

## The Extended Dagum Distribution: Properties and Application

Alisson de O. Silva<sup>1\*</sup>, Luana Cecília M. da Silva<sup>1</sup> and Gauss M. Cordeiro<sup>1</sup>

<sup>1</sup>*Universidade Federal de Pernambuco Departamento de Estatística*

*Abstract:* We study a new five-parameter model called the extended Dagum distribution. The proposed model contains as special cases the log-logistic and Burr III distributions, among others. We derive the moments, generating and quantile functions, mean deviations and Bonferroni, Lorenz and Zenga curves. We obtain the density function of the order statistics. The parameters are estimated by the method of maximum likelihood. The observed information matrix is determined. An application to real data illustrates the importance of the new model.

*Key words:* Dagum distribution, Exponentiated class, Maximum likelihood, Observed information matrix, Order statistic.

### 1. Introduction

The Dagum model pioneered by Camilo Dagum (1977, 1980) has been widely used in studies of income and wealth distributions. Its structural properties have been extensively investigated by several authors. For an excellent survey on the genesis and on empirical applications, see Kleiber and Kotz(2003) and Kleiber (2008). If a random variable  $Z$  has the Dagum distribution with positive parameters  $\beta$ ,  $\lambda$  and  $\delta$ , its cumulative distribution function (cdf) and probability density function (pdf) are given by (for  $z > 0$ )

$$G(z; \theta) = (1 + \lambda z^{-\delta})^{-\beta} \quad (1)$$

and

$$g(z; \theta) = \beta \lambda \delta z^{-\delta-1} (1 + \lambda z^{-\delta})^{-\beta-1}, \quad (2)$$

respectively, where  $\theta = (\beta, \lambda, \delta)$ . The parameter  $\lambda$  is a scale parameter, whereas  $\beta$  and  $\delta$  are shape parameters. Henceforth, the Dagum distribution with parameters  $\beta$ ,  $\lambda$  and  $\delta$  will be denoted by  $Da(\beta, \lambda, \delta)$ . It has positive asymmetry and it is unimodal for  $\beta\delta > 1$  and zero-modal for  $\beta\delta \leq 1$ .

The  $q$ th quantile and the  $r$ th ordinary moment of  $Z$  are given by  $z(q) = \lambda^{\frac{1}{\delta}}(q^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\delta}}$  and

$$E(Z^r) = \beta \lambda^{\frac{r}{\delta}} \beta \left( \beta + \frac{r}{\delta}, 1 - \frac{r}{\delta} \right), \quad (3)$$

respectively, for  $r < \delta$ , where  $B(\cdot, \cdot)$  is the beta function.

---

\* Corresponding author.

Recently, several authors have studied the Dagum distribution under some perspectives. Domma(2007) derived the asymptotic distribution of the maximum likelihood estimators (MLEs) of the parameters of the right-truncated Dagum distribution. Domma *et al.* (2009) determined the information matrix in doubly censored data. Domma *et al.* (2011a) calculated the observed information matrix in right censored samples and Domma *et al.* (2011b) discussed aspects of the maximum likelihood estimation for censored data. Shahzad and Asghar (2013) obtained the L-moments and TL-moments in closed-form. Domma *et al.* (2011c) studied the Dagum distribution from the perspective of reliability, whereas Domma and Perri (2009) investigated some developments of the log-Dagum distribution. Oluyede and Rajasoorya (2013) proposed the Mc-Dagum distribution and studied some mathematical properties. Doma and Condino (2013) extended the Dagum distribution based on the beta generator pioneered by Eugene *et al.* (2002). Several methods for generating new classes of distributions by extending well-known models and at the same time providing great flexibility in modeling real data have been proposed in the last years. The use of new generators of continuous distributions from classical ones has become more common in the last ten years or so. Some examples are the beta-generated (Eugene *et al.*, 2002), gamma-generated (Zografos and Balakrishnan, 2009) and generalized Kumaraswamy (Cordeiro and de Castro, 2011) classes of distributions.

For a continuous baseline cdf  $G(x)$ , Cordeiro *et al.* (2013) defined the exponentiated generalized (“EG” for short) class of distributions by

$$F(x) = [1 - \{1 - G(x)\}^a]^b, \quad (4)$$

where  $a > 0$  and  $b > 0$  are two extra shape parameters whose role is to add skewness and to vary tail weights. Because of its tractable cdf (4), the EG class can be used quite effectively even if the data are censored. The pdf corresponding to (4) is given by

$$f(x) = ab[1 - G(x)]^{a-1}\{1 - [1 - G(x)]^a\}^{b-1}g(x)$$

The baseline distribution  $G(x)$  is a basic exemplar of (4) when  $a = b = 1$ . Setting  $a = 1$  gives the *exponentiated-G* (exp-G) distribution. The case  $b = 1$  corresponds to the Lehman type II. So, the cdf (4) generalizes both the exponentiated and Lehman type II distributions.

In this paper, we propose a new lifetime model, named the extended Dagum (EDa) distribution with cdf obtained from equation (4) by taking  $G(x)$  to be the cdf of the Dagum( $\beta, \lambda, \delta$ ) distribution. We obtain some mathematical properties of the new distribution and discuss maximum likelihood estimation of its parameters. The rest of the paper is outlined as follows. In Section 2, we introduce the EDa distribution and provide a mixture representation for its density function. General expressions for the ordinary and incomplete moments are given in Section 3. In Section 4, we derive the moment generating function (mgf). In Section 5, we provide an expression for the quantile function (qf). The Rényi entropy is determined in Section 6. The mean deviations are calculated in Section 7. The Bonferroni, Lorenz and Zenga curves are obtained in Section 8. The density of the order statistics is determined in Section 9. The maximum likelihood estimation is addressed in Section 10. An application to real data is discussed in Section 11. Finally, concluding remarks are given in Section 12.

## 2. The Extended Dagum Distribution

Replacing (1) in equation (4), we obtain the EDa distribution defined by the cdf and pdf

$$F(x; \zeta) = \{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^b \quad (5)$$

and

$$f(x; \zeta) = ab\beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}[1 - (1 + \lambda x^{-\delta})^{-\beta}]^{a-1}\{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^{b-1}, \quad (6)$$

respectively, where  $\zeta = (\beta, \lambda, \delta, a, b)$  is the parameter vector. Henceforth, a random variable  $X$  having the EDa pdf (6) with parameter vector  $\zeta$  is denoted by  $X \sim EDa(\beta, \lambda, \delta, a, b)$ . The hazard rate function (hrf) and reverse hazard function of  $X$  are given, respectively, by

$$h(x; \zeta) = \frac{ab\beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}[1 - (1 + \lambda x^{-\delta})^{-\beta}]^{a-1}\{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^{b-1}}{1 - \{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^b},$$

and

$$r(x; \zeta) = \frac{ab\beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}[1 - (1 + \lambda x^{-\delta})^{-\beta}]^{a-1}\{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^{b-1}}{\{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^b}.$$

Plots of the EDa pdf and hrf are displayed in Figures 1 and 2, respectively. The pdf (6) includes several distributions as special models. In fact, the Dagum distribution (with parameters  $\beta, \lambda$  and  $\delta$ ) is clearly a basic exemplar for  $a = b = 1$ . The extended Fisk (EFisk) and extended Burr III (EBuIII) distributions are new models which arise for  $\beta = 1$  and  $\lambda = 1$ , respectively. The case  $b = 1$  corresponds to the Lehman type II Dagum distribution.

