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Estimation of lifetime distribution parameters with general progressive censoring from Imprecise data

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*Abstact:*The problem of estimating lifetime distribution parameters under general progressive censoring originated in the context of reliability. But traditionally it is assumed that the available data from this censoring scheme are performed in exact numbers. However, in many life testing and reliability studies, it is not possible to obtain the measurements of a statistical experiment exactly, but is possible to classify them into fuzzy sets. This paper deals with the estimation of lifetime distribution parameters under general progressive Type-II censoring scheme when the lifetime observations are reported by means of fuzzy numbers. A new method is proposed to determine the maximum likelihood estimates of the parameters of interest. The methodology is illustrated with two popular models in lifetime analysis, the Rayleigh and Lognormal lifetime distributions.

Keywords: General progressive censoring, Maximum likelihood estimation, EM algorithm, Missing information

1 Introduction

The scheme of progressive censoring is of great value in life-testing experiments. Its allowance for the removal of life units from the experiment at various stages is an attractive feature as it will potentially save a lot for the experimenter in terms of cost and time. Some of the early works on progressive censoring was conducted by Cohen (1963), Balakrishnan and Asgharzadeh (2005), Balakrishnan et. al. (2003) and Raqab and Madi (2011). This scheme of censoring was generalized by Balakrishnan and Sandhu (1996) as follows. Suppose that n randomly selected units are placed on a life test. The failure times of the first r units to fail are not observed; Immediately following the (r + 1) observed failure, Rr+1 number of surviving units are removed from the test randomly. Similarly, following the (r + 2) observed failure, Rr+2 number of surviving units are removed from the test. This process continue until, immediately following the *m*th observed failure, all the remaining $R_m = n - m - \sum_{j=r+1}^{m-1} R_j$ units are removed from the test. The R_i's, m and *r* are pre-specified integers which must satisfy the conditions: $0 \le r < m \le n$, $0 \le R_i \le n_{i-1} - 1$ for $i = r + 1,...,m - 1$ with $n_r = n - r$

and Rm = nm−1 − 1. Statistical analysis of lifetime distributions under this general progressive censoring scheme have been considered by several authors; See for example, Abdelrahman and Sultan (2007), Kim and Han (2009) and Fern¨andez (2004).

The above research results for estimating parameters of different lifetime distributions under general progressive Type-II censoring are limited to precise data. However, in many fields of application, it is sometimes impossible to obtain exact observations of lifetime. The obtained lifetime data may be imprecise most of the time. For instance, consider a life-testing experiment in which n identical batteries are placed on a test, and we are interested in the lifetime of these batteries. A tested battery may be considered as failed, or -strictly speaking- as nonconforming, when at least one value of its parameters falls beyond specification limits. In practice, however, the observer does not have the possibility to measure all parameters and is not able to define precisely the moment of failures, but rather he/she can only approximate them by means of the following imprecise quantities: "approximately lower than 1000 hours ", "approximately 1500 to 2000 hours", "approximately 2500 hours", "approximately 3000 hours", "approximately 3500 to 4000 hours", "approximately higher than 4500 hours", and so on. Classical statistical procedures and Bayesian inference are not appropriate to deal with such imprecise cases. In order to model imprecise lifetimes, a generalization of real numbers is necessary. These lifetimes can be represented by fuzzy numbers. A fuzzy number is a subset, denoted by $\tilde{x}$, of the set of real numbers (denoted by R) and is characterized by the so called membership function $\mu_{\tilde{x}}(.)$. Fuzzy numbers satisfy the following constraints (Dubois and Prade (1980)):

(1) $\mu_{\tilde{x}} : \mathrm{R} \longrightarrow [0,1]$ is Borel-measurable;

(2) $\exists x_0 \in \mathrm{R} : \mu_{\tilde{x}}(x_0) = 1$;

(3) The so-called λ−cuts $(0 < \lambda \le 1)$, defined as $B_\lambda(\tilde{x}) = \{x \in \mathrm{R} : \mu_{\tilde{x}}(x) \ge \lambda\}$, are

all closed interval, i.e., $B_\lambda(\tilde{x}) = [a_\lambda, b_\lambda]$, $\forall \lambda \in (0,1)$.

Among the various types of fuzzy numbers, LR-type fuzzy numbers (the triangular and trapezoidal fuzzy numbers are special cases of the LR-type fuzzy numbers) are most convenient and useful in describing fuzzy lifetime data.

