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Odds Generalized Exponential – Exponential Distribution

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Abstract: A new distribution, called Odds Generalized Exponential-Exponential distribution (OGEED) is proposed for modeling lifetime data. A comprehensive account of the mathematical properties of the new distribution including estimation and simulation issues is presented. A data set has been analyzed to illustrate its applicability.

Key words: Exponential distribution; Maximum likelihood estimation; Odds function; T-X family of distributions.

1. Introduction

There are several ways of adding one or more parameters to a distribution function. Such an addition of parameters makes the resulting distribution richer and more flexible for modeling data. Proportional hazard model (PHM), Proportional reversed hazard model (PRHM), Proportional odds model (POM), Power transformed model (PTM) are few such models originated from this idea to add a shape parameter. In these models, a few pioneering works are by Box and Cox (1964),Cox (1972), Mudholkar and Srivastava (1993), Shaked and Shantikumar (1994), Marshall and Olkin (1997), Gupta and Kundu (1999), Gupta and Gupta (2007) among others.

Many distributions have been developed in recent years that involves the logit of the beta distribution. Under this generalized class of beta distribution scheme, the cumulative distribution function (cdf) for this class of distributions for the random variable   is generated by applying the inverse of the cdf of   to a beta distributed random variable to obtain,

$$F(x) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1} dt; \alpha, \beta > 0,$$

where is the cdf of any other distribution. This class has not only generalized the beta distribution but also added parameter(s) to it. Among this class of distributions are, the beta-Normal [Eugene et al. (2002)]; beta-Gumbel [Nadarajah and Kotz (2004)]; beta-Exponential [Nadarajah and Kotz (2006)]; beta-Weibull [Famoye et al. (2005)]; beta-Rayleigh [Akinsete and Lowe (2009)]; beta-Laplace [Kozubowski and Nadarajah (2008)]; and beta-Pareto [Akinsete et al. (2008)], among a few others. Many useful statistical properties arising from these distributions and their applications to real life data have been discussed in the literature.

In the generalized class of beta distribution, since the beta random variable lies between 0 and 1, and the distribution function also lies between 0 and 1, to find out cdf of generalized distribution, the upper limit is replaced by cdf of the generalized distribution.

Alzaatreh et al. (2013) has proposed a new generalized family of distributions, called T-X family, and the cumulative distribution function (cdf) is defined as

$$F(x \mid \lambda, \theta) = \int_a^{W(F_\theta(x))} f_\lambda(t)dt, \tag{1.1}$$

where, the random variable , for and be a function of the cdf so that satisfies the following conditions:

(i). $W(F_\theta(x)) \in [a, b]$

(ii). $W(F_\theta(x))$ is differentiable and monotonically non-decreasing,

(iii). $W(F_\theta(x)) \to a$ as $x \to -\infty$ and $W(F_\theta(x)) \to b$ as $x \to \infty$.

We have defined a generalized class of any distribution having positive support. Taking $W(F_\theta(x)) = \dfrac{F_\theta(x)}{1 - F_\theta(x)}$, the odds function, the cdf of the proposed generalized class of distribution is given by

$$F(x \mid \lambda, \theta) = \int_0^{\frac{F_\theta(x)}{1-F_\theta(x)}} f_\lambda(t)dt. \tag{1.2}$$

The support of the resulting distribution will be that of $F_\theta(.)$. here,

$$\frac{F_\theta(x)}{1 - F_\theta(x)} = \frac{F_\theta(x)}{\overline{F}_\theta(x)} = \infty \text{ as } x \to \infty \text{ (assuming } \frac{1}{0} = \infty \text{).}$$

The resulting distribution is not only generalized but also added with some parameter(s) to the base distribution. We call this class of distributions as Odds Generalized family of distributions (OGFD).

Throughout this paper we use the following notations. We write upper incomplete gamma function and lower incomplete gamma function as $\Gamma(p, x) = \int_x^\infty w^{p-1}e^{-w}dw$ and $\gamma(p, x) = \int_0^x w^{p-1}e^{-w}dw$, for x ≥ 0, p > 0 respectively. The jth derivative with respect to p is denoted by $\Gamma^{(j)}(p, x) = \int_x^\infty (\ln w)^j w^{p-1}e^{-w}dw$ and $\gamma^{(j)}(p, x) = \int_0^x (\ln w)^j w^{p-1}e^{-w}dw$, for x ≥ 0, p > 0 respectively.

In the present paper, we choose particular choice of $F_{\lambda}(x)=1-e^{-\lambda x}$ i.e. the exponential distribution and $F_{\theta}(x)=1-e^{-\theta x}$ i.e. also the exponential distribution in (1. 2). Hence, we call this distribution as Odds Generalized Exponential-Exponential distribution (OGEED).

The paper is organized as follows. The distribution is developed in section 2. A comprehensive account of mathematical properties including structural and reliability of the new distribution is provided in section 3. Maximum likelihood method of estimation of parameters of the distribution is discussed in section 4. Simulation study results have been presented and dicussed in section 5.A real life data set has been analyzed and compared with other fitted distributions with respect to Akaike Information Criterion (AIC) in section 6. Section 7 concludes.

2. The Probability Density Function of the OGEED

The c.d.f. of the OGEED is given by the form as

$$F(x)=\int_0^{\frac{G(x)}{1-G(x)}} f(x)dx$$

where $G(x)=1-e^{-\theta x}$ and $f(x)=\lambda e^{-\lambda x}$, so that

$$F(X;\lambda,\theta)=\int_0^{(e^{\theta x}-1)}\lambda e^{-\lambda x}dx=1-e^{-\lambda(e^{\theta x}-1)} \tag{2.3}$$

Also the p.d.f. of the OGEED is given by the form as

$$f(x;\lambda,\theta)=\frac{dF(X;\lambda,\theta)}{dx}=\lambda\theta e^{\theta x}e^{-\lambda(e^{\theta x}-1)} \tag{2.4}$$

with range (0, ∞).