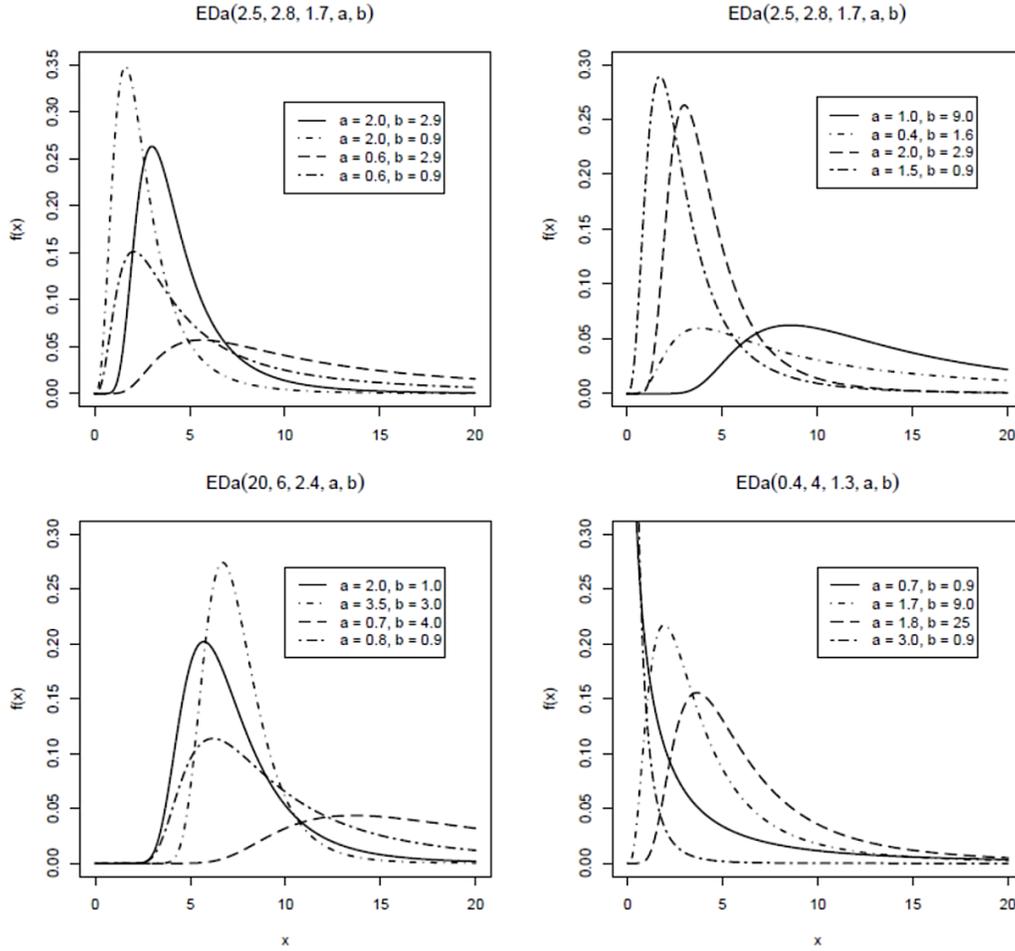


Figure 1: Plots of the EDa pdf.

For  $\rho > 0$  real non-integer and  $|z| < 1$ , we have

$$(1 - z)^\rho = \sum_{j=0}^{\infty} (-1)^j \binom{\rho}{j} z^j. \quad (7)$$

Expanding the binomials in (6) as in (7), we obtain  $f(x; \zeta)$  as a mixture of Dagum densities

$$f(x; \zeta) = \sum_{j=0}^{\infty} w_{j+1} g(x; (j+1)\beta, \lambda, \delta), \quad (8)$$

where

$$w_{j+1} = ab \sum_{k=0}^{\infty} \frac{(-1)^{j+k}}{(j+1)} \binom{b-1}{k} \binom{(k+1)a-1}{j}, \quad (9)$$

and

$$g(x; (j+1)\beta, \lambda, \delta) = (j+1)\beta\lambda\delta x^{-(\delta+1)}(1 + \lambda x^{-\delta})^{-(j+1)\beta-1}$$

denotes the  $Da((j + 1)\beta, \lambda, \delta)$  density function. We can demonstrate using Mathematica that  $\sum_{j=0}^{\infty} w_{j+1} = 1$ . The cdf corresponding to (8) is given by

$$F(x; \zeta) = \sum_{j=0}^{\infty} w_{j+1} G(x; (j + 1)\beta, \lambda, \delta). \tag{10}$$

Following Cordeiro *et al.* (2013), we can simulate observations from the  $EDa(\beta, \lambda, \delta, a, b)$  distribution from the relation

$$X = \lambda^{\frac{1}{\delta}} \{ [1 - (1 - U^{\frac{1}{b}})^{\frac{1}{a}}]^{-\frac{1}{\beta}} - 1 \}^{\frac{1}{\delta}}, \tag{11}$$

where  $U$  is uniform random variable.

