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Estimation for flexible Weibull extension under progressive Type-II censoring

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Abstract: : In this paper, we discussed classical and Bayes estimation procedures for estimating the unknown parameters as well as the reliability and hazard functions of the flexible Weibull distribution when observed data are collected under progressively Type-II censoring scheme. The performances of the maximum likelihood and Bayes estimators are compared in terms of their mean squared errors through the simulation study. For the computation of Bayes estimates, we proposed the use of Lindley's approximation and Markov Chain Monte Carlo (MCMC) techniques since the posteriors of the parameters are not analytically tractable. Further, we also derived the one and two sample posterior predictive densities of future samples and obtained the predictive bounds for future observations using MCMC techniques. To illustrate the discussed procedures, a set of real data is analysed.

Key words: Progressive Type-II censored data, reliability and hazard functions, maximum likelihood estimation, Bayes estimation, Bayes prediction

1. Introduction

The probability density function of the flexible Weibull distribution is defined as follows

$$f(x) = \left(\alpha + \frac{\beta}{x^2}\right) \exp\left(\alpha x - \frac{\beta}{x}\right) \exp\left(- exp(\alpha x - \frac{\beta}{x})\right) \; ; \; x > 0, \alpha > 0, \beta > 0. \tag{1}$$

The flexible Weibull distribution is proposed by [3] and has the nice closed forms of reliability and hazard functions are given by

$$R(x) = exp\left(exp\left(-e^{\alpha x - \frac{\beta}{x}}\right)\right); x > 0, \alpha, \beta > 0. \tag{2}$$

and

$$h(x) = \left(\alpha + \frac{\beta}{x^2}\right) exp\left(-e^{\alpha x - \frac{\beta}{x}}\right); \; x > 0, \alpha, \beta > 0. \tag{3}$$

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[3] has discussed the some statistical properties of the flexible Weibull distribution and shown its applicability to a set of real data of times between failures of secondary reactor pumps. They have also shown that this distribution is able to model various types of data with IFR (Increase Failure Rate), IFRA (Average Increase Failure Rate), and MBT (modified bathtub) shaped hazard rates. Therefore, this distribution provides an alternative platform to lifetime data analysis over other existing well- known lifetime distributions such exp, gamma, and Weibull etc. Recently, [19] discussed the inferential procedures for estimating the parameters and future order statistics of the flexible Weibull (FW) distribution under the Type-II censoring scheme.

There are many situations where the experimental units may damaged or lost accidentally or intentionally removed for some other purposes from the life-test at the time before completion of the experiment. For making the inference from such type of data thus obtained, the conventional Type-I and Type-II censoring scheme are found to be inappropriate. To overcome such a difficulty arise in survival studies, [6] introduced a methodology of the sampling mechanism for life-testing procedures which facilitate to incorporate the accidental breakage into analysis. Since then, the estimation procedures based on the progressively censored samples have been widely discussed in the literature. For a comprehensive review of progressive censoring, the readers may be referred to [2, 1]. The estimation of the parameters of various lifetime distribution based on the progressive Type-II censoring scheme can also be found in [7, 8, 12, 13, 18, 24, 15] and references cited therein.

[11] and [22] have discussed the prediction problems of future order statistics from Weibull distributions in case of progressive censoring scheme under symmetric and asymmetric loss functions respectively. Very recently, [20] developed the estimation and prediction procedures for the generalized Lindley distribution under the progressive Type-II censoring scheme with beta-binomial removals (PT- IICBBR).

In this paper, first we will discuss the estimation of the parameters, reliability and hazard function of FW distribution when only the progressive Type-II censored sample is available for the analysis purpose, and secondly, the predictive inference will be performed for the future samples censored due to the consideration of the progressive Type-II censoring scheme.

Under the above considerations, we organized the rest of the paper as follows: In section 2, the construction of the likelihood function of the FW distribution is described while the progressively Type II right censored sample is observed. Section 3 deals with maximum likelihood estimation (MLE) of the parameters as well as reliability and hazard functions. In section 4, Bayes estimation for the parameters, reliability and hazard functions have been discussed. In its subsections, Lindley's approximations and Markov Chain Monte Carlo (MCMC) techniques have been utilized to compute Bayes estimates. Section 5 deals with simulation study to compare classical and Bayes estimators. For illustration, a set of real data is discussed in Section 6. In sections 7 and 8, one and two sample prediction problems have been considered for FW distribution respectively. Section 9 provides the conclusions.

2. Sampling mechanism and likelihood function

Let an experiment starts with n identical experimental units at time 0. Here, the number of failures corresponds to the stages of the experiment, then, to reach at the m the stage of the experiment, we progressively proceeds in the following way. As soon as the first failure x_1 is observed, some units say r_1 are removed/lost from the remaining (n-1) surviving units at the first stage of the experiment. At the second stage, x_2 is observed and r_2 units are removed/lost from remaining $n-2-r_1$ units, and so on, and finally at mth stage, failure of the mth unit x_m is observed and experiment is stopped removing all the remaining $r_m = n-m-\sum_{j=1}^{m-1} r_j$ units from the experiment. Such a sampling mechanism is termed as the progressive Type-II censoring scheme. Here, $x_1, x_2, \cdots, x_m$ be a progressively Type II censored sample with progressive censoring scheme $\mathrm{R} = (\mathrm{r}_1, \mathrm{r}_2, \cdots, \mathrm{r}_\mathrm{m})$.

To get a progressive Type-II censored sample of size m from the flexible Weibull distribution, we used the algorithm given in [2], which can be described as follows: If $W_i \sim$ Uniform(0, 1); i = 1,2, ⋯,m. The PT-IICS from standard uniform distribution can be obtained by using the following relationship

$$U_i = 1 - V_m$$
$$U_i = 1 - (1 - U_{i-1})V_{m-i+1};\ i = 2,3, \ldots, m,$$

where, $V_i = W_i^{\left(1+\sum_{j=m-i+1}^{m} r_j\right)^{-1}}$

Using the uniform PT-IICS, the PT-IICS from the FW distribution is thus obtained using the probability integral transformation as defined below:

$$X_i = \frac{1}{2\alpha}\left[\log(-\log(1-U_i)) + \sqrt{(\log(-\log(1-U_i)))^2 + 4\alpha\beta}\right];\ i = 1,2, \ldots, m$$

Therefore, based on progressively Type-II censored sample (PT-IICS), $(x_1, r_1), (x_2, r_2), (x_3, r_3), \cdots, (x_m, r_m)$, the likelihood function can be defined as:

$$\ell(x, \alpha, \beta | r) = \mathrm{c}^* \prod_{i=1}^{m} f(x_i)[R(x_i)]^{r_i} \tag{4}$$

where $c^* = \prod_{j=1}^{m}\left(n - \sum_{i=1}^{j-1} r_i - j + 1\right)$, and $0 \le r_i \le \left(n - m - \sum_{j=1}^{i-1} r_j\right)$ for $i = 1,2,3, \ldots,$ $m-1$ and $r_0 = 0$, and $f(.)$ and R(.) given in the equation (1) and (2) respectively.