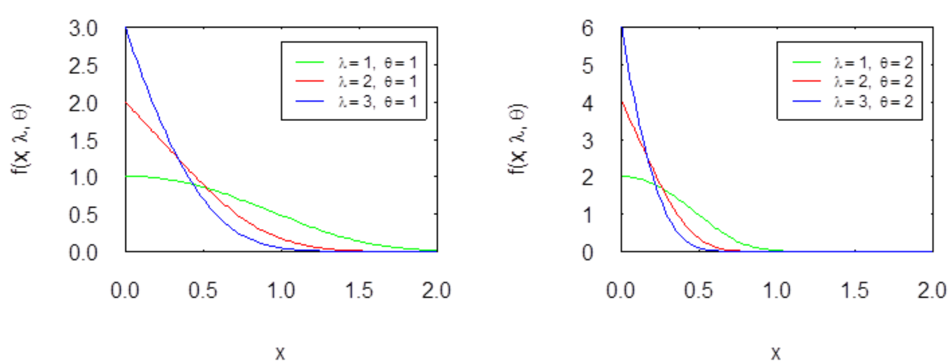

Figure 1: The probability density function of OGEED with θ=1 and 2 with λ= 1,2,3.

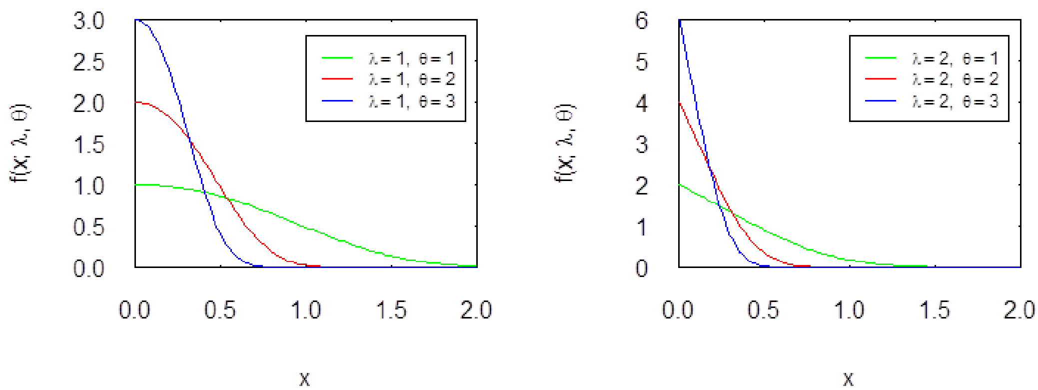

Figure 2: The probability density function of OGEED with λ =1 and 2 with θ = 1,2,3.

3. Statistical and Properties

3.1 Limit of the Probability Distribution Function

Since the c.d.f. of this distribution is

$$F(X;\lambda,\theta)=1-e^{-\lambda(e^{\theta x}-1)}$$

So,

$$\lim_{x\to 0}F(X;\lambda,\theta)=\lim_{x\to 0}(1-e^{-\lambda(e^{\theta x}-1)})=0 \text{ i.e. } F(0)=0$$

Also,

$$\lim_{x\to \infty}F(X;\lambda,\theta)=\lim_{x\to \infty}(1-e^{-\lambda(e^{\theta x}-1)})=1 \text{ i.e. } F(\infty)=1$$

3.2 Descriptive Statistics of OGEED

The mean of this OGEED is as follows:

$$\mu_1^{'}=E(X)=\lambda\theta\int_0^{\infty}xe^{\theta x}e^{-\lambda(e^{\theta x}-1)}dx$$

Put $u=e^{\theta x}-1$, we get

$$E(X)=\lambda\int_0^{\infty}\frac{1}{\theta}ln(1+u)e^{-\lambda u}du$$

$$= \frac{1}{\theta}\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{\lambda^{j+1}}$$

$$= \frac{e^{\lambda}}{\theta}\left[\Gamma^{(1)}(1,\lambda) - \ln\lambda.\Gamma(1,\lambda)\right]$$

So mean of the OGEED is

$$\frac{1}{\theta}\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{\lambda^{j+1}} \text{ or } \frac{e^{\lambda}}{\theta}\left[\Gamma^{(1)}(1,\lambda) - \ln\lambda.\Gamma(1,\lambda)\right].$$

The median of the OGEED is given by

$$0.5 = \int_0^m f(x;\lambda,\theta)dx$$

$$= 1 - e^{-\lambda(e^{\theta m}-1)}.$$

That gives,

$$m = \frac{ln(1+\frac{ln2}{\lambda})}{\theta}$$

Hence median of the OGEED is

$$\frac{ln(1+\frac{ln2}{\lambda})}{\theta}$$

The mode of the OGEED is given as:

$$mode = argmax(f(x))$$

Now ,

$$\frac{d}{dx}lnf(x) = \theta - \lambda\theta e^{\theta x} = 0 \quad \Rightarrow x = \frac{ln(\frac{1}{\lambda})}{\theta}.$$

So mode of OGEED is

$$\frac{ln(\frac{1}{\lambda})}{\theta}$$

The rth order raw moment of this OGEED is as follows:

$$E(X^r) = \lambda\theta\int_0^{\infty}x^r e^{\theta x} e^{-\lambda(e^{\theta x}-1)}dx$$

Put $u = e^{\theta x} - 1$, we get

$$E(X^r) = \frac{\lambda e^{\lambda}}{\theta^r} \int_1^{\infty} (\ln w)^r e^{\lambda w} dw$$

$$= \frac{e^{\lambda}}{\theta^r} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} (\ln \lambda)^{r-j} \Gamma^{(j)}(1, \lambda), \tag{3.5}$$

where $\Gamma^{(j)}(1,\lambda) = \frac{\partial^j \Gamma(p,\lambda)}{\partial p^j}|_{p=1}$ with $\Gamma(p,\lambda) = \int_{\lambda}^{\infty} x^{p-1} e^{-x} dx$.