Definition 1 (Zimmermann (1991) pp.62). Let L (and R) be decreasing, shape functions from R+ to [0,1] with L(0) = 1; L(x) < 1 for all x > 0; L(x) > 0 for all x < 1; L(1) = 0 or (L(x) > 0 for all x and L(+∞) = 0). Then a fuzzy number $\tilde{x}$ is called of LR-type if for c,α > 0, β > 0 in R,

$$\mu_{\tilde{x}}(x) = \begin{cases} L(\frac{c-x}{\alpha}) & x \le c \\ R(\frac{x-c}{\beta}) & x \ge c \end{cases}$$

where c is called the mean value of $\tilde{x}$ and α and β are called the left and right spreads, respectively. Symbolically, the LR-type fuzzy number is denoted by $\tilde{x} = (\alpha, c, \beta)$.

In recent years, several researchers pay attention to applying the fuzzy sets to estimation theory. Tanaka et al. (1979) determined the maximum possibility of system failure from the possibility of failure of each component within the system based on a fuzzy fault-tree model. Huang et al. (2006) proposed a new method to determine the membership function of the estimates of the parameters and the reliability function of multiparameter lifetime distributions. Coppi et al. (1991) presented some applications of fuzzy techniques in statistical analysis. Pak et al. (2013a),(2013b) conducted a series of studies to develop the inferential procedures for the lifetime distributions on the basis of fuzzy numbers. In this paper, our objective is to study the maximum likelihood estimation procedure for the lifetime distribution parameters when the general progressively censored data are reported in the form of fuzzy numbers. In Section 2, we introduce a generalization of the likelihood function under general progressive Type-II censoring and obtain the maximum likelihood estimates in general setting. Two popular models in lifetime analysis, via, Rayleigh and Lognormal distributions, are used to illustrate the proposed method, respectively, in Sections 3 and 4.

2 Maximum likelihood estimation

Suppose that n independent units are placed on a life-test with the corresponding lifetimes X1,...,Xn. As usual, it is assumed that Xi, i = 1,...,n are independent and identically distributed with probability density function fX(x;θ), where θ denotes the vector of parameters. Let X = (Xr+1,Xr+2,...,Xm) denotes a general progressively Type-II censored sample with (Rr+1,Rr+2,...,Rm) being the progressive censoring scheme. If a realization x of X was known exactly, we could obtain the likelihood function based on
this general progressively censored sample as

$$\ell(\boldsymbol{\theta}) = \binom{n}{r}\left(\prod_{i=r}^{m-1} n_i\right)\left[F_X(x_{r+1};\boldsymbol{\theta})\right]^r \prod_{i=r+1}^{m} f_X(x_i;\boldsymbol{\theta})\left[1-F_X(x_i;\boldsymbol{\theta})\right]^{R_i}, \quad (1)$$

in which FX(x;θ) is the cumulative distribution function. Now consider the problem where under a general progressive Type-II censoring scheme, failure times are not observed precisely and only partial information about them are available in the form of fuzzy numbers ˜xr+1 = (αr+1,cr+1,βr+1),...,x˜m = (αm,cm,βm) , with the corresponding membership functions µx˜r+1(.),...,µx˜m(.). Let c(r+1) ≤ c(r+2) ≤ ... ≤ c(m) denote the ordered values of the means of these fuzzy numbers. The lifetime of Rr+i, i = 1,...,m surviving units, which are removed from the test after the (r+i)th failure, can be encoded as fuzzy numbers ˜z(r+i)1,...,z˜(r+i)Rr+i with the membership

$$\mu_{\tilde{z}_{(r+i)j}}(z) = \begin{cases} 0 & z \le c_{(r+i)} \\ 1 & z > c_{(r+i)} \end{cases}, \qquad j = 1, ..., R_{r+i}.$$

Also the lifetimes of the first r missing units can be modeled by fuzzy numbers ˜y1,...,y˜r with the membership functions

$$\mu_{\tilde{y}_k}(y) = \begin{cases} 1 & y < c_{(k)} \\ 0 & y \le c_{(k)} \end{cases}, \qquad k = 1, ..., r.$$