3. Maximum likelihood estimation for α, β, R (.) and h (.)

In this section we obtain the maximum likelihood estimates (MLEs) of the parameter, reliability and hazard functions of the flexible Weibull distribution using progressively Type II censored sample. The likelihood function (4) can be written as

$$\ell(x, \alpha, \beta | r) = \mathrm{c}^* \prod_{i=1}^{m}\left[\left(\alpha + \frac{\beta}{x_i^2}\right)\exp\left(\alpha x_i - \frac{\beta}{x_i}\right)\right]\left[\exp\left(-\exp\left(\alpha x_i - \frac{\beta}{x_i}\right)\right)\right]^{r_i+1} \tag{5}$$

The log-likelihood function is given by

$$\ell(x, \alpha, \beta) = \log(c^*) + \sum_{i=1}^{m}\log\left(\alpha + \frac{\beta}{x_i^2}\right) + \sum_{i=1}^{m}\left(\alpha x_i - \frac{\beta}{x_i}\right) - \sum_{i=1}^{m}(r_i+1)\exp(\alpha x_i - \frac{\beta}{x_i}) \tag{6}$$

Let $\hat{\alpha}$ and $\hat{\beta}$denote the MLEs of the parameters α and β respectively, and can be obtained by solving the following log-likelihood equations:

$$\frac{\partial Log\ell(x,\alpha,\beta)}{\partial\alpha}\Big|_{\hat{\alpha},\hat{\beta}} = 0 \Longrightarrow \sum_{i=1}^{m}\frac{1}{\left(\hat{\alpha}+\frac{\hat{\beta}}{x_i^2}\right)} + \sum_{i=1}^{m} x_i - \sum_{i=1}^{m}(1+r_i)\, x_i \exp\left(\hat{\alpha}x_i - \frac{\hat{\beta}}{x_i^2}\right) = 0 \quad (7)$$

$$\frac{\partial Log\ell(x,\alpha,\beta)}{\partial\beta}\Big|_{\hat{\alpha},\hat{\beta}} = 0 \Longrightarrow \sum_{i=1}^{m}\frac{1}{(\hat{\alpha}x_i^2+\hat{\beta})} - \sum_{i=1}^{m}\frac{1}{x_i} + \sum_{i=1}^{m}\frac{(1+r_i)}{x_i}\exp\left(\hat{\alpha}x_i - \frac{\hat{\beta}}{x_i^2}\right) = 0 \quad (8)$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ cannot be obtained directly from (7) and (8). Thus, MLEs $\hat{\alpha}$ and $\hat{\beta}$ of αand β respectively can be obtained by using the numerical iterative procedure such as fixed point iteration method, for the complete algorithm of the fixed point iteration method, see [19]. Once, the MLEs $\hat{\alpha}$ and $\hat{\beta}$ are obtained, we can easily obtain the MLEs of reliability and hazard function at epoch t as follows:

$$\hat{R}(t) = \exp\left(-exp\left(\hat{\alpha}t - \frac{\hat{\beta}}{t}\right)\right) \quad (9)$$

$$\hat{h}(t) = \left(\hat{\alpha} + \frac{\hat{\beta}}{t^2}\right)\left(\exp\left(\hat{\alpha}t - \frac{\hat{\beta}}{t}\right)\right) \quad (10)$$

4. Bayes estimation

In this section, we discuss the Bayes estimation of the parameter, reliability and hazard functions of the flexible Weibull distribution using progressively Type-II censored sample while the unknown parameters α and β have the gamma priors with the density function of the following forms:

$$g(\alpha) \propto e^{-a\alpha}\alpha^{b-1};\ \alpha > 0,\ a,b > 0 \quad (11)$$

$$g(\beta) \propto e^{-c\beta}\beta^{d-1};\ \beta > 0,\ c,d > 0 \quad (12)$$

Using the likelihood function (5) and the priors in (11) and (12), the joint posterior distributions of α and β is then given by

$$\pi\left(\alpha,\beta\middle|\underset{\sim}{x}\right) \propto \alpha^{b-1}\beta^{d-1}\exp\left(-\beta\left(\sum_{i=1}^{m} x_i^{-1} + c\right)\right)\exp\left(-\alpha\left(a - \sum_{i=1}^{m} x_i\right)\right)$$
$$\exp\left(-(\mathrm{r_i}+1)\sum_{i=1}^{m}\exp\left(\alpha x_i - \frac{\beta}{x_i}\right)\right)\exp\left(\sum_{i=1}^{m}\log\left(\alpha + \frac{\beta}{x_i^2}\right)\right) \quad (13)$$

Since it is not easy to summarise the above posterior analytically, we proposed the use of the Lindley's approximation and Markov Chain Monte Cralo (MCMC) techniques for obtaining the Bayes estimates of parameters as well as reliability and hazard functions of flexible Weibull distribution in the next two subsections.

4.1 Lindley's approximation

The ratio of integrals often occurs in Bayesian analysis is given by

$$I\left(\underset{\sim}{x}\right)=\frac{\int_{\theta} u(\theta)\exp\{L(\theta)+\rho(\theta)\}d\underset{\sim}{\theta}}{\int_{\theta}\exp\{L(\theta)+\rho(\theta)\}d\underset{\sim}{\theta}} \tag{14}$$

Where,
$\theta = \{\theta_1, \theta_2, \dots, \theta_m\}$,
$u(\theta)$ = function of θ,
$L(\theta)$ = log-likelihood function, and
$\rho(\theta)$ = log of joint prior of θ,

In 1980, Lindley [14] provided the asymptotic solution for the ratio of two integrals that is encountered in Bayesian calculation as given in (14). Many authors have proposed the use of Lindley's approximation for obtaining the Bayes estimates of the parameters of various lifetime distributions, see [16, 17] and references cited therein. For large sample and under some regularity conditions, the above expression can be approximated by the following expansion

$$I\left(\underset{\sim}{x}\right)\approx u+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_{ij}+2u_i\rho_j\right)\sigma_{ij}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}L_{ijk}u_l\sigma_{ij}\sigma_{kl}. \tag{15}$$

In our case, $\theta = \{\alpha, \beta\}\ i.e.\ m = 2$, where,

$L_{111}=\frac{\partial^3 Log\ell(\alpha,\beta)}{\partial\alpha\partial\alpha\partial\alpha}=\sum_{i=1}^{m}\frac{2}{(\alpha+\frac{\beta}{x_i^2})^3}-\sum_{i=1}^{m}(1+r_i)x_i^3\, e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)}$,