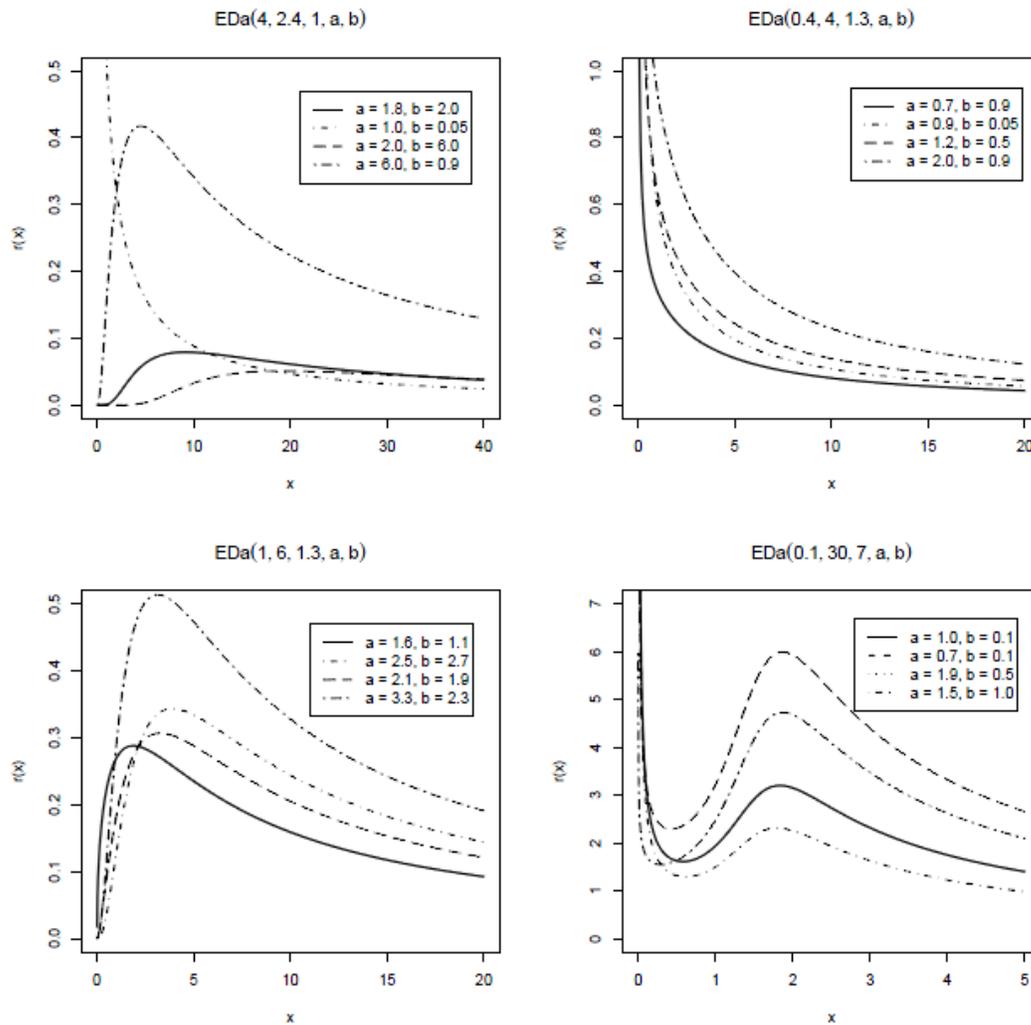


Figure 2: Plots of the EDa hrf.

### 3. Ordinary and Incomplete Moments

The  $r$ th ordinary moment of  $X$  for  $r < \delta$  comes from equations (3) and (8) as

$$\mu'_r = E(X^r) = \beta \lambda^{\frac{r}{\delta}} \sum_{j=0}^{\infty} (j+1) w_{j+1} B\left((j+1)\beta + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right). \quad (12)$$

Further, the central moments ( $\mu_p$ ) and cumulants ( $\kappa_p$ ) of  $X$  are easily obtained from equation (12) by

$$\mu_p = \sum_{k=0}^p \binom{p}{k} (-1)^k \mu_1'^k \mu'_{p-k} \quad \text{and} \quad \kappa_p = \mu'_p - \sum_{k=1}^{p-1} \binom{p-1}{k-1} \kappa_k \mu'_{p-k},$$

respectively, where  $\kappa_1 = \mu'_1$ . Thus,  $\kappa_2 = \mu'_2 - \mu_1'^2$ ,  $\kappa_3 = \mu'_3 - 3\mu_2'\mu_1' + 2\mu_1'^3$ , etc.

Finally, the  $r$ th incomplete moment  $X$  (for  $r < \lambda$ ) is given by

$$\begin{aligned} m_r(x) &= \sum_{j=0}^{\infty} w_{j+1} \int_0^x t^r g(t; (j+1)\beta, \lambda, \delta) dt \\ &= \beta \lambda^{\frac{r}{\delta}} \sum_{j=0}^{\infty} (j+1) w_{j+1} B\left(y^*; (j+1)\beta + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \end{aligned} \quad (13)$$

where  $y^* = (1 + \lambda x^{-\delta})^{-1} < 1$  and  $B(z; v_1; v_2) = \int_0^z w^{v_1-1} (1-w)^{v_2-1} dw$  is the incomplete beta function.

This result can be useful in the study of the income inequality measures.

### 4. Generating Function

Here, we derive an explicit expression for the mgf  $M(t; \zeta)$  of  $X$ . First, we have from (2), the mgf of the  $Da(\beta, \lambda, \delta)$  distribution

$$M_{Da}(t) = \beta \lambda \delta \int_0^{\infty} e^{tx} x^{-\delta-1} (1 + \lambda x^{-\delta})^{-\beta-1} dx. \quad (14)$$

Using the power series  $(ax + b)^{-k} = \sum_{j=0}^{\infty} \frac{a^j}{b^{j+k}} \binom{-k}{j} x^j$  in equation (14) for  $k > 0$  and the integral  $\int_0^{\infty} x^{\alpha-1} e^{-px} dx = \Gamma(\alpha) p^{-\alpha}$  (Prudnikove et al., 1986), we obtain (for  $t < 0$ )

$$M_{Da}(t) \sum_{l=0}^{\infty} \lambda^l \binom{-\beta-1}{l} \Gamma(-(l+1)\delta) (-t)^{-(l+1)\delta}.$$

Combining (8) and the last result, the mgf of  $X$  follows as

$$M(t, \zeta) = \beta \lambda \delta \sum_{j,l=0}^{\infty} (j+1) w_{j+1} \lambda^l \binom{-(j+1)\beta-1}{l} \Gamma(-(l+1)\delta) (-t)^{-(l+1)\delta}. \quad (15)$$

where  $w_j + 1$  is given by (9).

### 5. Quantile Function

By inverting equation (5), the qf of  $X$  is given by

$$Q(u) = \lambda^{\frac{1}{\delta}} \{ [1 - (1 - u^{\frac{1}{b}})^{\frac{1}{a}}]^{-\frac{1}{\beta}} - 1 \}^{-\frac{1}{\delta}}. \tag{16}$$

The effect of the shape parameters  $a$  and  $b$  on the skewness and kurtosis of the new distribution can be based on quantile measures. The shortcomings of the classical skewness and kurtosis measures are well-known. One of the earliest skewness measures to be suggested is the Bowley skewness (Kenney and Keeping, 1962) defined by the average of the quartiles minus the median divided by half the interquartile range, namely

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}.$$

The Moors kurtosis is based on octiles

$$M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

These measures are less sensitive to outliers and they exist even for distributions without moments.

In Figures 3 and 4, we plot the measures  $B$  and  $M$  for the EGDa distribution as functions of  $a$  and  $b$  for fixed values of the other parameter, respectively. These plots indicate that there is a great flexibility of the skewness and kurtosis curves of the new distribution.

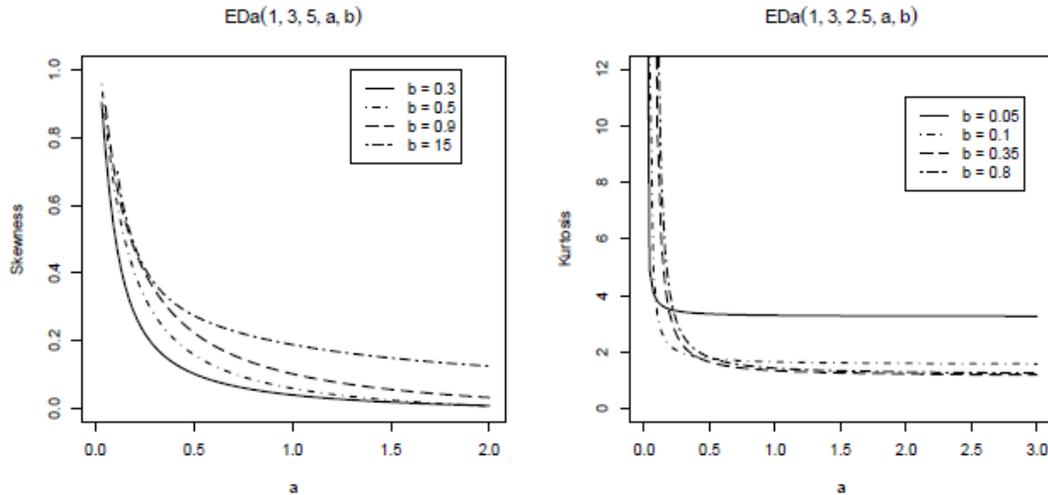


Figure 3: Skewness and kurtosis of  $X$  as a function of  $a$  for some values of  $b$ .