Now putting suitable values of r in the above equation, we get Variance, Skewness, Kurtosis and Coefficients of variation of the Odds Generalized Exponential - Exponential Distribution(OGEED).

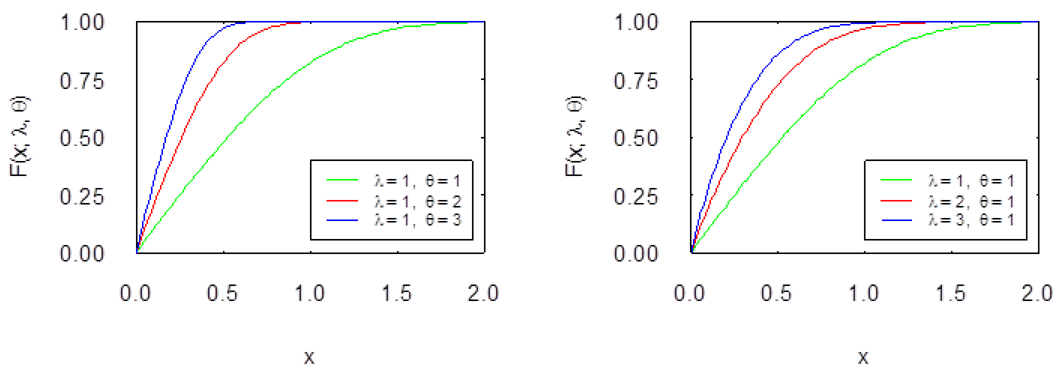

Figure 3: The distribution function of OGEED

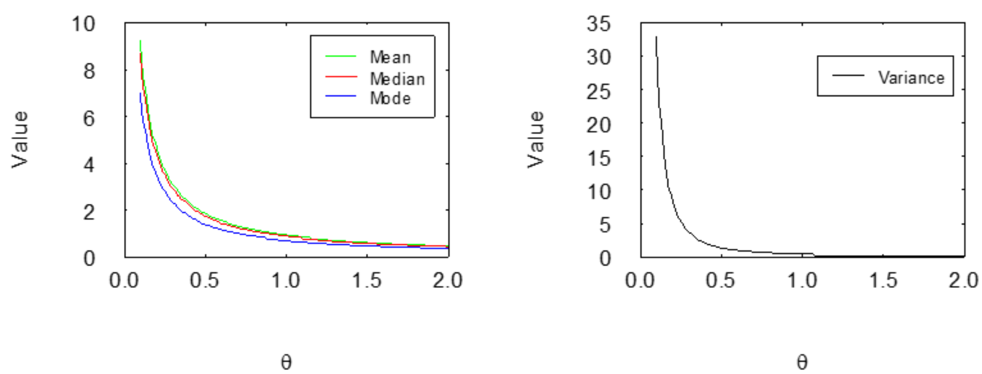

Figure 4: The mean, median, mode and variance of OGEED with λ = 0.5 and different values of θ.

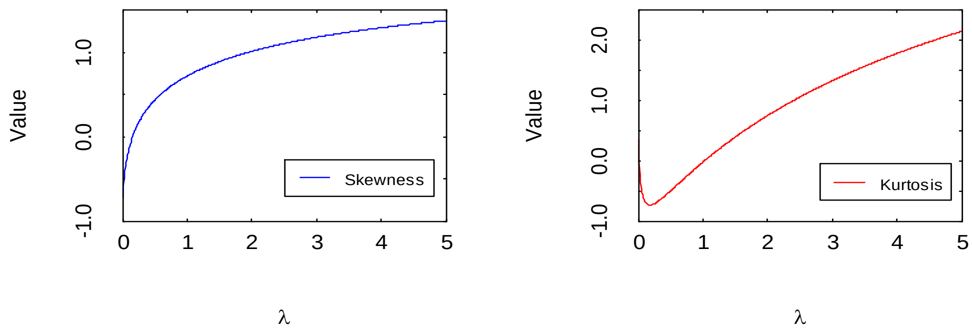

Figure 5: The Skewness and Kurtosis of OGEED with θ =1 and different values of λ.

Moment Generating Function(MGF):

$$M_X(t) = E(e^{tX}) \;= \sum_{r=0}^{\infty} \frac{t^r}{r!}\frac{e^{\lambda}}{\theta^r} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} (\ln\lambda)^{r-j} \Gamma^{(j)}(1,\lambda) \tag{3.6}$$

Characteristic Function(CF):

$$\Psi_X(t) = E(e^{itX}) \;= \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\frac{e^{\lambda}}{\theta^r} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} (\ln\lambda)^{r-j} \Gamma^{(j)}(1,\lambda) \tag{3.7}$$

Cumulant Generating Function(CGF):

$$K_X(t) = \ln_e(M_X(t)) \;= \ln_e\left[\sum_{r=0}^{\infty} \frac{t^r}{r!}\frac{e^{\lambda}}{\theta^r} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} (\ln\lambda)^{r-j} \Gamma^{(j)}(1,\lambda)\right] \tag{3.8}$$

Mean Deviation:

The **mean deviation** about the mean and the mean deviation about the **median** is defined by

$$MD_{\mu} = \int_0^{\infty} |x-\mu| f(x)dx$$

and

$$MD_M = \int_0^{\infty} |x-M| f(x)dx$$

where, $\mu = E(X)$ and $M = Median(X)$ denotes the mean and median respectively.