$L_{221}=L_{212}=L_{122}=\frac{\partial^3 Log\ell(\alpha,\beta)}{\partial\beta\partial\beta\partial\alpha}=\sum_{i=1}^{m}\frac{2x_i^2}{(\alpha x_i^2+\beta)^3}-\sum_{i=1}^{m}\frac{(1+r_i)}{x_i}e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)}$,

$L_{222}=\sum_{i=1}^{m}\frac{2}{\left(\alpha x_i^2+\beta\right)^3}+\sum_{i=1}^{m}\frac{(1+r_i)}{x_i^3}e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)}, L_{112}=L_{121}=L_{211}=\frac{\partial^3 Log\ell(\alpha,\beta)}{\partial\alpha\partial\alpha\partial\beta}=$
$\sum_{i=1}^{m}\frac{2x_i^4}{\left(\alpha x_i^2+\beta\right)^3}+\sum_{i=1}^{m}(1+r_i)\,x_i e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)}, \rho_1=\frac{b-1}{\alpha}-a, \rho_2=\frac{d-1}{\beta}-c, u_1=\frac{\partial u}{\partial\alpha}, u_2=\frac{\partial u}{\partial\beta}$,
$u_{12}=u_{21}=\frac{\partial u}{\partial\alpha\partial\beta}, u_{22}=\frac{\partial^2 u}{\partial\beta\partial\beta}, u_{11}=\frac{\partial^2 u}{\partial\alpha\partial\alpha}$ and $\sigma_{i,j}=(i,j)$th element of the matrix Δ^{-1} defined as

$$\Delta^{-1}=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}=\begin{pmatrix}\frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\alpha\partial\beta}\\ \frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\beta\partial\alpha} & \frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\beta^2}\end{pmatrix}^{-1}$$

with

$$\frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\alpha^2}=-\sum_{i-1}^{m}\frac{1}{(\alpha-\frac{\beta}{x_i^2})^2}-\sum_{i=1}^{m}(1+r_i)x_i^2 e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)} \tag{16}$$

$$\frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\beta^2}=-\sum_{i-1}^{m}\frac{1}{(\alpha x_i^2+\beta)^2}-\sum_{i=1}^{m}\frac{(1+r_i)}{x_i^2}e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)} \tag{17}$$

$$\frac{\partial^2 Log\ell(x,\alpha,\beta)}{\partial\beta\partial\alpha}=-\sum_{i=1}^{m}\frac{x_i}{(\alpha x_i^2+\beta)}+\sum_{i=1}^{m}(1+r_i)e^{\left(\alpha x_i-\frac{\beta}{x_i}\right)} \tag{18}$$

Note that all the above terms and derivatives are calculated at the maximum likelihood estimates of α and β. Thus, the value of $I(\underset{\sim}{x})$ is calculated by $\hat{I}\left(\underset{\sim}{x}\right) = \mathrm{I} - \left(\underset{\sim}{x}\right)_{\widehat{\alpha,B}}$

4.2 Bayes estimates of α and β using Lindley's approximation

If $u(\alpha,\beta) = \alpha$, then $u_1 = 1, u_2 = u_{12} = u_{21} = u_{22} = u_{11} = 0$. Therefore, the Bayes estimate of α is then defined as follows:

$$\hat{\alpha}_{LD} = \hat{\alpha}_{ML} + \sigma_{11}\rho_1 + \sigma_{12}\rho_1 + \sigma_{22}\rho_2 + 0.5\{L_{111}\sigma_{11}^2 + 3L_{112}\sigma_{11}\sigma_{12} + L_{122}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\} \quad (19)$$

If $u(\alpha,\beta) = \beta$, then $u_2 = 1, u_1 = u_{12} = u_{21} = u_{22} = u_{11} = 0$. Therefore, the Bayes estimate of β can be calculated by the following formula:

$$\hat{\beta}_{LD} = \hat{\beta}_{ML} + \sigma_{21}\rho_1 + \sigma_{22}\rho_2 + 0.5\{L_{222}\sigma_{22}^2 + L_{112}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2 + 3L_{122}\sigma_{12}\sigma_{22} + L_{111}\sigma_{11}\sigma_{12})\} \quad (20)$$

4.3 Bayes estimates of R(.) and h(.) using Lindley's approximation

If $u(\alpha,\beta\ ) = \exp\left(\exp\left(-\mathrm{e}^{\alpha x - \frac{\beta}{x}}\right)\right)$, Bayes estimate of $R(t)$ is given by

$$\hat{R}_{LD}(t) \approx u^r + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(u_{ij}^r + 2u_i^r\rho_j\right)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} L_{ijk}u_l^r\sigma_{ij}\,\sigma_{kl}. \quad (21)$$

Here,

$$u_1^r = -t(u^r)\,exp\left(\alpha t - \frac{\beta}{t}\right),$$
$$u_2^r = -\frac{(u^r)}{t}exp\left(\alpha t - \frac{\beta}{t}\right),$$
$$u_{12}^r = u_{21}^r = -\,exp\left(\alpha t - \frac{\beta}{t}\right)((u_2^r)t - u^r),$$
$$u_{22}^r = \frac{1}{t^2}exp\left(\alpha t - \frac{\beta}{t}\right)(u_2^r t - u^r),$$
$$u_{11}^r = -t\,exp\left(\alpha t - \frac{\beta}{t}\right)(tu^r - u_1^r),$$

In the same way, we can also obtain the Bayes estimate of the hazard function as follows:

$$\hat{h}_{\mathrm{Ld}}(t) \approx u^r + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(u_{ij}^r + 2u_i^h\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} L_{ijk}u_l^r\sigma_{ij}\sigma_{kl}. \quad (22)$$

By expending the summation terms, we have

$$\hat{h}_{LD}(t) \approx u^h + 0.5[\sigma_{11}\left(u_{11}^h + 2u_1^h\rho_1\right) + \sigma_{12}\left(u_{12}^h + 2u_1^r\rho_2\right) + \sigma_{21}\left(u_{21}^h + 2u_2^h\rho_1\right) + \ \sigma_{22}(u_{22}^h$$
$$+2u_2^h\rho_2)] + 0.5\left[L_{111}\left(u_1^h\sigma_{11}^2 + u_2^h\sigma_{11}\sigma_{12}\right) + L_{112}\left(3u_1^h\sigma_{11}\sigma_{12} + u_2^h(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\right)\right]$$
$$+0.5[L_{222}\left(u_2^h\sigma_{22}^2 + u_1^h\sigma_{12}\sigma_{22}\right) + L_{122}(3u_2^h\sigma_{12}\sigma_{22} + u_1^h(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2))\ ] \quad (23)$$

where,

$$u^h = \left(\alpha + \frac{\beta}{t^2}\right)ex\,p\left(\alpha t - \frac{\beta}{t}\right),$$
$$u_1^h = exp\left(\alpha t - \frac{\beta}{t}\right) + u^h t,$$