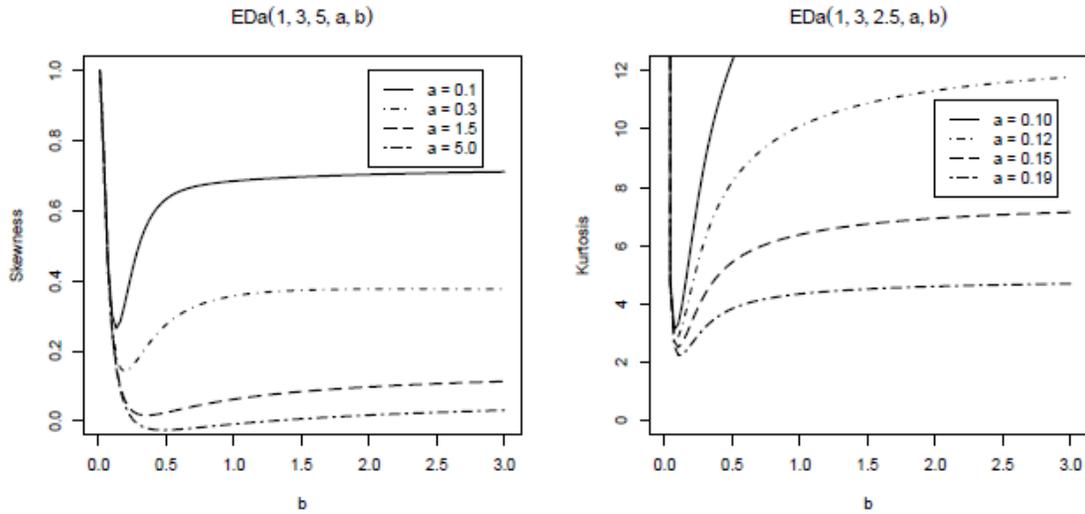


Figure 4: Skewness and kurtosis of  $X$  as a function of  $b$  for some values of  $a$ .

## 6. Rényi Entropy

Entropy has been used in various situations in science and engineering. Numerous entropy measures have been proposed and studied in the literature. The entropy of a random variable  $X$  with pdf  $f(x)$  is a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_R(\rho) = \frac{1}{1-\rho} \log\{ \int f(x)^\rho dx \}, \quad (17)$$

where  $\rho > 0$  and  $\rho \neq 1$ . For further details, see Song (2001).

The entropy of  $X$  can be express as

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ (ab\beta\lambda\delta)^\rho \int_0^\infty x^{-\rho(\delta+1)} (1 + \lambda x^{-\delta})^{-\rho(\beta+1)} [1 - (1 + \lambda x^{-\delta})^{-\beta}]^{\rho(a-1)} \times \{1 - [1 - (1 + \lambda x^{-\delta})^{-\beta}]^a\}^{\rho(b-1)} dx \right\}. \quad (18)$$

By using the power series (7) twice in equation (18), we have

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ (ab\beta\lambda\delta)^\rho \sum_{j,k=0}^{\infty} (-1)^{j+k} \binom{\rho(b-1)}{k} \binom{(k+\rho)a-\rho}{j} \times \int_0^\infty x^{-\rho(\delta+1)} (1 + \lambda x^{-\delta})^{-[\beta(j+\rho)+\rho]} dx \right\}.$$

Setting  $y = (1 + \lambda x^{-\delta})^{-1}$ , such that  $x = \left(\frac{1-y}{\lambda y}\right)^{-\frac{1}{\delta}}$ , and after some algebra, we obtain

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ \kappa(\rho) \sum_{j=0}^{\infty} v_j B \left( \beta(j+\rho) - \frac{(\rho-1)}{\delta}, \frac{\rho}{\delta}(\delta+1) - \frac{1}{\delta} \right) \right\},$$

for  $0 < \rho < 1$ , where  $\kappa(\rho) = (ab\beta)^\rho \delta^{\rho-1} \lambda^{\rho-\delta^{-1}[\rho(\delta+1)-1]}$ ,  
and

$$v_j = v_j(a, b, \rho) = \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\rho(b-1)}{k} \binom{(k+\rho)a-\rho}{j}.$$

## 7. Mean Deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviations about the mean and about the median defined by

$$\delta_1 = \int_0^{\infty} |x - \mu'_1| f(x) dx \text{ and } \delta_2 = \int_0^{\infty} |x - M| f(x) dx,$$

respectively, where  $\mu'_1 = E(X)$  and  $M = \text{Median}(X)$  denotes the median. From equation (16)

$$M = \lambda^{\frac{1}{\delta}} \left\{ \left[ 1 - \left( 1 - 2^{-\frac{1}{b}} \right)^{\frac{1}{a}} \right]^{-\frac{1}{\beta}} - 1 \right\}^{-\frac{1}{\delta}}.$$

These measures can be determined using the following relationships

$$\delta_1 = 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1) \text{ and } \delta_2 = \mu'_1 - 2m_1(M),$$

where  $F(\mu'_1)$  is calculated from (5) and  $m_1(q)$  is the first incomplete moment of  $X$  obtained from (13) with  $r = 1$ .

Finally, we have (for  $\delta > 1$ )

$$\delta_1 = 2\mu'_1 F(\mu'_1) - 2\beta\lambda^{\frac{1}{\delta}} \sum_{j=0}^{\infty} (j+1) w_{j+1} B \left( (1 + \lambda\mu'^{-\delta})^{-1}; (j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta} \right),$$

and

$$\delta_2 = \mu'_1 - 2\beta\lambda^{\frac{1}{\delta}} \sum_{j=0}^{\infty} (j+1) w_{j+1} B \left( (1 + \lambda M^{-\delta})^{-1}; (j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta} \right).$$

## 8. Bonferroni, Lorenz and Zenga Curves

Bonferroni and Lorenz curves have applications in several fields, principally economics and engineering. They are defined by  $B(p) = m_1(q)/(p\mu'_1)$  and  $L(p) = m_1(q)/\mu'_1$ , respectively, where  $\mu'_1 = E(X)$  and  $q = F^{-1}(p)$ . In addition, Zenga (2007) proposed an alternative curve based on the ratio of lower upper means defined by

$$A(p) = 1 - \frac{\mu^-(p)}{\mu^+(p)},$$

where  $\mu^-(p) = \frac{m_1(q)}{p}$  and  $\mu^+(p) = [\mu'_1 - m_1(q)]/(1-p)$  are the lower and upper means, respectively.