Thus

$$MD_{\mu} = 2\mu\left[1-e^{-\lambda(e^{\theta\mu}-1)}\right] - 2\mu + 2\theta\sum_{r=0}^{\infty}\frac{(-1)^r}{r+1}\frac{\Gamma(r+2,\lambda(e^{\theta\mu}-1))}{\lambda^{r+1}}$$

$$= 2\left[\theta\sum_{r=0}^{\infty}\frac{(-1)^r}{r+1}\frac{\Gamma(r+2,\lambda(e^{\theta\mu}-1))}{\lambda^{r+1}} - \mu e^{-\lambda(e^{\theta\mu}-1)}\right] \tag{3.9}$$

and

$$MD_M = -\mu + 2\theta\sum_{r=0}^{\infty}\frac{(-1)^r}{r+1}\frac{\Gamma(r+2,\lambda(e^{\theta M}-1))}{\lambda^{r+1}} \tag{3.10}$$

Conditional Moments:

The residual life and the reversed residual life play an important role in reliability theory and other branches of statistics. Here, the r-th order raw moment of the residual life is given by

$$\mu_r'(t) = E[(X-t)^r \mid X > t] = \frac{1}{\bar{F}(t)}\int_t^{\infty}(x-t)^r f(x)dx$$

$$= \frac{\lambda\theta}{e^{-\lambda(e^{\theta t}-1)}}\int_t^{\infty}(x-t)^r e^{\theta x} e^{-\lambda(e^{\theta x}-1)}dx$$

$$= \frac{\lambda}{e^{-\lambda e^{\theta t}}}\sum_{j=0}^{r}\frac{(-1)^j}{\theta^j}\binom{r}{j}t^{r-j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(\ln\lambda)^{j-k}\Gamma^{(k)}(1,\lambda e^{\theta t})$$

The r-th order raw moment of the **reversed residual life** is given by

$$m_r(t) = E[(t-X)^r \mid X < t] = \frac{1}{F(t)}\int_0^t (t-x)^r f(x)dx$$

$$= \frac{\lambda e^{\lambda}}{1-e^{-\lambda(e^{\theta t}-1)}}\sum_{j=0}^{r}\frac{(-1)^j}{\theta^j}\binom{r}{j}t^{r-j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(\ln\lambda)^{j-k}\left[\gamma^{(k)}(1,\lambda e^{\theta t}) - \gamma^{(k)}(1,\lambda)\right]$$

L- Moments:

Define $X_{k:n}$ be the k^{th} smallest observation in a sample of size n. The L-moments of X are defined by

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left[X_{r-k:r}\right], \quad r = 1,2,..$$

Now for OGEED with parameter λ and θ, we have

$$E\left[X_{j:r}\right] = \frac{r!}{(j-1)!(r-j)!}\int_0^{\infty}x[F(x)]^{j-1}[1-F(x)]^{r-j}dF(x)$$

$$= \frac{r!}{(j-1)!(r-j)!}\lambda\theta\int_0^{\infty} x e^{\theta x} e^{-\lambda(r-j+1)(e^{\theta x}-1)}[1-e^{-\lambda(e^{\theta x}-1)}]^{j-1}dx$$

So the first four **L- Moments** are,

$$\lambda_1 = E[X_{1:1}] = \frac{1}{\theta}\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{\lambda^{j+1}}$$

$$\lambda_2 = \frac{1}{2}E[X_{2:2} - X_{1:2}] = \frac{1}{\theta}\left[\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{\lambda^{j+1}} - \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{(2\lambda)^{j+1}}\right]$$

$$\lambda_3 = \frac{1}{3}E[X_{3:3} - 2X_{2:3} + X_{1:3}]$$

$$= \frac{1}{\theta}\left[\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{\lambda^{j+1}} - 3\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{(2\lambda)^{j+1}} + 2\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{(3\lambda)^{j+1}}\right]$$

$$\lambda_4 = \frac{1}{4}E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]$$

$$= \frac{1}{\theta}\left[\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{\lambda^{j+1}} - 6\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{(2\lambda)^{j+1}} + 10\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{(3\lambda)^{j+1}} - 5\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1)}{(4\lambda)^{j+1}}\right]$$

Quantile function:

Let X denote a random variable with the probability density function 2.4. The quantile function, say $Q(p)$, defined by $F(Q(p)) = p$ is the root of the equation

$$1 - e^{-\lambda(e^{\theta Q(p)}-1)} = p$$

So,

$$Q(p) = \frac{ln(1 - \frac{ln(1-p)}{\lambda})}{\theta} \tag{3.11}$$

3.3 Bonferroni curve, Lorenz curve and Ginis index

The Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx \tag{3.12}$$

and

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx \tag{3.13}$$

respectively, or equivalently by

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)dx \tag{3.14}$$

and

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)dx \tag{3.15}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. The Bonferroni and Gini indices are defined by

$$B = 1 - \int_0^1 B(p)dp \tag{3.16}$$

and

$$G = 1 - 2\int_0^1 L(p)dp \tag{3.17}$$

By using Eq. 3.11, we calculate Eq. 3.14 and 3.15 as

$$\int_0^p F^{-1}(x)dx = \frac{1}{\theta}\int_0^p ln(1 - \frac{ln(1-x)}{\lambda})dx$$
$$= -\frac{1}{\theta}\sum_{r=0}^{\infty}\frac{1}{\lambda^{r+1}(r+1)}\int_0^p \left[ln(1-x)\right]^{r+1}dx$$

After some algebraic simpification, we have,

$$B(p) = \frac{1}{p\mu\theta}\sum_{r=0}^{\infty}\frac{1}{\lambda^{r+1}(r+1)}\gamma(r+2, ln(1-p)) \tag{3.18}$$

and

$$L(p) = \frac{1}{\mu\theta}\sum_{r=0}^{\infty}\frac{1}{\lambda^{r+1}(r+1)}\gamma(r+2, ln(1-p)) \tag{3.19}$$

where $\gamma(,)$ represents lower incomplete Gamma function.