$u_{11}^{h} = t\ exp\left(\alpha t - \frac{\beta}{t}\right) + u_{1}^{h}t,$

$u_{2}^{h} = \frac{\exp\left(\alpha t - \frac{\beta}{t}\right)}{t^{2}} - \frac{u^{h}}{t},$

$u_{22}^{h} = -\frac{\exp\left(\alpha t - \frac{\beta}{t}\right)}{t^{3}} - \frac{u_{2}^{h}}{t},$

$u_{12}^{h} = u_{21}^{h} = -\frac{\exp\left(\alpha t - \frac{\beta}{t}\right)}{t} + u_{2}^{h}t,$

4.4 MCMC techniques

The MCMC techniques are the most widely used computation tools in the Bayesian analysis. Basically, we draw a sample based inference to the posteriors of the parameters of interest. So doing this, we need to have a posterior sample and in many situations it not an easy job to draw a sample from the posterior, specially in the multivariate case. Therefore, the reduction of the dimensionality of the problem is the only possible solution. In such away, we can generate the sample from the joint posterior using the algorithm known as the Gibbs sampler which reduces the k dimensions problem into a single dimension. If we are interested to draw a sample from $\pi(\alpha, \beta | \underset{\sim}{x})$, two stage Gibbs ~algorithm can be utilized as follows:

Step1. Set starting points, say $\alpha^{(0)}$, $\beta^{(0)}$ and then at *i*th stage

Step2. Generate $\alpha_i \sim \pi_1(\alpha | \beta^{(i-1)}, \underset{\sim}{x})$

Step3. Generate $\beta_i \sim \pi_2(\beta | \alpha_i, \underset{\sim}{x})$

Step4. Repeat steps 2 and 3, M(=1500) times to get the samples of sizes M from the correspondings

Step5. Obtain the Bayes estimates of α and β using the formulae $\hat{\alpha}_{MC} = \frac{1}{M-M_0}$

Step6. Similarly, the Bayes estimates of $R(t)$ and $h(t)$ are

$$\hat{R}_{MC}(t) = \frac{1}{M - M_0} \sum_{J=M_0+1}^{M} \exp\left(-\exp\left(\alpha_j t - \frac{\beta_j}{t}\right)\right)$$

and

$$\hat{h}_{MC}(t) = \frac{1}{M - M_0} \sum_{J=M_{0+1}}^{M} \left(\alpha_j + \frac{\beta_j}{t^2}\right) \exp\left(\alpha_j t - \frac{\beta_j}{t}\right)$$

respectively.

Here, $\pi_1(\alpha | \beta, \underset{\sim}{x})$ and $\pi_2(\beta | \alpha, \underset{\sim}{x})$ are the full conditional posteriors of α and β respectively and given by

$$\pi_1\left(\alpha \middle| \beta, \underset{\sim}{x}\right) \propto \alpha^{\mathrm{b}-1} \exp\left(-\alpha\left(a - \sum_{i=1}^{m} x_i\right)\right) \exp\left(-(r_i + 1) \sum_{i=1}^{m} \exp\left(\alpha x_i - \frac{\beta}{x_i}\right)\right) \exp\left(\sum_{i=1}^{m} \log(\alpha + \frac{\beta}{x_i^2})\right) \tag{24}$$

$$\pi_2\left(\beta \middle| \alpha, \underset{\sim}{x}\right) \propto \beta^{d-1} \exp\left(-\beta\left(\sum_{i=1}^{m} x_i^{-1} + c\right)\right) \exp\left(-(r_i+1)\sum_{i=1}^{m} exp\left(\alpha x_i - \frac{\beta}{x_i}\right)\right) \exp\left(\sum_{i=1}^{m} log\left(\alpha + \frac{\beta}{x_i^2}\right)\right) \tag{25}$$

The full conditionals (24) and (25) are not the standard distributions from which the simulation is easy. Therefore, we use the independent Metropolis Hastings (MH) algorithm to generate the samples from (24) and (25) in Steps 2 and 3 of the above algorithm. For more detail on the implications of the MCMC techniques i.e. Gibbs sampler and MH algorithm, readers may be referred to [4, 10, 9, 21].

5. Simulation study

In this section, we studied the behaviour of the proposed estimators on the basis of simulated sample from FW distribution. Using the algorithm given in section 2, different progressive Type-II censored samples have been generated from the FW distribution for given values of n = 20, 30, 50, α = 2 and β = 2. For this study, we considered different removal patterns for given n and m as follows:

Scheme 0. No censoring i.e. complete study,

Scheme 1. All (n − m) items are removed at first stage,

Scheme 2. All (n − m) items are removed at *m*th stage,

Scheme 3. Items are removed at some beginning stages,

Scheme 4. Items are removed at some last stages, Scheme 5. Items are removed at some middle stages.

All considered censoring schemes are summarised in Table 1. In Table 1, (a ∗r) represents the vector of containing r times a. For example, consider a censoring scheme (0 ∗ 2, 3 ∗ 2, 0 ∗ 3) gives (0, 0, 3, 3, 0, 0, 0).

Applying the procedures discussed in the previous sections on each sample, we calculated the maximum

Table 1: Progressive censoring schemes used in simulation study

n	m	Scheme				
		(1)	(2)	(3)	(4)	(5)
20	12	(8 ∗ 1, 0 ∗ 11)	(0 ∗ 11, 8 ∗ 1)	(2 ∗ 4, 0 ∗ 8)	(0 ∗ 8, 2 ∗ 4)	(0 ∗ 4, 2 ∗ 4, 0 ∗ 4)
	16	(4 ∗ 1, 0 ∗ 15)	(0 ∗ 15, 4 ∗ 1)	(1 ∗ 4, 0 ∗ 12)	(0 ∗ 12, 1 ∗ 4)	(0 ∗ 4, 1 ∗ 4, 0 ∗ 4)
30	18	(12 ∗ 1, 0 ∗ 17)	(0 ∗ 17, 12 ∗ 1)	(2 ∗ 6, 0 ∗ 12)	(0 ∗ 12, 2 ∗ 6)	(0 ∗ 6, 2 ∗ 6, 0 ∗ 6)
	24	(6 ∗ 1, 0 ∗ 23)	(0 ∗ 23, 6 ∗ 1)	(1 ∗ 6, 0 ∗ 18)	(0 ∗ 18, 1 ∗ 6)	(0 ∗ 9, 1 ∗ 6, 0 ∗ 9)
50	30	(20 ∗ 1, 0 ∗ 29)	(0 ∗ 29, 20 ∗ 1)	(2 ∗ 10, 0 ∗ 20)	(0 ∗ 20, 2 ∗ 10)	(0 ∗ 10, 2 ∗ 10, 0 ∗ 10)
	40	(10 ∗ 1, 0 ∗ 39)	(0 ∗ 39, 10 ∗ 1)	(1 ∗ 10, 0 ∗ 30)	(0 ∗ 30, 1 ∗ 10)	(0 ∗ 15, 1 ∗ 10, 0 ∗ 15)