The fuzzy data w˜ = (y˜1,...,y˜r,x˜r+1,...,x˜m,˜zr+1,...,˜zm), where ˜zl is a 1×Rr+i vector with
˜zl = (z˜(r+i)1,z˜(r+i)2,...,z˜(r+i)Rr+i), for l = r+1,...,m, is thus the set of observed lifetimes. The corresponding observed-data log-likelihood function can be obtained, using Zadeh's
definition of the probability of a fuzzy event (Zadeh (1968)), as

$$\begin{aligned} L_O(\tilde{\mathbf{w}};\boldsymbol{\theta}) &= \sum_{k=1}^{r}\log\left\{\int \mu_{\tilde{y}_k}(y) f(y;\boldsymbol{\theta})dy\right\} + \sum_{l=r+1}^{m}\log\left\{\int \mu_{\tilde{x}_l}(x) f(x;\boldsymbol{\theta})dx\right\} \\ &+ \sum_{l=r+1}^{m}\sum_{j=1}^{R_l}\log\left\{\int \mu_{\tilde{z}_{lj}}(z) f(z;\boldsymbol{\theta})dz\right\}. \end{aligned} \quad (2)$$

Since the observed fuzzy data w˜ can be seen as an incomplete specification of a complete data vector w, the EM algorithm is applicable to obtain the maximum likelihood estimates (MLE) of the parameters. The EM algorithm, introduced by Dempster et al. (1977), is a very popular tool to handle any missing or incomplete data situation. This algorithm is an iterative method which has two steps. In the E-step, it replaces any missing data by its expected value and in the M-step the log-likelihood function is maximized with the observed data and expected value of the incomplete data, producing an update of the parameter estimates. In the following, we use the EM algorithm to determine the MLE of θ.

First of all, denote the lifetime of the missing, failed and censored units by Y = (Y1,...,Yr), X = (Xr+1,...,Xm) and Z = (Zr+1,...,Zm), respectively, where Zl is a 1 × Rr+i vector with Zl = (Z(r+i)1,...Z(r+i)Rr+i), for l = r + 1,...,m. The combination of (Y,X,Z) = W forms the complete lifetimes and the corresponding log-likelihood function is denoted by L(W;θ). The E-step of EM algorithm requires the calculation of

$$E_{(h)}\left(L(\mathbf{W};\theta) \mid \mathbf{w}^{\sim}\right), \tag{3}$$

which mainly involves the computation of the conditional expectation of functions of Y, X and Z conditional on the observed values y˜ = (y˜1,...,y˜r), x˜ = (x˜r+1,...,x˜m) and ˜z = (˜zr+1,...,˜zm), respectively, and the current value of the parameters. To this end, we need to determine the conditional probability of Y, X and Z given y˜, x˜ and ˜z,

respectively, from the following formula:

$$g(\mathbf{u} \mid \tilde{\mathbf{u}}; \boldsymbol{\theta}^{(h)}) = \frac{\mu_{\tilde{\mathbf{u}}}(\mathbf{u}) f(\mathbf{u}; \boldsymbol{\theta}^{(h)})}{\int \mu_{\tilde{\mathbf{u}}}(\mathbf{u}) f(\mathbf{u}; \boldsymbol{\theta}^{(h)}) d\mathbf{u}}. \tag{4}$$

In the M-step on the (h+1)th iteration of the algorithm, the value of θ which maximizes E (h) (L(W;θ) | w˜) will be used as the next estimate of θ(h+1). The MLE of θ can be θ obtained by repeating the E- and M-step until convergence occurs.

3 Rayleigh lifetime data

The Rayleigh distribution provides a population model which is useful in several areas of statistics, including life testing and reliability. Polovko (1968) and Dyer and Whisenand (1973) demonstrated the importance of this distribution in electro vacuum devices and communication engineering. The probability density function (p.d.f.) of the Rayleigh distribution is defined as

$$f(y;\sigma) = \frac{y}{\sigma^2}\exp(-\frac{y^2}{2\sigma^2}) \ , \qquad y > 0, \ \sigma > 0. \tag{5}$$

In this case the log-likelihood function based on the complete lifetimes W becomes proportional to

$$L(\mathbf{W};\sigma) \propto -2n\log\sigma - \frac{1}{2\sigma^2}\left[\sum_{k=1}^{r} y_k^2 + \sum_{l=r+1}^{m} x_l^2 + \sum_{l=r+1}^{m}\sum_{j=1}^{R_l} z_{lj}^2\right]. \tag{6}$$