We can write the Bonferroni, Lorenz and Zenga curves of X from (12) and (13) as (for  $\delta > 1$ )

$$B(p) = \frac{\sum_{j=0}^{\infty} B((1 + \lambda_q^{-\delta})^{-1}; (j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta})}{p \sum_{j=0}^{\infty} B((j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta})},$$

$$L(p) = \frac{\sum_{j=0}^{\infty} B((1 + \lambda_q^{-\delta})^{-1}; (j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta})}{\sum_{j=0}^{\infty} B((j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta})},$$

and

$$A(p) = 1 - \frac{(1-p) \sum_{j=0}^{\infty} B((1 + \lambda_q^{-\delta})^{-1}; (j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta})}{p \sum_{j=0}^{\infty} K_j},$$

where  $K_j = B((j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta}) - B((1 + \lambda_q^{-\delta})^{-1}; (j+1)\beta + \frac{1}{\delta}, 1 - \frac{1}{\delta})$ .

## 9. Order statistics

Order statistics appear in many areas of statistics and play an important role in practice. The density  $f_{i:n}(x)$  of the  $i$ th order statistic, for  $i = 1, \dots, n$ , from independent and identically distributed random variables  $X_1, \dots, X_n$  is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i; n-i+1)} F(x)^{i-1} [1 - F(x)]^{n-i}.$$

The binomial expansion yields

$$f_{i:n}(x) = \frac{f(x)}{B(i; n-i+1)} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F(x)^{k+i-1}. \quad (19)$$

Substituting (5) and (6) in equation (19), we can write

$$f_{i:n}(x) = \frac{ab\beta\lambda\delta}{B(i; n-i+1)} x^{-\delta-1} (1 + \lambda x^{-\delta})^{-\beta-1} \left[ 1 - (1 + \lambda x^{-\delta})^{-\beta} \right]^{\alpha-1} \\ \times \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left\{ 1 - \left[ 1 - (1 + \lambda x^{-\delta})^{-\beta} \right]^{\alpha} \right\}^{b(k+i)-1}.$$

Applying the binomial expansion twice in the previous equation, we obtain

$$f_{i:n}(x) = \frac{ab\beta\lambda\delta}{B(i; n-i+1)} x^{-\delta-1} \sum_{l=0}^{\infty} \sum_{k=0}^{n-i} \sum_{r=0}^{\infty} (-1)^{l+k+r} \binom{(r+1)a-1}{l} \binom{n-i}{k} \binom{b(k+i)-1}{r} \\ \times (1 + \lambda x^{-\delta})^{-(l+1)\beta-1}.$$

Finally, we have

$$f_{i:n}(x) = \sum_{l=0}^{\infty} s_{l+1} g(x; (l+1)\beta, \lambda, \delta),$$

where

$$s_{l+1} = \frac{ab}{(l+1)B(i; n-i+1)} \sum_{k=0}^{n-i} \sum_{r=0}^{\infty} (-1)^{l+k+r} \binom{(r+1)a-1}{l} \binom{n-i}{k} \binom{b(k+i)-1}{r}$$

Then, the density of the EDA order statistics is a linear combination of Dagum densities. So, some mathematical properties of the EDA order statistics follow from the Dagum distribution.

## 10. Maximum likelihood estimation

We consider estimation of the parameters of the EDA distribution by the method of maximum likelihood. Let  $x = (x_1, \dots, x_n)^\top$  be a sample of size  $n$  from the EDA distribution with unknown parameter vector  $\zeta = (\beta, \lambda, a, b)$ . The total log-likelihood function for  $\zeta$  is

$$\ell(\zeta) = n \log(a) + n \log(b) + n \log(\beta) + n \log(\lambda) + n \log(\delta) - (\delta + 1) \sum_{i=1}^n \log(x_i) + (1 \\ + \frac{1}{\beta}) \sum_{i=1}^n \log(z_i) + (a-1) \sum_{i=1}^n \log(\bar{z}_i) + (b-1) \sum_{i=1}^n \log(\bar{z}_i^a),$$

where  $z_i = z_i(\beta, \lambda, \delta) = (1 + \lambda x_i^{-\delta})^{-\beta}$  and  $\bar{z}_i = 1 - z_i$ , for  $i = 1, \dots, n$ . By taking the partial derivative of the above log-likelihood function with respect to  $\beta, \lambda, \delta, a$  and  $b$ , we obtain the components of the score vector  $U_\zeta = (U_\beta, U_\lambda, U_\delta, U_a, U_b)^\top$

$$U_\beta = \frac{n}{\beta} + \frac{1}{\beta} \sum_{i=1}^n \log(z_i) - \frac{(a-1)}{\beta} \sum_{i=1}^n \frac{\log(z_i)}{z_i^{-1} - 1} - \frac{a(b-1)}{\beta} \sum_{i=1}^n \frac{\bar{z}_i^a \log(z_i)}{(z_i^{-1} - 1)(\bar{z}_i^a - 1)}, \\ U_\lambda = \frac{n}{\lambda} - (\beta + 1) \sum_{i=1}^n \frac{1}{\lambda + x_i^\delta} + (a-1)\beta \sum_{i=1}^n \frac{1}{(\lambda + x_i^\delta)(z_i^{-1} - 1)} \\ + a(b-1)\beta \sum_{i=1}^n \frac{\bar{z}_i^a}{(\lambda + x_i^\delta)(z_i^{-1} - 1)(\bar{z}_i^a - 1)},$$

$$\begin{aligned}
U_\delta &= \frac{n}{\delta} - \sum_{i=1}^n \log(x_i) - (\beta + 1)\lambda \sum_{i=1}^n \frac{\log(x_i)}{\lambda + x_i^\delta} - (a - 1)\beta\lambda \sum_{i=1}^n \frac{\log(x_i)}{(\lambda + x_i^\delta)(z_i^{-1} - 1)} \\
&\quad - a(b - 1)\beta\lambda \sum_{i=1}^n \frac{\bar{z}_i^a \log(x_i)}{(\lambda + x_i^\delta)(z_i^{-1} - 1)(\bar{z}_i^a - 1)}, \\
U_a &= \frac{n}{a} + \sum_{i=1}^n \log(\bar{z}_i) + (b - 1) \sum_{i=1}^n \frac{\bar{z}_i^a \log(\bar{z}_i)}{\bar{z}_i^a - 1}, \\
U_b &= \frac{n}{b} + \sum_{i=1}^n \log(1 - \bar{z}_i^a).
\end{aligned}$$

The maximum-likelihood estimate (MLE)  $\hat{\zeta} = (\hat{\beta}, \hat{\lambda}, \hat{\delta}, \hat{a}, \hat{b})^\top$  of  $\zeta = (\beta, \lambda, \delta, a, b)^\top$  is obtained by solving the nonlinear equations  $U_\beta = 0, U_\lambda = 0, U_\delta = 0, U_a = 0$  and  $U_b = 0$ . These equations cannot be solved analytically, but we can use iterative techniques such as a Newton-Raphson type algorithm to calculate  $\hat{\zeta}$ . The Broyden-Fletcher-Goldfarb-Shanno method (see e.g., Nocedal and Wright, 1999; Press et al., 2007) with analytical derivatives has been used for maximizing the log-likelihood function  $\ell(\zeta)$ .