likelihood and Bayes estimates of the parameters, reliability and hazard function at t = 1. This process is repeated 1000 of times and, average estimates and corresponding mean square errors of all estimators are reported in Tables [2,3,4,5,6]. Where A(.) and M (.) represent the average estimate and mean squared error respectively, if $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_{1000}$ are the 1000 estimates of θ obtained from 1000 different samples, then average estimate and mean square error can be calculated by the following formulae,

$$\mathrm{A}\left(\hat{\phi}(\theta)\right) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\phi}_i(\theta)\right), M\left(\hat{\phi}(\theta)\right) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\phi}_i(\theta) - \phi(\theta)^2\right),$$

where $\hat{\varphi}\ (\theta)$ is an estimate of $\varphi(\theta)$. For Bayes estimation, we take two combinations (0, 0, 0, 0) and (3, 6, 3, 6) of the hyper-parameters (a, b, c, d) respectively. The values (0, 0, 0, 0) of the hyper parameters corresponds to the vague priors of the unknown parameters of the flexible Weibull distribution while the combination (3, 6, 3, 6) is taken under the consideration of the informative gamma priors by setting the prior means equal to the true values of the parameters. From the above study, we can made the following conclusions.

- In all considered censoring schemes, as sample size increases, the mean square error (mse) of all estimators decrease.
- Bayes estimators obtained under informative priors show smaller mean square error than maximum likelihood estimators. However, Bayes estimators obtained under non-informative behave more or less same as maximum likelihood estimators.
- When the non-informative prior is considered, Bayes estimators obtained by using MCMC techniques are superior than Bayes estimators obtained by using Lindley's approximation.
- Under the assumption of informative gamma priors, Bayes estimators obtained by using Lindley's approximation perform well than Bayes estimators obtained by using MCMC techniques.
- Lindley's approximation is useful for obtaining the Bayes estimates when sample size is large enough.
- The maximum likelihood and Bayes estimators of reliability function show relatively small mean square errors. The maximum likelihood and Bayes estimators of hazard function show large mean square errors. That signals, we should take care in estimating the hazard function.
- Finally, we can conclude that Bayes estimates can be superiorly preferred for estimating the parameters, reliability and hazard functions of flexible Weibull distribution.

Table 2: The ML estimates and MSE of the estimators of the parameters, reliability and hazard function for $\alpha = 2$, $\beta = 2$

n, m, s	$A(\hat{\alpha})$	$M(\hat{\alpha})$	$A(\hat{\beta})$	$M(\hat{\beta})$	$A(\hat{R}(1))$	$M(\hat{R}(1))$	$A(\hat{h}(1))$	$M(\hat{h}(1))$
20,12,1	2.25266	0.34166	2.21985	0.34933	0.35798	0.01438	4.96236	8.80980
20,12,2	2.43746	0.84740	2.31816	0.51574	0.33270	0.01818	6.47918	9.37716
20,12,3	2.26296	0.35820	2.21498	0.32104	0.35312	0.01421	5.09433	9.62653
20,12,4	2.41985	0.78754	2.30682	0.48919	0.33441	0.01765	6.31333	8.84607
20,12,5	2.32482	0.49926	2.24605	0.35855	0.34388	0.01523	5.56478	4.13324
20,16,1	2.18544	0.24703	2.16360	0.26398	0.36070	0.01009	4.63203	3.99218
20,16,2	2.24335	0.38300	2.19820	0.31589	0.35319	0.01083	4.96261	7.68584
20,16,3	2.18677	0.24867	2.16045	0.25348	0.35919	0.01002	4.66186	4.23171
20,16,4	2.23833	0.36792	2.19450	0.30714	0.35360	0.01072	4.93591	7.31618
20,16,5	2.20783	0.28792	2.17222	0.26169	0.35618	0.01019	4.78246	5.45588
20,20,0	2.15162	0.18648	2.13706	0.21609	0.36302	0.00830	4.47909	2.68446
30,18,1	2.15440	0.19029	2.13006	0.19731	0.35979	0.00862	4.55158	3.28057
30,18,2	2.26265	0.44804	2.18457	0.26486	0.34267	0.01139	5.29759	13.1861
30,18,3	2.15919	0.19635	2.12351	0.17529	0.35593	0.00853	4.63018	3.88814
30,18,4	2.25205	0.41096	2.17790	0.24956	0.34384	0.01093	5.22086	12.1460
30,18,5	2.19538	0.25836	2.14199	0.18539	0.35017	0.00924	4.85471	6.41431
30,30,0	2.11434	0.11701	2.11492	0.14729	0.36816	0.00537	4.29018	1.18537
30,24,1	2.12637	0.13953	2.11600	0.16163	0.36434	0.00688	4.38711	1.86424
30,24,2	2.16460	0.21462	2.13755	0.18727	0.35868	0.00716	4.58647	3.41353
30,24,3	2.12606	0.13887	2.11201	0.15251	0.36306	0.00678	4.40540	1.95890
30,24,4	2.16057	0.20386	2.13445	0.18061	0.35898	0.00710	4.56828	3.23454
30,24,5	2.13959	0.15919	2.11898	0.15520	0.36080	0.00682	4.47595	2.41710
50,30,1	2.10040	0.10326	2.09349	0.11999	0.36550	0.00507	4.29265	1.18729
50,30,2	2.16710	0.23313	2.12333	0.14675	0.35275	0.00655	4.69285	3.87615
50,30,3	2.10145	0.10420	2.08565	0.10208	0.36234	0.00493	4.33955	1.37906
50,30,4	2.15931	0.21135	2.11844	0.13792	0.35370	0.00624	4.64553	3.43057
50,30,5	2.12241	0.13399	2.09505	0.10549	0.35830	0.00524	4.45423	1.99738
50,40,1	2.07076	0.06835	2.06710	0.08519	0.36662	0.00378	4.19790	0.73444
50,40,2	2.09104	0.10283	2.07750	0.09362	0.36311	0.00381	4.29792	1.17785
50,40,3	2.07020	0.06783	2.06394	0.07953	0.36569	0.00369	4.20845	0.76426
50,40,4	2.08900	0.09758	2.07602	0.09098	0.36330	0.00377	4.28848	1.11511
50,40,5	2.07783	0.07672	2.06789	0.08000	0.36437	0.00367	4.24376	0.88827
50,50,0	2.05830	0.05280	2.05873	0.07169	0.36805	0.00327	4.14933	0.55219