In the E-step, one needs to compute

$$-2n\log\sigma - \frac{1}{2\sigma^2}\left[\sum_{k=1}^{r} E_{\sigma^{(h)}}(Y_k^2 \mid \tilde{y}_k) + \sum_{l=r+1}^{m} E_{\sigma^{(h)}}(X_l^2 \mid \tilde{x}_l) + \sum_{l=r+1}^{m}\sum_{j=1}^{R_l} E_{\sigma^{(h)}}(Z_{lj}^2 \mid \tilde{z}_{lj})\right], \quad (7)$$

where $\sigma^{(h)}$ denotes the current fit of σ at iteration h.The conditional expectations $\gamma_k^{(h)} = E_{\sigma^{(h)}}(Y_k^2 \mid \tilde{y}_k),\ \alpha_l^{(h)} = E_{\sigma^{(h)}}(X_l^2 \mid \tilde{x}_l)$ and $\beta_{lj}^{(h)} = E_{\sigma^{(h)}}(Z_{lj}^2 \mid \tilde{z}_{lj})$ can be computed using

$$E_{\sigma^{(h)}}(U^2 \mid \tilde{u}) = \frac{\int u^2 \mu_{\tilde{u}}(u) f(u;\sigma^{(h)})du}{\int \mu_{\tilde{u}}(u) f(u;\sigma^{(h)})du}. \quad (8)$$

Hence, in the $(h + 1)th$ iteration, the values of $\sigma^{(h+1)}$ are computed by the following

formula:

$$\hat{\sigma}^{(h+1)} = \left\{\frac{1}{2n}\left(\sum_{k=1}^{r}\gamma_k^{(h)} + \sum_{l=r+1}^{m}\alpha_l^{(h)} + \sum_{l=r+1}^{m}\sum_{j=1}^{R_l}\beta_{lj}^{(h)}\right)\right\}^{1/2}. \quad (9)$$

In order to assess the accuracy of the MLEs computed through the procedure described above, we have carried out a simulation study. First, for different choices of n , m , σ and (R1,...,Rm), we have generated progressively censored sample x1,...,xm from Rayleigh distribution using the method proposed by Balakrishnan and Sandhu (1996). Then we have defined fuzzy numbers ˜x1,...,x˜m with the corresponding membership functions

$$\mu_{\tilde{x}_i}(x) = \begin{cases} \frac{x-(x_i-h_i)}{h_i} & x_i - h_i < x \le x_i \\ \frac{x_i+h_i-x}{h_i} & x_i < x \le x_i + h_i \end{cases}, i = 1, ..., m.$$

where hi = 0.05xi. This procedure simulates the situation where the observer has only approximate knowledge of the failure times, and can only provide a guess xi and an interval of plausible values [xi − hi,xi + hi]. From these fuzzy numbers, we obtain the MLE of σ, using the iterative algorithm (9). We have used the initial estimate to be $\sigma^{(0)} = (\frac{1}{2m}\sum_{i=1}^{m} x_i^2)^{1/2}$. The iterative process stops when the relative change of
the log-likelihood becomes less than 10−6. The average values (AV) and mean squared errors(MSE) of the estimates based on 1000 replication are presented in Table 1. From this table we observe that, the performance of the MLEs are satisfactory in terms of AVs and MSEs. For fixed n as m increases, the MSEs decrease for all cases as expected.

Example 2. A general progressively Type-II censored sample from the data on the failure times of 23 ball bearings in endurance test (Table 2) is used to demonstrate the above estimation procedure. For this data set, Pak et al. (2013) considered doubly type II censoring scheme to determine the maximum likelihood estimate of σ from a Rayleigh distribution. However, in practice, measuring the lifetime of a ball bearing may not yield an exact result. A ball bearing may work perfectly over a certain period but be braking for some time, and finally be unusable at a certain time. So, such data may be reported as imprecise quantities. Assume that imprecision of the failure times of ball bearings is formulated by fuzzy numbers ˜xi = (hi,xi), where hi = 0.005xi, i = 1,...,16, with membership functions

$$\mu_{\tilde{x}_i}(x) = \begin{cases} \frac{x-(x_i-h_i)}{h_i} & x_i - h_i \le x \le x_i \\ 0 & x > x_i \end{cases} \quad , i = 1, ..., 16.$$

From these fuzzy data and using the starting value $\sigma^{(0)} = (\frac{1}{32}\sum_{i=1}^{16} x_i^2)^{1/2}$=40.0687, which is the estimate of the parameter computed over the centers of each fuzzy numbers, the

Table 1: The average values (AV) and mean squared errors (MSE) for the MLE of σ for different sample sizes and different sampling schemes.