Integrating Eqs. 3.18 and 3.19 with respect to p, we can calculate the Bonferroni and Gini indices given by Eqs. 3.16 and 3.17 , respectively, as

$$B = 1 - \frac{1}{\mu\theta}\sum_{r=0}^{\infty}\frac{1}{\lambda^{r+1}(r+1)}\int_0^1 \frac{\gamma(r+2, ln(1-p))}{p}dp \tag{3.20}$$

and

$$G = 1 - \frac{2}{\mu\theta}\sum_{r=0}^{\infty}\frac{1}{\lambda^{r+1}(r+1)}\int_0^1 \gamma(r+2, ln(1-p))dp \tag{3.21}$$

3.4 Order Statistics

Suppose $X_1, X_2, X_3, ..., X_n$ is a random sample from the distribution in (2.4). Let $X_{(1)}, X_{(2)}, X_{(3)}, ..., X_{(n)}$, denote the corresponding order statistics. Hence the probability density function and the cumulative distribution function of the k^{th} order statistic, say $Y = X_{(k)}$, are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!}F^{k-1}(y)[1-F(y)]^{n-k} f(y)$$

$$= \frac{n!}{(k-1)!(n-k)!}\lambda\theta e^{\theta y} e^{-\lambda(n-k+1)(e^{\theta y}-1)}\left[1-e^{-\lambda(e^{\theta y}-1)}\right]^{k-1} \tag{3.22}$$

and

$$F_Y(y) = \sum_{j=k}^{n}\binom{n}{j}F^{j}(y)[1-F(y)]^{n-j}$$

$$= \sum_{j=k}^{n}\binom{n}{j}e^{-\lambda(n-j)(e^{\theta y}-1)}\left[1-e^{-\lambda(e^{\theta y}-1)}\right]^{j} \tag{3.23}$$

respectively.

3.5 Entropies

An entropy of a random variable X is a measure of variation of the uncertainty. A popular entropy measure is **Renyi entropy** (Renyi 1961). If X has the probability density function f(x), then Renyi entropy is defined by

$$H_R(\gamma) = \frac{1}{1-\gamma}\ln\left\{\int_0^{\infty} f^{\gamma}(x)dx\right\} \tag{3.24}$$

where $\gamma > 0$ and $\gamma \neq 1$. Suppose X has the probability density function 2.4. Then, one can calculate

$$\int_0^{\infty} f^{\gamma}(x)dx = \int_0^{\infty}(\lambda\theta)^{\gamma} e^{\gamma\theta x} e^{-\gamma\lambda(e^{\theta x}-1)}dx$$

$$= (\lambda\theta)^{\gamma-1}\sum_{r=0}^{\infty}\frac{(\gamma-1)(\gamma-2)(\gamma-3)....(\gamma-r)}{\lambda^r\gamma^{r+1}}$$

So **Renyi entropy** is

$$H_R(\gamma) = \frac{1}{1-\gamma}\ln\left\{(\lambda\theta)^{\gamma-1}\sum_{r=0}^{\infty}\frac{(\gamma-1)(\gamma-2)(\gamma-3)....(\gamma-r)}{\lambda^r\gamma^{r+1}}\right\}$$

$$= -\ln\lambda\theta + \frac{1}{1-\gamma}\ln\sum_{r=0}^{\infty}\frac{(\gamma-1)(\gamma-2)(\gamma-3)....(\gamma-r)}{\lambda^r\gamma^{r+1}} \quad (3.25)$$

Shannon measure of entropy is defined as

$$H(f) = E[-\ln f(x)] = -\int_0^{\infty} f(x)\ln f(x)dx$$

$$= -\ln\lambda\theta - \theta\int_0^{\infty} xf(x)dx + \lambda\int_0^{\infty}(e^{\theta x}-1)f(x)dx$$

After some algebraic simplification, we have

$$H(f) = 1-\ln\lambda\theta - \sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(r+1)}{\lambda^{r+1}} \quad (3.26)$$

3.6 Loss reserves data for Queensland, Australia

The **Reliability** function of OGEED is given by the form as:

$$R(x) = 1-F(x) = e^{-\lambda(e^{\theta x}-1)} \quad (3.27)$$

and the **Hazard rate** of OGEED is given by the form as:

$$r(t) = \frac{f(t)}{1-F(t)} = \lambda\theta e^{\theta t} \quad (3.28)$$

Now

$$\frac{d^2}{dx^2}\ln f(x) = -\lambda\theta^2 e^{\theta x}$$

For $\lambda > 0$, $\theta > 0$ and $x > 0$, $\frac{d^2}{dx^2} \ln f(x) < 0$.

So, the distribution is **log-concave**. Therefore, the distribution posses Increasing failure rate (IFR) and Decreasing Mean Residual Life (DMRL) property.