Table 3: Using Lindley's approximation, Bayes estimates and MSE of the estimators of the parameters, reliability and hazard function for $\alpha = 2$, $\beta = 2$ and hyper-parameters $a = b = c = d = 0$

n, m, s	$A(\hat{\alpha})$	$M(\hat{\alpha})$	$A(\hat{\beta})$	$M(\hat{\beta})$	$(\hat{R}(1))$	$M(\hat{R}(1))$	$(\hat{h}(1))$	$M(\hat{h}(1))$
20,12,1	2.12017	0.26618	2.13781	0.29181	0.37593	0.01216	4.63746	7.47696
20,12,2	2.20115	0.61132	2.17535	0.38911	0.36728	0.01371	6.13164	7.69653
20,12,3	2.11953	0.27518	2.12943	0.26553	0.37392	0.01191	4.73518	8.93865
20,12,4	2.18945	0.56703	2.16721	0.36964	0.36776	0.01346	5.90890	7.92908
20,12,5	2.13988	0.36555	2.13424	0.28082	0.37031	0.01225	5.10797	8.43906
20,16,1	2.07808	0.20079	2.08746	0.22477	0.37197	0.00870	4.38813	3.34746
20,16,2	2.10437	0.30308	2.10277	0.25945	0.36925	0.00904	4.66375	6.61522
20,16,3	2.07772	0.20185	2.08501	0.21592	0.37138	0.00862	4.41092	3.54149
20,16,4	2.10050	0.29077	2.09954	0.25231	0.36938	0.00896	4.63451	6.23980
20,16,5	2.08335	0.22883	2.08609	0.21796	0.37017	0.00865	4.49594	4.52603
20,20,0	2.06037	0.15483	2.06653	0.18684	0.37051	0.00729	4.28407	2.29119
30,18,1	2.06834	0.16040	2.07643	0.17444	0.37159	0.00763	4.34669	2.83915
30,18,2	2.11276	0.35883	2.09550	0.21870	0.36556	0.00923	5.02792	12.5473
30,18,3	2.06661	0.16407	2.06914	0.15437	0.36995	0.00748	4.40565	3.33973
30,18,4	2.10561	0.32798	2.09053	0.20609	0.36595	0.00891	4.93311	11.3401
30,18,5	2.07679	0.20812	2.07127	0.15706	0.36784	0.00784	4.57091	5.46686
30,24,1	2.05604	0.11965	2.06583	0.14406	0.37166	0.00623	4.23196	1.63050
30,24,2	2.07450	0.18088	2.07598	0.16284	0.36923	0.00634	4.39673	3.01195
30,24,3	2.05488	0.11910	2.06293	0.13612	0.37112	0.00614	4.24608	1.71072
30,24,4	2.07112	0.17147	2.07307	0.15698	0.36935	0.00629	4.37689	2.83560
30,24,5	2.05866	0.13434	2.06318	0.13627	0.37003	0.00611	4.29531	2.08840
30,30,0	2.05432	0.10163	2.06832	0.13192	0.37300	0.00495	4.16576	1.05155
50,30,1	2.04899	0.09174	2.06045	0.11021	0.37219	0.00472	4.17502	1.06642
50,30,2	2.07972	0.20081	2.07154	0.12930	0.36613	0.00574	4.53185	3.54798
50,30,3	2.04678	0.09229	2.05353	0.09382	0.37063	0.00457	4.21151	1.22994
50,30,4	2.07381	0.18149	2.06751	0.12149	0.36664	0.00549	4.47590	3.09028
50,30,5	2.05299	0.11582	2.05370	0.09454	0.36869	0.00474	4.29439	1.74942
50,40,1	2.02911	0.06194	2.03715	0.07915	0.37089	0.00356	4.10826	0.67198
50,40,2	2.03834	0.09228	2.04170	0.08575	0.36935	0.00354	4.18920	1.07303
50,40,3	2.02828	0.06153	2.03513	0.07407	0.37049	0.00348	4.11666	0.69810
50,40,4	2.03659	0.08742	2.04022	0.08332	0.36943	0.00351	4.17868	1.01184
50,40,5	2.03021	0.06884	2.03516	0.07372	0.36985	0.00343	4.14020	0.80311
50,50,0	2.02300	0.04841	2.03148	0.06713	0.37098	0.00312	4.07674	0.51262

Table 4: Using MCMC techniques, Bayes estimates and MSE of the estimators of the parameters, reliability and hazard function for $\alpha = 2$, $\beta = 2$ and hyper-parameters $a = b = c = d = 0$

n, m, s	A $(\hat{\alpha})$	M $(\hat{\alpha})$	A $\hat{\beta}$	M $\hat{\beta}$	A $\hat{R}(1)$	M $\hat{R}(1)$	A $\hat{h}(1)$	M $\hat{h}(1)$
20,12,1	2.05406	0.23278	2.04054	0.25330	0.36471	0.01151	4.55647	6.05007
20,12,2	2.04782	0.50703	2.04297	0.31780	0.37086	0.01300	5.13310	7.10088
20,12,3	2.06476	0.24572	2.04871	0.23506	0.36425	0.01139	4.64358	7.21650
20,12,4	2.05925	0.47764	2.04946	0.30799	0.36898	0.01275	5.10771	6.06234
20,12,5	2.08396	0.32955	2.06325	0.25148	0.36381	0.01182	4.89140	3.47467
20,16,1	2.03112	0.18342	2.01590	0.20542	0.36317	0.00839	4.35827	2.99348
20,16,2	2.02819	0.26947	2.01649	0.23103	0.36503	0.00855	4.46430	5.05216
20,16,3	2.03428	0.18545	2.01843	0.19839	0.36302	0.00834	4.37965	3.16671
20,16,4	2.03161	0.26039	2.01898	0.22640	0.36466	0.00852	4.46401	4.88167
20,16,5	2.04116	0.21187	2.02535	0.20185	0.36329	0.00838	4.43823	3.94036
20,20,0	2.02415	0.14355	2.01015	0.17389	0.36333	0.00711	4.27195	2.12003
30,18,1	2.02808	0.14732	2.01358	0.15981	0.36341	0.00738	4.32396	2.57472
30,18,2	2.02191	0.31919	2.01395	0.19409	0.36729	0.00886	4.60838	8.19453
30,18,3	2.03531	0.15298	2.01892	0.14352	0.36294	0.00730	4.37860	3.02401
30,18,4	2.03007	0.29507	2.01893	0.18498	0.36603	0.00859	4.60023	7.77448
30,18,5	2.04691	0.19560	2.02874	0.14719	0.36280	0.00769	4.49739	4.68978
30,24,1	2.02733	0.11216	2.01974	0.13464	0.36542	0.00608	4.22698	1.54817
30,24,2	2.02678	0.16601	2.02022	0.14917	0.36610	0.00610	4.29427	2.58840
30,24,3	2.02885	0.11210	2.02056	0.12767	0.36521	0.00601	4.24033	1.62522
30,24,4	2.02892	0.15842	2.02171	0.14474	0.36584	0.00608	4.29368	2.48643
30,24,5	2.03390	0.12699	2.02494	0.12833	0.36508	0.00599	4.27634	1.95643
30,30,0	2.03216	0.09623	2.03221	0.12431	0.36800	0.00484	4.16759	1.01726
50,30,1	2.02630	0.08680	2.02322	0.10360	0.36692	0.00461	4.17331	1.02808
50,30,2	2.02856	0.18543	2.02425	0.11884	0.36702	0.00556	4.34079	2.88596
50,30,3	2.03046	0.08854	2.02499	0.08937	0.36614	0.00450	4.20823	1.18814
50,30,4	2.03278	0.16912	2.02699	0.11280	0.36642	0.00534	4.33119	2.60397
50,30,5	2.03851	0.11161	2.03035	0.09067	0.36532	0.00469	4.27245	1.65420
50,40,1	2.01299	0.05954	2.01040	0.07609	0.36703	0.00350	4.10978	0.65327
50,40,2	2.01132	0.08773	2.00948	0.08154	0.36739	0.00345	4.13891	0.99007
50,40,3	2.01403	0.05937	2.01091	0.07148	0.36686	0.00343	4.11785	0.67928
50,40,4	2.01329	0.08347	2.01102	0.07958	0.36722	0.00344	4.13998	0.94519
50,40,5	2.01695	0.06657	2.01357	0.07132	0.36679	0.00339	4.13492	0.77641
50,50,0	2.01067	0.04699	2.01070	0.06504	0.36793	0.00307	4.08021	0.50373