n	m	Censoring Scheme	$\sigma = 1$		$\sigma = 2$	
			AV	MSE	AV	MSE
20	5	(0,0,0,0,15)	0.9793	0.0497	1.9548	0.2108
20	5	(15,0,0,0,0)	0.9777	0.0474	1.9520	0.1994
20	5	(0,15,0,0,0)	0.9836	0.0511	1.9561	0.2049
20	10	(0,...,0,10)	0.9865	0.0258	1.9634	0.0996
20	10	(10,0,...,0)	0.9849	0.0234	1.9754	0.0963
20	10	(0,10,0,...,0)	0.9934	0.0231	1.9735	0.0911
20	15	(0,...,0,5)	0.9923	0.0162	1.9942	0.0662
20	15	(5,0,...,0)	0.9909	0.0174	1.9782	0.0662
20	15	(0,5,0,...,0)	0.9934	0.0160	1.9751	0.0672
30	10	(0,...,0,20)	0.9830	0.0239	1.9823	0.0945
30	10	(20,0,...,0)	1.0138	0.0233	1.9714	0.0923
30	10	(0,20,0,...,0)	0.9799	0.0227	1.9853	0.0910
30	15	(0,...,0,15)	0.9890	0.0154	1.9966	0.0642
30	15	(15,0,...,0)	0.9890	0.0166	1.9872	0.0627
30	15	(0,15,0,...,0)	0.9886	0.0152	1.9873	0.0661
30	20	(0,...,0,10)	0.9947	0.0130	1.9944	0.0519
30	20	(10,0,...,0)	0.9933	0.0127	1.9949	0.0509
30	20	(0,10,0,...,0)	1.0034	0.0131	1.9920	0.0478
50	20	(0,...,0,30)	0.9916	0.0120	1.9892	0.0491
50	20	(30,0,...,0)	0.9880	0.0122	1.9789	0.0486
50	20	(0,30,0,...,0)	0.9972	0.0124	1.9837	0.0472
50	25	(0,...,0,25)	0.9927	0.0102	1.9967	0.0404
50	25	(25,0,...,0)	0.9936	0.0099	1.9893	0.0407
50	25	(0,25,0,...,0)	1.0012	0.0100	1.9869	0.0407
50	30	(0,...,0,20)	0.9956	0.0074	2.0021	0.0330
50	30	(20,0,...,0)	0.9939	0.0084	1.9901	0.0345
50	30	(0,20,0,...,0)	0.9991	0.0080	1.9913	0.0315

Table 2: Progressively censored sample for Example 2

i	1	2	3	4	5	6	7	8
x_i	17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84
R_i	2	0	0	1	0	0	1	0
i	9	10	11	12	13	14	15	16
x_i	51.96	54.12	55.56	67.80	68.64	68.88	84.12	93.12
R_i	0	0	1	0	0	0	0	2

final MLE of σ is found from (9) to be $\hat{\sigma}$ = 48.8245. Fig. 1 shows a plot of the observeddata log-likelihood as a function of $\sigma^{(h)}$. We can check that the MLE corresponds in this case to a global maximum of the observed-data log-likelihood.

4 Lognormal lifetime data

Lognormal distribution is another commonly used lifetime distribution model in lifetime data analysis. Let X be the original lifetime variable that follows a Lognormal distribution with parameters μ and σ. The density of X is given by

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{(2\pi)}\sigma x} \exp\left[-\frac{1}{2\sigma^2}(\log x - \mu)^2\right] \quad ,x > 0 \tag{10}$$

where μ and σ are the scale and shape parameters, respectively. Let θ = (μ,σ) denotes the vector of parameters. The log-likelihood function based on the complete lifetime and

$$E_{\boldsymbol{\theta}^{(h)}}(\log U \mid \tilde{u}) = \frac{\int (\log u)\, \mu_{\tilde{u}}(u) f(u;\mu^{(h)},\sigma^{(h)})du}{\int \mu_{\tilde{u}}(u) f(u;\mu^{(h)},\sigma^{(h)})du}. \tag{11}$$

From the usual results for complete data maximum likelihood estimation for lognormal distribution, the explicit formulas for the MLE of μ and σ are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log w_i ,$$

