both contraceptive users and never contraceptive users. Thus the variance is almost same among both the groups as well as for whole population. With the application of proposed procedure the estimates of parameter θ and ϕ are obtained. The estimated value of incidence parameter θ is found to be 1.13, 0.70 and 0.86 respectively for ever contraceptive users, never contraceptive users and total population.

The estimated value of reduction parameter ϕ is found to be 1.07, 0.30 and 0.59 in contraceptive users, never contraceptive users and total population respectively, which indicates that in contraceptive users the fertility is almost negligible however in non users it is on average 0.40. In the whole population reduction parameter is 0.59 and hence resulting incidence as 0.27 only. The expected frequencies obtained by proposed model are close to the observed frequencies and corresponding chi square and p-value indicates the model suitability. Since the data size is small so that there is a need of application of model somewhere else. Thus it seems likely that if more extensive data on multiple births and covering a longer period of time are available, some extension of this model using variability in incidence parameters will describe the situation more properly than this model. Thus there is a need to do simulation study and also to study the performance of parameters. It is expected to do soon separately.

Table 1: Distribution of number of births in last seven years prior to the survey date for the females aged 30-35 in Uttar Pradesh according to the contraceptive use

Number of births	Ever Users		Never Users		Total	
	Observed Number of females	Expected Number of females	Observed Number of females	Expected Number of females	Observed Number of females	Expected Number of females
1	55	54.84	49	46.75	104	100.66
2	34	34.28	29	35.06	63	70.46
3	13	13.63	18	14.10	31	27.76
4	5	4.06 }5.25	5	4.01 }5.09	10	7.93 }10.12
5	1	1.19	0	1.08	1	2.19
Total	108	108.00	101	101.00	209	209.00
θ	1.13		0.70		0.86	
ϕ	1.07		0.30		0.59	
χ^2	0.14		2.24		1.36	
p-value	0.71		0.14		0.24	
Mean	1.73		1.79		1.76	

Variance	0.83	0.83	0.82
θ-ϕ	0.06	0.40	0.27

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Appendix

The moments are calculated as follows

$$E(x)=\sum_{x=1}^{\infty} x\, p(x)$$

$$E(x)=\sum_{x=1}^{\infty} x\, \frac{(2\theta-\phi)^x-(\theta-\phi)^x}{(e^\theta-1)e^{(\theta-\phi)}x!}$$

$$E(x)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\sum_{x=1}^{\infty}\frac{(2\theta-\phi)^x-(\theta-\phi)^x}{(x-1)!}$$

$$E(x)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\left[\frac{(2\theta-\phi)^1-(\theta-\phi)^1}{0!}+\frac{(2\theta-\phi)^2-(\theta-\phi)^2}{1!}+\frac{(2\theta-\phi)^3-(\theta-\phi)^3}{2!}+\ldots\right]$$

$$E(x)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\left[\left\{\frac{(2\theta-\phi)^1}{0!}+\frac{(2\theta-\phi)^2}{1!}+\frac{(2\theta-\phi)^3}{2!}+\ldots\right\}-\left\{\frac{(\theta-\phi)^1}{0!}+\frac{(\theta-\phi)^2}{1!}+\frac{(\theta-\phi)^3}{2!}+\ldots\right\}\right]$$

$$E(x)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\left[(2\theta-\phi)\left\{1+\frac{(2\theta-\phi)^1}{1!}+\frac{(2\theta-\phi)^2}{2!}+\ldots\right\}-(\theta-\phi)\left\{1+\frac{(\theta-\phi)^1}{1!}+\frac{(\theta-\phi)^2}{2!}+\ldots\right\}\right]$$

$$E(x)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\left[(2\theta-\phi)e^{(2\theta-\phi)}-(\theta-\phi)e^{(\theta-\phi)}\right]$$

Thus

$$E(x)=\frac{1}{(e^\theta-1)}\left[(2\theta e^\theta-\theta)-\phi(e^\theta-1)\right]=\frac{(2\theta e^\theta-\theta)}{(e^\theta-1)}-\phi$$

$$E(x)=\theta\left[\frac{e^\theta+e^\theta-1}{(e^\theta-1)}\right]-\phi\Rightarrow E(x)=\frac{\theta e^\theta}{(e^\theta-1)}+(\theta-\phi)$$