Table 5: Using Lindley's approximation, Bayes estimates and MSE of the estimators of the parameters, reliability and hazard function for $\alpha = 2$, $\beta = 2$ and hyper-parameters $a = c = 3,\ b = d = 6$

n, m, s	$A(\hat{\alpha})$	$M(\hat{\alpha})$	$A\ \hat{\beta}$	$M\ \hat{\beta}$	$A\ \hat{R}(1)$	$M\ \hat{R}(1)$	$A\ \hat{h}(1)$	$M\ \hat{h}(1)$
20,12,1	1.91595	0.04732	1.91364	0.04890	0.36825	0.00856	4.09157	1.65609
20,12,2	1.54841	0.49202	1.65268	0.31947	0.39552	0.00901	0.61262	1.96028
20,12,3	1.92254	0.04749	1.92831	0.04641	0.37097	0.00872	4.05651	1.76788
20,12,4	1.61176	0.36403	1.69863	0.24581	0.39123	0.00881	1.35939	1.27013
20,16,5	1.85552	0.06599	1.88024	0.05940	0.37608	0.00877	3.51519	1.79063
20,16,1	1.95602	0.04738	1.94967	0.04624	0.36632	0.00673	4.15189	1.37231
20,16,2	1.88484	0.04907	1.89150	0.04508	0.37082	0.00656	3.91233	1.48658
20,16,3	1.95976	0.04942	1.95521	0.04714	0.36699	0.00678	4.15572	1.39999
20,16,4	1.89527	0.04742	1.90046	0.04344	0.37033	0.00660	3.94842	1.27192
20,16,5	1.94446	0.04904	1.94338	0.04470	0.36825	0.00677	4.10624	1.30581
20,20,0	1.97777	0.04921	1.97050	0.05480	0.36573	0.00587	4.14927	1.18557
30,18,1	1.98833	0.05538	1.98752	0.05443	0.36826	0.00629	4.18090	1.40070
30,18,2	1.86810	0.04845	1.90037	0.03762	0.37906	0.00626	3.65245	3.23093
30,18,3	1.99197	0.06134	1.99301	0.05514	0.36907	0.00635	4.19804	1.50777
30,18,4	1.89211	0.04847	1.91765	0.03759	0.37696	0.00631	3.76872	2.88262
30,18,5	1.97258	0.06118	1.97791	0.04834	0.37067	0.00643	4.14711	1.37119
30,24,1	2.00401	0.05199	2.00560	0.05912	0.36874	0.00534	4.14926	1.06879
30,24,2	1.98238	0.05228	1.98594	0.04749	0.36966	0.00532	4.12939	1.09373
30,24,3	2.00554	0.05402	2.00745	0.05893	0.36890	0.00533	4.15679	1.10539
30,24,4	1.98607	0.05329	1.98917	0.04926	0.36948	0.00532	4.13651	1.10350
30,24,5	2.00109	0.05633	2.00282	0.05615	0.36893	0.00533	4.16096	1.15783
30,30,0	2.01445	0.05164	2.02072	0.06240	0.37028	0.00431	4.11711	0.81259
50,30,1	2.01804	0.05189	2.02477	0.05879	0.37050	0.00424	4.12912	0.83947
50,30,2	1.99185	0.06375	2.00034	0.04256	0.37155	0.00451	4.16139	1.13017
50,30,3	2.01944	0.05581	2.02490	0.05505	0.37014	0.00420	4.15271	0.93276
50,30,4	1.99823	0.06548	2.00514	0.04514	0.37097	0.00448	4.17313	1.15699
50,30,5	2.01590	0.06332	2.01964	0.05164	0.36967	0.00429	4.18058	1.10494
50,40,1	2.01197	0.04121	2.01689	0.05057	0.36977	0.00329	4.08659	0.58178
50,40,2	2.00902	0.05207	2.01273	0.04774	0.36945	0.00330	4.11900	0.76238
50,40,3	2.01230	0.04190	2.01688	0.04880	0.36967	0.00325	4.09294	0.60054
50,40,4	2.00952	0.05095	2.01316	0.04785	0.36941	0.00328	4.11616	0.74162
50,40,5	2.01167	0.04545	2.01542	0.04760	0.36941	0.00323	4.10475	0.65837
50,50,0	2.01135	0.03499	2.01734	0.04723	0.37009	0.00290	4.06390	0.46182