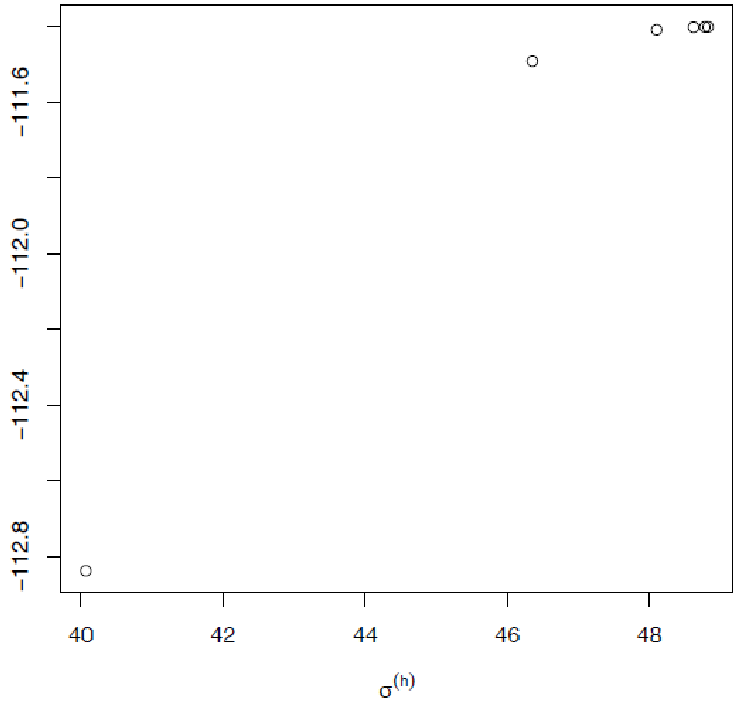

Figure 1: Plot of the observed-data log-likelihood as a function of $\sigma^{(h)}$

$$\hat{\sigma} = \left[\frac{1}{n}\sum_{i=1}^{n}(\log w_i - \hat{\mu})^2\right]^{1/2}.$$

Therefore, in the $(h+1)$th iteration of algorithm, the value of $\mu^{(h+1)}$ and $\sigma^{(h+1)}$ are computed by the following formulas:

$$\hat{\mu}^{(h+1)} = \frac{1}{n}\left[\sum_{k=1}^{r}\zeta_k^{(h)} + \sum_{l=r+1}^{m}\gamma_l^{(h)} + \sum_{l=r+1}^{m}\sum_{j=1}^{R_l}\eta_{lj}^{(h)}\right], \tag{12}$$

$$\hat{\sigma}^{(h+1)} = \left\{\frac{1}{n}\left[\sum_{k=1}^{r}\delta_k^{(h)} + \sum_{l=r+1}^{m}\alpha_l^{(h)} + \sum_{l=r+1}^{m}\sum_{j=1}^{R_l}\beta_{lj}^{(h)}\right] - \left(\hat{\mu}^{(h+1)}\right)^2\right\}^{1/2}. \tag{13}$$

Example 3. To illustrate experimentally the method presented in this section, we perform the following experiment. We first generated a general progressive Type-II censored sample of size $m = 15$ from standard lognormal distribution. The data are presented in Table 3. Then, each realization of lifetimes was fuzzified by fuzzy numbers

Table 3: Simulated progressively censored sample from standard lognormal distribution

i	1	2	3	4	5	6	7	8
x_i	0.2721	0.2910	0.2929	0.3882	0.5594	0.5831	0.7041	0.8272
R_i	1	0	0	1	0	0	1	0
i	9	10	11	12	13	14	15	
x_i	0.8454	0.8963	1.1084	1.7867	2.0148	2.2213	4.2340	
R_i	0	0	0	0	0	0	2	

$\tilde{x}_1,...,\tilde{x}_m$ with the corresponding membership function

$$\mu_{\tilde{x}_i}(x) = \begin{cases} \exp\left(-(x_i - x)^2\right) & x \le x_i \\ \exp\left(-(x - x_i)^2\right) & x > x_i \end{cases} \quad , i = 1, ..., m.$$

Since the mean and standard deviation of the 15 observed sample points equal to −0.2015 and 0.8192, respectively, thus we can put $\mu(0) = -0.2015$ and $\sigma(0) = 0.8192$ as the starting values of the EM algorithm. After a few iterations, the estimates in (15) and(16) converge to the values $\hat{\mu} = 0.1276$ and $\hat{\sigma} = 1.0161$.

5 Conclusion

In this paper, we have proposed a new method to determine maximum likelihood estimates of the parameters of lifetime distributions regarding a life-test from which the general progressive Type-II censored data are reported in the form of fuzzy numbers. We analyze the data under the assumptions that the lifetimes of the test units are independent identically distributed Rayleigh and Lognormal random variables. For the two cases, the subsequent guesses of the parameters are in explicit expression. The study of the applicability of the proposed approach in estimating the parameters of lifetime distributions under the other censoring schemes such as Hybrid Type-II and Hybrid progressive Type-II censoring are possible topics for further research.

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