The normal approximation of the MLE of  $\zeta$  can be used for constructing approximate confidence intervals and for testing hypotheses on the parameters  $\beta, \lambda, \delta, a$  and  $b$ . The elements of the observed information matrix  $J(\zeta) = -\partial^2 \ell(\zeta) / \partial \zeta \partial \zeta^\top$  are given in the Appendix. We can construct approximate confidence regions for the parameters based on the multivariate normal  $N_5(0, J(\hat{\zeta})^{-1})$  distribution.

Further, the likelihood ratio (LR) statistic can be used for comparing this distribution with some of its special models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain LR statistics for testing some sub-models of the EDa distribution.

## 11. Application

We use a real data set to compare the fits of the EDa distribution with three of its sub-models and with two other non-nested models, namely:

- the Kumaraswamy Burr XII (KwBuXII) distribution (Paranaíba et al., 2012). Its pdf (for  $x > 0$ ) is given by

$$f_{KwBuXII}(x) = abcks^{-c} x^{c-1} \left[ 1 + \left( \frac{x}{s} \right)^c \right]^{-k-1} \left\{ 1 - \left[ 1 + \left( \frac{x}{s} \right)^c \right]^{-k} \right\}^{a-1} \left[ 1 - \left\{ 1 + \left( \frac{x}{s} \right)^c \right\}^a \right]^{b-1}.$$

- the beta Dagum (BDa) distribution (Doma and Condimo, 2013). Its pdf (for  $x > 0$ ) is given by

$$f_{BDa}(x) = \frac{\beta\lambda\delta x^{-\delta-1}}{B(a, b)} (1 + \lambda x^{-\delta})^{-\beta-1} [(1 + \lambda x^{-\delta})^{-\beta}]^{a-1} [1 - (1 + \lambda x^{-\delta})^{-\beta}]^{b-1}.$$

The parameters are estimated by maximum likelihood as described in Section 10 using the R software. The data set was studied by Andrews and Herzberg (1985), Cooray and Ananda (2008) and, more recently, by Paranaíba et al. (2012). The  $n = 101$  observations represent the stress-rupture lives of kevlar 49/epoxy strands subjected to constant sustained pressure at the 90% stress level until all had failed such that we obtain a complete data with exact failure times.

Table 1 lists the MLEs (and their standard errors) of the parameters and the Akaike information criterion (AIC) for the fitted models. These results indicate that the EDa distribution has the lowest AIC value among all fitted models, and so it could be chosen as the best model. In order to assess if the model is appropriate Figure 5(a) and 5(b) display the histogram of the data and the plots of the fitted EDa, exp-Da and BDa distributions and the empirical and their estimated survival functions, respectively. These plots indicate that the EDa distribution provides the best fit to these data.

Table 1: MLEs and AIC.

Distribution	$\beta$	$\lambda$	$\delta$	$a$	$b$	AIC
EDa	0.3561 (0.2811)	210.1025 (1.6179)	12.8190 (1.2985)	0.1472 (0.0530)	0.1452 (0.0894)	<b>206.1175</b>
EBuIII	0.1328 (0.0857)	1 -	6.4632 (2.8710)	0.2808 (0.1582)	0.6704 (0.4702)	210.2956
exp-Da	0.1225 (0.0568)	5.7257 (5.0095)	4.8931 (1.4760)	0.5693 (0.2092)	1 -	206.6514
Da	0.2110 (0.0548)	9.0354 (6.9954)	3.4267 (0.6903)	1 -	1 -	206.2137
BDa	10.7435 (12.8560)	18.2728 (22.7181)	4.6932 (1.9385)	0.0144 (0.0138)	0.4119 (0.2770)	208.2504
Distribution	$c$	$k$	$s$	$a$	$b$	AIC
KwBuXII	5.6326 (0.0228)	0.2391 (0.0223)	2.0255 (0.0158)	0.1341 (0.0160)	1.4374 (0.2065)	209.268

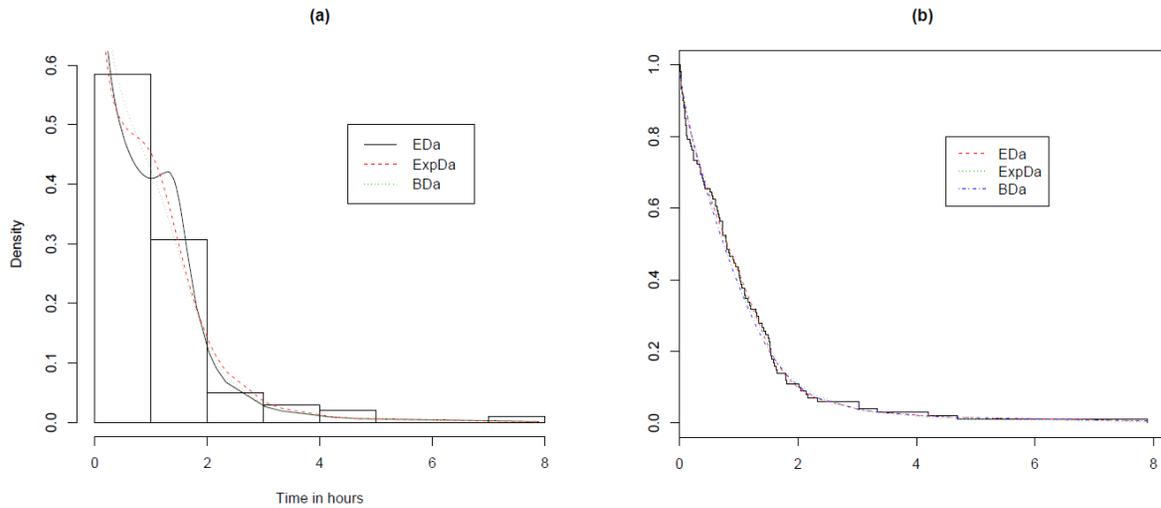


Figure 5: (a) The estimated densities of the EDa, exp-Da and BDa models for stress data. (b) The estimated survival functions from the fitted EDa, exp-Da and BDa distributions and the empirical survival for stress data.

In addition, we use the formal goodness-of-fit tests in order to verify which distribution fits better to the current data. We consider the Cramér-von Mises ( $W^*$ ) and Anderson-Darling ( $A^*$ ) statistics (Chen and Balakrishnan, 1995). In general, the smaller the values of the statistics  $W^*$  and  $A^*$ , the better the fit to the data. The values of the statistics  $W^*$  and  $A^*$  for all models are listed in Table 2. Based on the values of these statistics, we conclude that the EDa model yields the best fit to the current data.

Table 2: Goodness-of-fit tests.