Mean Residual Life (MRL) function is defined as

$$e_x(t) = \frac{\lambda}{e^{-\lambda e^{\theta t}}} \sum_{j=0}^{1} \frac{(-1)^j}{\theta^j} \binom{1}{j} t^{1-j} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (\ln \lambda)^{j-k} \Gamma^{(k)}(1, \lambda e^{\theta t})$$

$$= \frac{\lambda}{e^{-\lambda e^{\theta t}}} \left[\left(t + \frac{\ln \lambda}{\theta} \right) \Gamma(1, \lambda e^{\theta t}) + \frac{1}{\theta} \Gamma^{(1)}(1, \lambda e^{\theta t}) \right]. \quad (3.29)$$

Reversed Hazard rate:

$$\mu_F(x) = \frac{f(x)}{F(x)}$$

$$= \frac{\lambda \theta e^{\theta x}}{e^{\lambda(e^{\theta x}-1)} - 1} \quad (3.30)$$

Expected Inactivity Time (EIT) or Mean Reversed Residual Life (MRRL) function is defined as

$$\bar{e}_x(t) = E(t - X \mid X < t)$$

$$= \frac{\lambda e^{\lambda}}{1 - e^{-\lambda(e^{\theta t}-1)}} \sum_{j=0}^{1} \frac{(-1)^j}{\theta^j} \binom{1}{j} t^{1-j} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} (\ln \lambda)^{j-k} \left[\gamma^{(k)}(1, \lambda e^{\theta t}) - \gamma^{(k)}(1, \lambda) \right]$$

$$= \frac{\lambda e^{\lambda}}{1 - e^{-\lambda(e^{\theta t}-1)}} \left[\left(t + \frac{\ln \lambda}{\theta} \right) \{ \gamma(1, \lambda e^{\theta t}) - \gamma(1, \lambda) \} + \frac{1}{\theta} \{ \gamma^{(1)}(1, \lambda e^{\theta t}) - \gamma^{(1)}(1, \lambda) \} \right].. \quad (3.31)$$

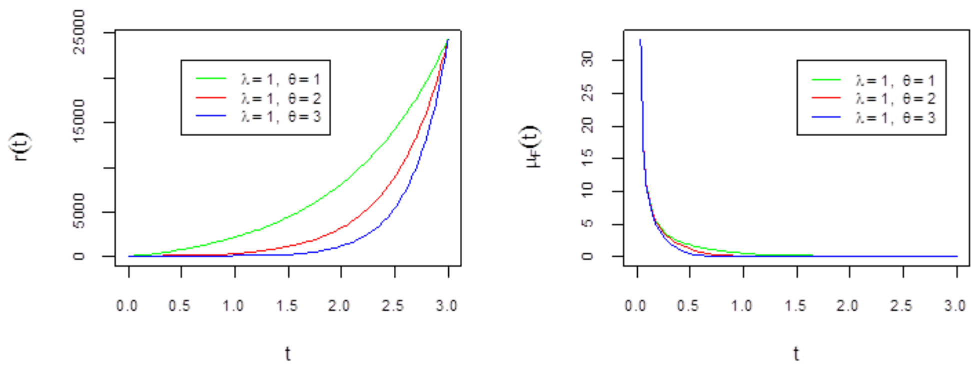

Figure 6: The Hazard rate am Reversed Hazard rate of OGEED with λ=1 and different values of θ.

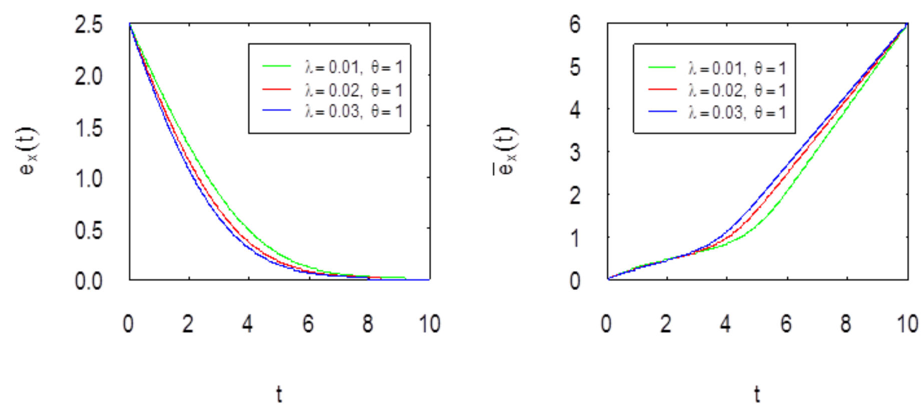

Figure 7: The Mean Residual Life and Mean Reversed Residual Life of OGEED with θ =1 and different values of λ

3.7 Stress_Strength Reliability

The Stress-Strength model describes the life of a component which has a random strength X that is subjected to a random stress Y. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X > Y$. So, Stress-Strength Reliability is $R = Pr(Y < X)$.

Let $X \sim OGEED(\lambda_1, \theta_1)$ and $Y \sim OGEED(\lambda_2, \theta_2)$ be independent random variables. Then Stress-Strength Reliability

$$R = Pr(Y < X)$$

$$= 1 - \lambda_1\theta_1\int_0^\infty e^{\theta_1 x} e^{-\lambda_1(e^{\theta_1 x}-1)} e^{-\lambda_2(e^{\theta_2 x}-1)} dx$$

If $\theta_1 = \theta_2 = \theta$, then

$$R = 1 - \lambda_1\theta\int_0^\infty e^{\theta x} e^{-(\lambda_1+\lambda_2)(e^{\theta x}-1)} dx$$

$$= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

4. Maximum Likelihood Method of Estimation of the Parameters

Using the method of Maximum Likelihood, we estimate the parameter of the OGEED. Since

$$f(x;\lambda,\theta) = \lambda\theta e^{\theta x} e^{-\lambda(e^{\theta x}-1)}$$

The likelihood function is given by

$$L(x;\lambda,\theta) = \prod_{i=1}^{n} f(x_i)$$

$$= \lambda^n\theta^n e^{\theta\sum_{i=1}^{n} x_i} e^{-\lambda\sum_{i=1}^{n}(e^{\theta x_i}-1)} \tag{4.32}$$

The MLEs of λ and θ are the roots of

$$\frac{\partial \ln L(x;\lambda,\theta)}{\partial\lambda} = 0 \text{ and } \frac{\partial \ln L(x;\lambda,\theta)}{\partial\theta} = 0.$$

From these equations, we have

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n}(e^{\theta x_i} - 1)} \tag{4.33}$$

And

$$\frac{n}{\theta}+\sum_{i=1}^{n}x_i-\frac{n\sum_{i=1}^{n}x_i e^{\theta x_i}}{\frac{n}{\sum_{i=1}^{n}(e^{\theta x_i}-1)}}=0. \tag{4.34}$$

Estimation of two parameters λ and θ are obtained by solving the two equations numerically.