Now the second moment is

$$E(x^2)=\sum_{x=1}^{\infty} x^2\, p(x)=\sum_{x=1}^{\infty} x^2\, \frac{(2\theta-\phi)^x-(\theta-\phi)^x}{(e^\theta-1)e^{(\theta-\phi)}x!}$$

$$E(x^2)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\sum_{x=1}^{\infty}\left\{x(x-1)+x\right\}\frac{(2\theta-\phi)^x-(\theta-\phi)^x}{x!}$$

$$E(x^2)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\sum_{x=1}^{\infty}x(x-1)\frac{(2\theta-\phi)^x-(\theta-\phi)^x}{x!}+\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\sum_{x=1}^{\infty}x\frac{(2\theta-\phi)^x-(\theta-\phi)^x}{x!}$$

$$E(x^2)=\frac{1}{(e^\theta-1)e^{(\theta-\phi)}}\sum_{x=2}^{\infty}\frac{(2\theta-\phi)^x-(\theta-\phi)^x}{(x-2)!}+E(x)$$

$$E(x^2) = I + E(x)$$

$$I = \frac{1}{(e^\theta - 1)e^{(\theta-\phi)}} \sum_{x=2}^{\infty} \frac{(2\theta-\phi)^x - (\theta-\phi)^x}{(x-2)!}$$

$$I = \frac{1}{(e^\theta - 1)e^{(\theta-\phi)}} \left\{ \frac{(2\theta-\phi)^2 - (\theta-\phi)^2}{0!} + \frac{(2\theta-\phi)^3 - (\theta-\phi)^3}{1!} + \frac{(2\theta-\phi)^4 - (\theta-\phi)^4}{2!} + \ldots \right\}$$

$$I = \frac{1}{(e^\theta - 1)e^{(\theta-\phi)}} \left[\left\{ \frac{(2\theta-\phi)^2}{0!} + \frac{(2\theta-\phi)^3}{1!} + \frac{(2\theta-\phi)^4}{2!} + \ldots \right\} - \left\{ \frac{(\theta-\phi)^2}{0!} + \frac{(\theta-\phi)^3}{1!} + \frac{(\theta-\phi)^4}{2!} + \ldots \right\} \right]$$

$$I = \frac{1}{(e^\theta - 1)e^{(\theta-\phi)}} \left[\left\{ \frac{(2\theta-\phi)^2}{0!} + \frac{(2\theta-\phi)^3}{1!} + \frac{(2\theta-\phi)^4}{2!} + \ldots \right\} - \left\{ \frac{(\theta-\phi)^2}{0!} + \frac{(\theta-\phi)^3}{1!} + \frac{(\theta-\phi)^4}{2!} + \ldots \right\} \right]$$

$$I = \frac{1}{(e^\theta - 1)e^{(\theta-\phi)}} \left[(2\theta-\phi)^2 \left\{ 1 + \frac{(2\theta-\phi)^1}{1!} + \frac{(2\theta-\phi)^2}{2!} + \ldots \right\} - (\theta-\phi)^2 \left\{ 1 + \frac{(\theta-\phi)^1}{1!} + \frac{(\theta-\phi)^2}{2!} + \ldots \right\} \right]$$

$$I = \frac{1}{(e^\theta - 1)e^{(\theta-\phi)}} \left[(2\theta-\phi)^2 e^{(2\theta-\phi)} - (\theta-\phi)^2 e^{(\theta-\phi)} \right]$$

$$I = \frac{1}{(e^\theta - 1)} \left[(2\theta-\phi)^2 e^\theta - (\theta-\phi)^2 \right]$$

$$\therefore \; E(x^2) = \frac{1}{(e^\theta - 1)} \left[(2\theta-\phi)^2 e^\theta - (\theta-\phi)^2 \right] + E(x)$$

$$\therefore \; E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[(2\theta-\phi)^2 e^\theta - (\theta-\phi)^2 \right]$$

$$E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[e^\theta \{\theta + (\theta-\phi)\}^2 - (\theta-\phi)^2 \right]$$

$$E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[e^\theta \{\theta^2 + (\theta-\phi)^2 + 2\theta(\theta-\phi)\} - (\theta-\phi)^2 \right]$$

$$E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[e^\theta \theta^2 + e^\theta (\theta-\phi)^2 + 2e^\theta \theta(\theta-\phi) - (\theta-\phi)^2 \right]$$

$$E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[e^\theta \theta^2 + 2e^\theta \theta(\theta-\phi) + (\theta-\phi)^2 (e^\theta - 1) \right]$$

$$E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[e^\theta \theta^2 + 2e^\theta \theta^2 - 2e^\theta \theta\phi + (\theta-\phi)^2 (e^\theta - 1) \right]$$

$$E(x^2) - E(x) = \frac{1}{(e^\theta - 1)} \left[\theta e^\theta (3\theta - 2\phi) + (\theta-\phi)^2 (e^\theta - 1) \right]$$

$$E(x^2)-E\left(x\right)=\frac{\theta e^{\theta}(3\theta-2\phi)}{(e^{\theta}-1)}+(\theta-\phi)^2$$