Distribution	Statistic	
	$W^*$	$A^*$
EDa	<b>0.03521</b>	<b>0.27928</b>
EBuIII	0.10101	0.67370
exp-Da	0.04246	0.35267
Da	0.07418	0.49800
BDa	0.06472	0.42938
KwBuXII	0.07959	0.50013

## 12. Concluding remarks

We propose a new five-parameter lifetime model referred to as the extended Dagum (EDa) distribution, which is based on the exponentiated generalized class of distributions recently introduced by Cordeiro *et al.* (2013). The new distribution contains as special models the extended Dagum, extended Burr III, extended log-logistic, log-logistic, Burr III and Dagum

distributions. We demonstrate that the EDa density function can be expressed as a linear combination of the Dagum densities. Based on this result, we derive some structural properties of the proposed distribution and provide the ordinary and incomplete moments, generating function and quantile function. We also obtain the Rényi entropy, mean deviations, Bonferroni, Lorenz and Zenga curves and the density of  $i$ th order statistic. The estimation of the model parameters is approached by maximum likelihood and the observed information matrix is derived. The usefulness of the new distribution is illustrated by means of a real data set.

## Acknowledgment

The authors are grateful to the editor and an anonymous referee for their helpful comments and suggestions. We also gratefully acknowledge grants from CNPq (Brazil).

## Appendix

The observed information matrix for the parameter vector  $\hat{\zeta} = (\hat{\beta}, \hat{\lambda}, \hat{\delta}, \hat{a}, \hat{b})^\top$  is given by

$$J(\zeta) = \frac{\partial^2 \ell(\zeta)}{\partial \zeta \partial \zeta^\top} = \{-U_{rs}\},$$

Where  $r, s \in \{\beta, \lambda, \delta, a, b\}$  and the elements are given by

$$\begin{aligned} U_{\beta\beta} &= -\frac{n}{\beta^2} - \frac{(a-1)}{\beta^2} \sum_{i=1}^n \frac{z_i^{-1} \log^2(z_i)}{(z_i^{-1} - 1)^2} - \frac{a(b-1)}{\beta^2} \sum_{i=1}^n \frac{\log^2(z_i) \bar{z}_i^a [z_i^{-1}(\bar{z}_i^a - 1) + a]}{(z_i^{-1} - 1)^2 (\bar{z}_i^a - 1)^2}, \\ U_{\lambda\lambda} &= -\frac{n}{\lambda^2} + (\beta + 1) \sum_{i=1}^n \frac{1}{(\lambda + x_i^\delta)^2} - (a-1)\beta \sum_{i=1}^n \frac{\beta z_i^{-1} + z_i^{-1} - 1}{(\lambda + x_i^\delta)^2 (z_i^{-1} - 1)^2} \\ &\quad - a(b-1)\beta \sum_{i=1}^n \frac{\bar{z}_i^a [(\beta z_i^{-1} + z_i^{-1} - 1)(\bar{z}_i^a - 1)_{\alpha\beta}]}{(\lambda + x_i^\delta)^2 (z_i^{-1} - 1)^2 (\bar{z}_i^a - 1)^2}, \\ U_{\delta\delta} &= -\frac{n}{\delta^2} - (\beta + 1)\lambda \sum_{i=1}^n \frac{x_i^\delta \log^2(x_i)}{(\lambda + x_i^\delta)^2} - (a-1)\beta\lambda \sum_{i=1}^n \frac{\log^2(x_i) [\beta\lambda z_i^{-1} - x_i^\delta (z_i^{-1} - 1)]}{(\lambda + x_i^\delta)^2 (z_i^{-1} - 1)^2} \\ &\quad - a(b-1)\beta\lambda \sum_{i=1}^n \frac{\log^2(x_i) \bar{z}_i^a \{\beta\lambda [z_i^{-1}(\bar{z}_i^a - 1) + a] - x_i^\delta (z_i^{-1} - 1)(\bar{z}_i^a - 1)\}}{(\lambda + x_i^\delta)^2 (z_i^{-1} - 1)^2 (\bar{z}_i^a - 1)^2}, \\ U_{aa} &= -\frac{n}{a^2} - (b-1) \sum_{i=1}^n \frac{\bar{z}_i^a \log^2 \bar{z}_i}{(\bar{z}_i^a - 1)^2}, \\ U_{bb} &= -\frac{n}{b^2}, \\ U_{\beta\lambda} &= -\sum_{i=1}^n \frac{1}{\lambda + x_i^\delta} + (a-1) \sum_{i=1}^n \frac{z_i^{-1} + z_i^{-1} \log(z_i) - 1}{(\lambda + x_i^\delta)(z_i^{-1} - 1)^2} \end{aligned}$$

$$\begin{aligned}
& + a(b-1) \sum_{i=1}^n \frac{\bar{z}_i^a \{ \log(z_i) [z_i^{-1}(\bar{z}_i^a - 1) + a] + (z_i^{-1} - 1)(\bar{z}_i^a - 1) \}}{(\lambda + x_i^\delta)(z_i^{-1} - 1)^2(\bar{z}_i^a - 1)^2}, \\
U_{\beta\delta} &= \lambda \sum_{i=1}^n \frac{\log(x_i)}{\lambda + x_i^\delta} + (a-1)\lambda \sum_{i=1}^n \frac{\log(x_i)[-z_i^{-1} - z_i^{-1}\log(z_i) + 1]}{(\lambda + x_i^\delta)(z_i^{-1} - 1)^2} \\
& - a(b-1)\lambda \sum_{i=1}^n \frac{\log(x_i) \bar{z}_i^a \{ \log(z_i) [z_i^{-1}(\bar{z}_i^a - 1) + a] + (z_i^{-1} - 1)(\bar{z}_i^a - 1) \}}{(\lambda + x_i^\delta)(z_i^{-1} - 1)^2(\bar{z}_i^a - 1)^2}, \\
U_{\beta a} &= -\frac{1}{\beta} \sum_{i=1}^n \frac{\log(z_i)}{z_i^{-1} - 1} - \frac{(b-1)}{\beta} \sum_{i=1}^n \frac{\log(z_i) \bar{z}_i^a [\bar{z}_i^a - a \log(\bar{z}_i) - 1]}{(z_i^{-1} - 1)(\bar{z}_i^a - 1)^2} \\
U_{\beta b} &= -\frac{\alpha}{\beta} \sum_{i=1}^n \frac{\log(z_i) \bar{z}_i^a}{(z_i^{-1} - 1)(\bar{z}_i^a - 1)}, \\
U_{\lambda\delta} &= (\beta + 1) \sum_{i=1}^n \frac{x_i^\delta \log(x_i)}{(\lambda + x_i^\delta)^2} + (a-1)\beta \sum_{i=1}^n \frac{\log(x_i) [\beta \lambda z_i^{-1} - x_i^\delta (z_i^{-1} - 1)]}{(\lambda + x_i^\delta)^2 (z_i^{-1} - 1)^2} \\
& + a(b-1)\beta \sum_{i=1}^n \frac{\log(x_i) \bar{z}_i^a \{ \beta \lambda [z_i^{-1}(\bar{z}_i^a - 1) + a] - x_i^\delta (z_i^{-1} - 1)(\bar{z}_i^a - 1) \}}{(\lambda + x_i^\delta)^2 (z_i^{-1} - 1)^2 (\bar{z}_i^a - 1)^2}, \\
U_{\lambda a} &= \beta \sum_{i=1}^n \frac{1}{(\lambda + x_i^\delta)(z_i^{-1} - 1)} + (b-1)\beta \sum_{i=1}^n \frac{\bar{z}_i^a [\bar{z}_i^a - a \log(\bar{z}_i) - 1]}{(\lambda + x_i^\delta)(z_i^{-1} - 1)(\bar{z}_i^a - 1)^2}, \\
U_{\lambda b} &= \alpha\beta \sum_{i=1}^n \frac{\bar{z}_i^a}{(\lambda + x_i^\delta)(z_i^{-1} - 1)(\bar{z}_i^a - 1)}, \\
U_{\delta a} &= -\beta\lambda \sum_{i=1}^n \frac{\log(x_i)}{(\lambda + x_i^\delta)(z_i^{-1} - 1)} - (b-1)\beta\lambda \sum_{i=1}^n \frac{\log(x_i) \bar{z}_i^a [\bar{z}_i^a - a \log(\bar{z}_i) - 1]}{(\lambda + x_i^\delta)(z_i^{-1} - 1)(\bar{z}_i^a - 1)^2}, \\
U_{\delta b} &= -\alpha\beta\lambda \sum_{i=1}^n \frac{\log(x_i) \bar{z}_i^a}{(\lambda + x_i^\delta)(z_i^{-1} - 1)(\bar{z}_i^a - 1)},
\end{aligned}$$