5. Simulation Study

Here we use the inversion method for generating random data from the Odds Generalized Exponential-Exponential Distribution.

Algorithm:

1. Generate U from Uniform (0, 1)
2. Set $X=\dfrac{ln(1-\frac{ln(1-U)}{\lambda})}{\theta}$

A Monte-Carlo simulation study was carried out considering N=1000 times for selected values of n, λ and θ. Samples of sizes 20, 40 and 100 were considered and values of θ were taken as 0.1, 1.0 and 2 for λ=0.01 and 0.1 respectively. The required numerical evaluations are carried out using R 3.1.1 software. The following two measures were computed:

(i) Bias of the simulated estimates $\hat{\theta}_i$ and $\hat{\lambda}_i$, i = 1, 2… N:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i-\theta) \text{ and } \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda)$$

(ii) Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i$ and $\hat{\lambda}_i$, i = 1, 2… N:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2 \text{ and } \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda)^2$$

The result of the simulation study has been tabulated in Table 1 and Table 2 below. In Table 1 we take λ=0.01 for θ = 0.1, 1.0 and 2. In Table 2 we take λ=0.1 for θ = 0.1, 1.0 and 2. The resulting values relating to Odds Generalized Exponential-Exponential Distribution (OGEED) have been presented in first row and that relating to Gamma Distribution (GD), Exponentiated Exponential Distribution (EED), Weibull Distribution (WD) and Pareto Distribution (PD) in second, third, fourth and fifth row.

Observations:

(i) Table 1 and 2 shows that the bias is positive in case Odds Generalized Exponential-Exponential Distribution (OGEED). Table 1 and 2 also shows that bias and MSE decreases as n increases.

(ii) In terms of bias and MSE, the parameter λ and θ of the Odds Generalized Exponential -Exponential Distribution is efficiently estimated compared to that of the other distribution.

Table 1: Bias and MSE of the ML estimators for different λ and θ

n	Distribution	λ=0.01		θ=0.1		λ=0.01		θ=1		λ=0.01		θ=2	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	OGEED	0.0023	0.0002	0.0069	0.0005	0.0015	0.0001	0.0884	0.0616	0.0014	0.0002	0.1674	0.2198
	GD	5.1592	36.1188	11.6464	193.215	0.5068	0.3844	11.0461	185.850	0.2503	0.0797	10.1703	164.990
	EED	0.0899	0.0090	27.8047	1004.70	0.6252	0.3999	9.0556	104.321	1.3674	1.9238	10.8877	172.184
	WD	44.8709	2014.68	4.2598	19.3979	4.4738	20.0278	3.4252	13.1618	2.2320	4.9870	2.4375	7.2533
	PD	14.8955	282.151	1.0744	1.5639	1.4848	2.8102	-0.4632	0.4266	0.7469	0.7092	-1.4604	2.1829
40	OGEED	0.0010	0.0001	0.0037	0.0002	0.0011	0.0001	0.0320	0.0211	0.0008	0.0001	0.0690	0.0828
	GD	5.0948	32.9013	9.6462	114.497	0.4948	0.2954	8.8368	98.4042	0.2453	0.0741	7.7433	80.2829
	EED	0.0896	0.0086	24.266	708.384	0.6129	0.381	7.8582	74.1156	1.3105	1.7545	8.4246	97.445
	WD	44.7605	2006.96	4.1021	17.4211	4.4655	19.9742	3.1976	10.7738	2.2309	4.9856	2.1979	5.3946
	PD	10.378	148.676	0.7056	0.6544	1.0557	1.4978	-0.1788	0.2614	0.5145	0.3606	-1.1895	1.5625
100	OGEED	0.0000	0.0000	0.0022	0.0001	0.0003	0.0000	0.0164	0.0082	0.0005	0.0000	0.0188	0.0288
	GD	5.0032	28.2834	8.6213	81.2385	0.4926	0.2794	7.5851	64.1084	0.2318	0.0687	6.5573	49.8967
	EED	0.0889	0.0081	22.1086	534.215	0.5969	0.3590	6.8282	51.732	1.2708	1.6342	6.8808	57.6388
	WD	44.7128	2005.88	4.0395	16.5405	4.4611	19.9664	3.1127	9.9061	2.2293	4.9851	2.0881	4.5685
	PD	6.1093	55.4026	0.4433	0.2481	0.5876	0.5117	0.1716	0.1805	0.2908	0.1292	-0.8131	1.0700