We know that

$$E(x)=\frac{\theta e^{\theta}}{(e^{\theta}-1)}+(\theta-\phi) \qquad \Rightarrow (\theta-\phi)=E(x)-\frac{\theta e^{\theta}}{(e^{\theta}-1)}$$

Putting the value of (θ-ϕ) in equation

$$E(x^2)-E\left(x\right)=\frac{\theta e^{\theta}(3\theta-2\phi)}{(e^{\theta}-1)}+(\theta-\phi)^2$$

We get

$$E(x^2)-E\left(x\right)=\frac{\theta e^{\theta}(3\theta-2\phi)}{(e^{\theta}-1)}+\left[E(x)-\frac{\theta e^{\theta}}{(e^{\theta}-1)}\right]^2$$

$$E(x^2)-E\left(x\right)=\frac{\theta e^{\theta}(3\theta-2\phi)}{(e^{\theta}-1)}+\left\{E(x)\right\}^2+\left\{\frac{\theta e^{\theta}}{(e^{\theta}-1)}\right\}^2-\frac{2E(x)\theta e^{\theta}}{(e^{\theta}-1)}$$

$$E(x^2)-\left\{E(x)\right\}^2-E\left(x\right)=\frac{\theta e^{\theta}(3\theta-2\phi)}{(e^{\theta}-1)}+\left\{\frac{\theta e^{\theta}}{(e^{\theta}-1)}\right\}^2-\frac{2E(x)\theta e^{\theta}}{(e^{\theta}-1)}$$

$$Var(x)-E\left(x\right)=\frac{\theta e^{\theta}}{(e^{\theta}-1)}\left[(3\theta-2\phi)+\frac{\theta e^{\theta}}{(e^{\theta}-1)}-2E(x)\right]$$

$$\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]=\left[(3\theta-2\phi)+\frac{\theta e^{\theta}}{(e^{\theta}-1)}-2E(x)\right]$$

$$\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]+2E(x)-3\theta-\frac{\theta e^{\theta}}{(e^{\theta}-1)}=-2\varphi$$

Now putting the value of ϕ in equation of mean, we get

$$2E(x)=\frac{2\theta e^{\theta}}{(e^{\theta}-1)}+2\theta+\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]+2E(x)-3\theta-\frac{\theta e^{\theta}}{(e^{\theta}-1)}$$

$$\Rightarrow \frac{2\theta e^{\theta}}{(e^{\theta}-1)}-\frac{\theta e^{\theta}}{(e^{\theta}-1)}-3\theta+2\theta+\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]=0$$

$$\Rightarrow \frac{\theta e^{\theta}}{(e^{\theta}-1)}-\theta+\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]=0$$

$$\Rightarrow \theta\left\{\frac{e^{\theta}}{(e^{\theta}-1)}-1\right\}+\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]=0$$

$$\Rightarrow \theta\left\{\frac{e^{\theta}-e^{\theta}+1}{(e^{\theta}-1)}\right\}+\left[Var(x)-E\left(x\right)\right]\left[\frac{(e^{\theta}-1)}{\theta e^{\theta}}\right]=0$$

$$\Rightarrow \frac{\theta}{(e^{\theta}-1)}+\frac{(e^{\theta}-1)}{\theta e^{\theta}}\left[Var(x)-E\left(x\right)\right]=0$$

$$\Rightarrow \frac{\theta}{(e^{\theta}-1)}=-\frac{(e^{\theta}-1)}{\theta e^{\theta}}\left[Var(x)-E\left(x\right)\right]$$

$$\Rightarrow \frac{\theta}{(e^{\theta}-1)}=\frac{(e^{\theta}-1)}{\theta e^{\theta}}\left[E\left(x\right)-Var(x)\right]$$

$$\Rightarrow \left[E\left(x\right)-Var(x)\right]=\frac{\theta^{2}e^{\theta}}{(e^{\theta}-1)^{2}} \Rightarrow \left[E(x)-Var(x)\right]-\frac{\theta^{2}e^{\theta}}{(e^{\theta}-1)^{2}}=0$$