and

$$U_{ab} = \sum_{i=1}^n \frac{\bar{z}_i^a \log(\bar{z}_i)}{\bar{z}_i^a - 1}$$

where  $z_i$  and  $\bar{z}_i$  are defined in Section 10.

## References

- [1] Andrews, D. F. and Herzberg, A. M. (1985). Data: A Collection of Problems from Many Fields for the Student and Research Worker. New York: Springer Series in Statistics.

- 
- [2] Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test, *J. Qual. Technol.*, **27**, 154-161.
- [3] Cooray, K. and Ananda, M.M.A. (2008). A generalization of the half-normal distribution with applications to lifetime data, *Commun. Stat. Theory Meth.*, **37**, 1323-1337.
- [4] Cordeiro, G. M. and de Castro M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, **81**, 883-898.
- [5] Cordeiro, G.M., Ortega, E.M.M. and da Cunha, D.C.C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, **11**, 1-27.
- [6] Dagum, C. (1977). A new model of personal income distribution: specification and estimation. *Economie Appliquée*, **30**, 413-437.
- [7] Dagum, C. (1980). The generation and distribution of income, the Lorenz curve and the Gini ratio. *Economie Appliquée*, **33**, 327-367.
- [8] Domma, F. (2007) Asymptotic distribution of the maximum likelihood estimators of the parameters of the right-truncated Dagum distribution, *Comm. Statist. Simulation Comput.*, **36**, 1187-1199.
- [9] Domma, F. and Condino, F. (2013) The Beta-Dagum Distribution: Definition and Properties, *Communications in Statistics-Theory and Methods*, in press.
- [10] Domma, F., Giordano, S. and Zenga, M. (2009). The Fisher information matrix in doubly censored data from the Dagum distribution, Working Paper No. 8, Department of Economics and Statistics, University of Calabria, Italy.
- [11] Domma, F., Giordano, S. and Zenga, M. (2011a). The Fisher information matrix in right censored samples from the Dagum distribution, Working Paper No. 4, Department of Economics and Statistics, University of Calabria, Italy.
- [12] Domma, F., Giordano, S. and Zenga, M. (2011b). Maximum Likelihood Estimation in Dagum Distribution with Censored Samples, *Journal of Applied Statistics*, **38**, 2971-2985.

- [13] Domma, F., Latorre, G. and Zenga, M. (2011c). Reliability studies of the Dagum distribution, Working Paper No. 207, Department of Quantitative Methods for Economics and Business, University of Milan - Bicocca, Italy.
- [14] Domma, F. and Perri, P.F. (2009). Some developments on the log-Dagum distribution, *Statist. Methods Appl.*, **18**, 205-209.
- [15] Eugene N., Lee C., Famoye F.(2002). Beta-Normal distribution and its application. *Communication in Statistics - Theory and Methods*, **4**, 497-512.
- [16] Kenney, J. F. and Keeping, E. S. (1962). *Mathematics of Statistics, Part 1*. Princeton, NJ: Van Nostrand.
- [17] Kleiber, C.(2008). A guide to the Dagum distribution. In: Duangkamon, C. (ed.) *Modeling Income Distributions and Lorenz Curves Series: Economic Studies in Inequality, Social Exclusion and Well-Being*, 5. Springer, New York.
- [18] Kleiber, C. and Kotz, S. (2003). *Statistical Size Distribution in Economics and Actuarial Sciences*. John Wiley & Sons.
- [19] Nocedal, J. and Wright, S.J. (1999). *Numerical Optimization*, Springer, NewYork.
- [20] Oluyede, B. O. and Rajasooriya, S. (2013). The Mc-Dagum Distribution and its Statistical Properties with Applications. *Asian Journal of Mathematics and Applications*, 1-16.
- [21] Paranaíba, P. F., Ortega, E.M.M., Cordeiro, G.M. and de Pascoa, M. A. R. (2012). The Kumaraswamy Burr XII distribution: theory and practice. *Journal of Statistical Computation and Simulation*, **82**, 1-27.
- [22] Press, W.H., Teulosky, S.A., Vetterling, W.T. and Flannery, B.P. (2007) *Numerical Recipes in C: The Art of Scientific Computing*, 3rd ed.. Cambridge University Press, NewYork.
- [23] Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I. (1986). *Integrals and Series*, Vol. 1, Gordon and Breach Science Publishers, Amsterdam.
- [24] Shahzad, M. N. and Asghar, Z. (2013). Comparing TL-Moments, L-Moments and Conventional Moments of Dagum Distribution by Simulated data. *Revista Colombiana de Estadística*, **36**, 79-93.

- [25] Song, K.S. (2001). Rényi information, loglikelihood and an intrinsic distribution measure, *J. Statist. Plann. Inference.*, **93**, 51-69.
- [26] Zografos K. and Balakrishnan N. (2009). On families of beta and generalized gamma-generated distributions and associated inference, *Statistics*, **23**, 344-362.
- [27] Zenga Mi. (2007). Inequality curve and inequality index based on the ratios between lower and upper arithmetic means. *Statistica & Applicazioni*, **5**, 3-27.

Received December 3, 2013; accepted July 26, 2014.

Alisson de O. Silva  
Universidade Federal de Pernambuco Departamento de Estatística  
Av. Prof. Moraes Rego, 1235 - Cidade Universitária, Recife - PE - CEP: 50670-901  
Allysson\_jlr@yahoo.com.br

Luana Cecília M. da Silva  
Universidade Federal de Pernambuco Departamento de Estatística  
Av. Prof. Moraes Rego, 1235 - Cidade Universitária, Recife - PE - CEP: 50670-901  
Ceciliameireles2006@hotmail.com

Gauss M. Cordeiro  
Universidade Federal de Pernambuco Departamento de Estatística  
Av. Prof. Moraes Rego, 1235 - Cidade Universitária, Recife - PE - CEP: 50670-901  
gauss@de.ufpe.br