Table 2:Bias and MSE of the ML estimators for different λ and θ

n	Distribution	λ=0.1		θ=0.1		λ=0.1		θ=1		λ=0.1		θ=2	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
20	OGEED	0.0109	0.0089	0.0091	0.0008	0.0142	0.0113	0.0968	0.0900	0.0043	0.0088	0.2399	0.3698
	GD	6.4779	47.3296	3.8234	19.3784	0.5563	0.3714	3.0069	14.4719	0.2331	0.0645	2.1133	10.6514
	EED	-0.0042	0.0001	3.6532	15.4715	0.8735	0.8060	3.3264	19.6426	1.8994	3.8220	2.7413	22.5686
	WD	22.4485	505.783	2.3216	5.7473	2.1519	4.6539	1.4417	2.4798	1.0275	1.0626	0.4792	0.6208
	PD	3.4305	19.4624	0.5365	0.3926	0.2528	0.1443	-0.6744	0.4660	0.0842	0.0279	-1.6827	2.8421
40	OGEED	0.0030	0.0032	0.0047	0.0003	0.0049	0.0043	0.0518	0.0399	0.0013	0.0037	0.1207	0.1631
	GD	6.4086	45.4298	3.3578	12.9083	0.5406	0.3351	2.5377	8.2686	0.2234	0.0613	1.5165	4.1257
	EED	-0.0036	0.0001	3.4795	13.1433	0.8322	0.7132	2.6479	9.7009	1.7770	3.2532	1.7028	6.0153
	WD	22.4368	504.935	2.2533	5.2403	2.1507	4.6358	1.3721	2.0667	1.0247	1.0618	0.3763	0.3275
	PD	1.9536	6.7810	0.3572	0.1635	0.1070	0.0446	-0.5426	0.3336	-0.0568	0.0076	-1.5515	2.4429
100	OGEED	0.0001	0.0012	0.0023	0.0001	0.0024	0.0014	0.0188	0.0141	0.0012	0.0012	0.0388	0.0490
	GD	6.2616	43.7819	3.1037	10.1464	0.5341	0.3268	2.2053	5.3924	0.2104	0.0588	1.1575	1.8542
	EED	-0.0021	0.0000	3.3967	11.9616	0.8009	0.6488	2.1977	5.4942	1.7058	2.9456	1.1861	2.1255
	WD	22.2459	500.125	2.2079	4.9371	2.1348	4.6170	1.3058	1.7730	1.0241	1.0526	0.2924	0.1473
	PD	0.8388	1.4223	0.2264	0.0627	-0.0068	0.0071	-0.3683	0.2376	-0.0002	0.0048	-1.3475	1.9361

6. Data Analysis

In this section, we fit the exponential exponential model to a real data set obtained from Smith and Naylor (1987). The data are the strengths of 1.5 cm glass fibres, measured at the National

Physical Laboratory, England and have been shown in Table 3. Histogram shows that the data set is negatively skewed. We have fitted this data set with the Odds Generalized Exponential - Exponential distribution. We have also fitted this data set for some other probability distributions with two parameters like Gamma, Exponentiated Exponential, Weibull and Pareto. The summarized results have been presented Table 4 and it is noticed that the OGEED is the better fit for minimum Akaike Information Criterion (AIC). Histogram and fitted Odds Generalized Exponential-Exponential curve to data set have been shown in Figure 8.

Table 3: Strengths of glass fibres data set

0.55 0.93 1.25 1.36 1.49 1.52 1.58 1.61 1.64 1.68 1.73 1.81 2.00 0.74 1.04 1.27 1.39 1.49 1.53 1.59 1.61
1.66 1.68 1.76 1.82 2.01 0.77 1.11 1.28 1.42 1.50 1.54 1.60 1.62 1.66 1.69 1.76 1.84 2.24 0.81 1.13 1.29
1.48 1.50 1.55 1.61 1.62 1.66 1.70 1.77 1.84 0.84 1.24 1.30 1.48 1.51 1.55 1.61 1.63 1.67 1.70 1.78 1.89

Table 4: summarized results of fitting differnet distributions to data set of Smith and Naylor(1987)

Distribution	Estimate of the parameter	Log-likelihood	AIC
OGEED	$\hat{\lambda} = 0.002418, \hat{\theta} = 3.647411$	-14.81	33.616
Gamma Distribution	$\hat{\lambda} = 0.08640267, \hat{\theta} = 17.43957$	-23.95	51.903
Exponentiated Exponential Distribution	$\hat{\lambda} = 2.231604, \hat{\theta} = 19.89626$	-32.70	69.409
Weibull Distribution	$\hat{\lambda} = 1.6281131, \hat{\theta} = 5.780701$	-15.21	34.414
Pareto Distribution	$\hat{\lambda} = 0.55, \hat{\theta} = 1.021557$	-85.66	175.326

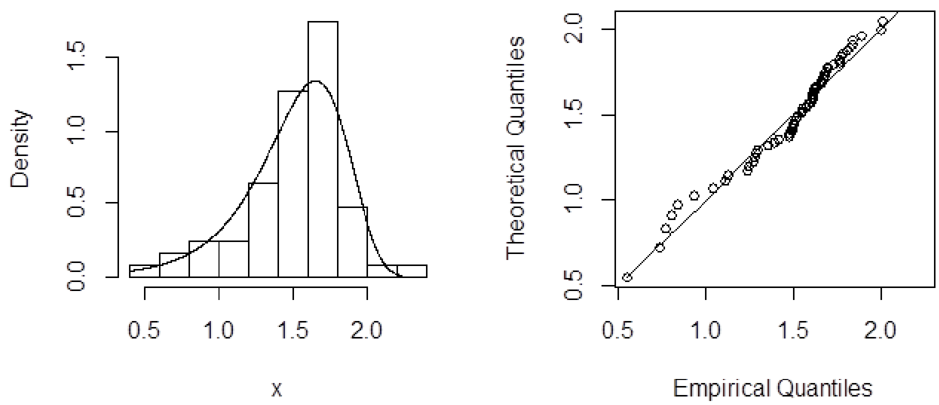

Figure 8: Plots of the fitted paf and estimated quantiles versus observed quantiles of the OGEED.

7. Concluding Remark

In this article, we have studied a new probability distribution called Odds Generalized Exponential **-** Exponential Distribution. This is a particular case of T-X family of distributions proposed by Alzaatreh et al. (2013). The structural and reliability properties of this distribution have been studied and inference on parameters have also been mentioned. The proposed distribution has been compared with some standard distributions with two parameters through simulation study and the supiriority of the proposed distribution has been established. The appropriateness of fitting the odds generalized exponential **-** exponential distribution has also been established by analyzing a real life data set.

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