Table 6: Using MCMC techniques, Bayes estimates and MSE of the estimators of the parameters, reliability and hazard function for $\alpha = 2$, $\beta = 2$ and hyper-parameters $a = c = 3,\ b = d = 6$

n, m, s	A $(\hat{\alpha})$	M $(\hat{\alpha})$	A $\hat{\beta}$	M $\hat{\beta}$	A $\hat{R}(1)$	M $\hat{R}(1)$	A $\hat{h}(1)$	M $\hat{h}(1)$
20,12,1	2.01656	0.09246	1.99611	0.09977	0.36200	0.00972	4.41594	3.00817
20,12,2	1.98576	0.11138	1.98318	0.08373	0.36873	0.00965	4.39817	3.52810
20,12,3	2.02251	0.09658	2.00235	0.09519	0.36223	0.00969	4.44625	3.23334
20,12,4	1.99292	0.11010	1.98738	0.08437	0.36770	0.00966	4.41677	3.56846
20,12,5	2.02322	0.10518	2.00532	0.08940	0.36327	0.00972	4.48338	3.64233
20,16,1	2.00728	0.08429	1.98742	0.09004	0.36149	0.00732	4.29842	1.92499
20,16,2	1.99483	0.09892	1.98150	0.08489	0.36402	0.00737	4.29882	2.27404
20,16,3	2.00968	0.08591	1.98984	0.08859	0.36153	0.00732	4.30897	1.98957
20,16,4	1.99802	0.09763	1.98380	0.08486	0.36369	0.00737	4.30488	2.26143
20,16,5	2.01109	0.09098	1.99283	0.08562	0.36218	0.00735	4.32498	2.16039
20,20,0	2.00671	0.07396	1.98861	0.08662	0.36188	0.00628	4.24065	1.48163
30,18,1	2.01083	0.07898	1.99280	0.08305	0.36208	0.00660	4.27718	1.74894
30,18,2	1.98911	0.11146	1.98392	0.07326	0.36710	0.00698	4.29950	2.58040
30,18,3	2.01650	0.08320	1.99812	0.07761	0.36205	0.00659	4.30714	1.93830
30,18,4	1.99641	0.10829	1.98837	0.07323	0.36607	0.00693	4.31493	2.56344
30,18,5	2.01986	0.09402	2.00276	0.07362	0.36269	0.00675	4.34786	2.33091
30,24,1	2.01361	0.06560	2.00269	0.07686	0.36425	0.00552	4.20422	1.18565
30,24,2	2.00677	0.08233	1.99887	0.07495	0.36548	0.00552	4.21848	1.51249
30,24,3	2.01526	0.06661	2.00411	0.07450	0.36418	0.00549	4.21289	1.22906
30,24,4	2.00929	0.08042	2.00070	0.07446	0.36522	0.00552	4.22247	1.49026
30,24,5	2.01752	0.07213	2.00662	0.07291	0.36433	0.00548	4.22999	1.36068
30,30,0	2.01906	0.06041	2.01569	0.07516	0.36681	0.00441	4.15434	0.85148
50,30,1	2.01640	0.05788	2.01089	0.06700	0.36607	0.00428	4.15958	0.86659
50,30,2	2.00827	0.09427	2.00609	0.06241	0.36753	0.00474	4.21830	1.56054
50,30,3	2.02068	0.06067	2.01388	0.06047	0.36565	0.00423	4.18469	0.97032
50,30,4	2.01326	0.09001	2.00929	0.06184	0.36686	0.00466	4.22466	1.49788
50,30,5	2.02493	0.07156	2.01694	0.05869	0.36531	0.00435	4.22210	1.20425
50,40,1	2.00791	0.04345	2.00381	0.05417	0.36650	0.00331	4.10548	0.58578
50,40,2	2.00388	0.05799	2.00134	0.05385	0.36712	0.00329	4.11755	0.77895
50,40,3	2.00903	0.04385	2.00461	0.05175	0.36639	0.00326	4.11173	0.60515
50,40,4	2.00588	0.05611	2.00286	0.05335	0.36694	0.00328	4.12022	0.75680
50,40,5	2.01101	0.04809	2.00666	0.05096	0.36643	0.00324	4.12328	0.66776
50,50,0	2.00707	0.03611	2.00590	0.04907	0.36750	0.00291	4.07837	0.46358

6. Real data study

In this section, we provided a real data illustration applying the methodologies discussed in the above sections to a set of real data representing the times between failures of the secondary reactor pumps. This data set is originally reported by [23] in the literature. After that, [3] have modelled this data set by assuming that the waiting times between failures of the secondary reactor pumps follow the flexible Weibull distribution. [19] have used this data set for estimating the parameters of the flexible Weibull distribution under Type-II censoring scheme.

We estimated the parameters, reliability and hazard functions of the flexible Weibull distribution under the different progressive Type-II censoring schemes. Table 7 shows the estimates of α and β based on the different progressive Type-II censored samples artificially generated from the given set of real data. The estimates of the reliability and hazard rate are presented in Table 8. For Bayesian estimation, the non-informative priors are considered for the parameters α and β. The density and trace plots for simulated MCMC samples of α and β are shown in Figure 1.

Table 7: The ML and Bayes (Lindley's approximation and MCMC) estimates of α and β for

m	Scheme	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{LD}$	$\hat{\alpha}_{MC}$	$\hat{\beta}_{ML}$	$\hat{\beta}_{LD}$	$\hat{\alpha}_{MC}$
23	$0*23$	0.2071	0.1974	0.1984	0.2588	0.2583	0.2473
18	$5*1, 0*17$	0.2012	0.1902	0.1903	0.4400	0.4361	0.4111
18	$0*17, 5*1$	0.2751	0.2412	0.2304	0.2557	0.2577	0.2458
18	$1*5, 0*13$	0.2000	0.1895	0.1902	0.3408	0.3396	0.3193
18	$0*13, 1*5$	0.1474	0.1295	0.1238	0.2577	0.2592	0.2490
18	$0*6, 1*5, 0*7$	0.1878	0.1775	0.1775	0.2364	0.2355	0.2224
16	$7*1, 0*15$	0.1992	0.1884	0.1890	0.4328	0.4287	0.3986
16	$0*15, 7*1$	0.3329	0.2644	0.1512	0.2557	0.2588	0.2544
16	$1*7, 0*9$	0.1923	0.1817	0.1811	0.3260	0.3246	0.3050
16	$0*9, 1*7$	0.1360	0.1182	0.1098	0.2460	0.2473	0.2363
16	$0*6, 2*3, 1*1, 0*6$	0.2156	0.2024	0.2024	0.2270	0.2259	0.2138

Table 8: The ML and Bayes (Lindley's approximation and MCMC) estimates of $R\ (t = \bar{x} = 1.57787)$ and $h(\bar{x})$ for different progressive Type-II censored sample from real data.

m	Scheme	$\hat{R}(.)_{ML}$	$\hat{R}(.)_{LD}$	$\hat{R}(.)_{MC}$	$\hat{h}(.)_{ML}$	$\hat{h}(.)_{LD}$	$\hat{h}(.)_{MC}$
23	$0*23$	0.3083	0.3140	0.3107	0.3660	0.3523	0.3499
18	$5*1, 0*17$	0.3537	0.3592	0.3533	0.3928	0.3752	0.3707
18	$0*17, 5*1$	0.2691	0.2902	0.2928	0.4960	0.4539	0.4217
18	$1*5, 0*13$	0.3313	0.3373	0.3321	0.3721	0.3566	0.3524
18	$0*13, 1*5$	0.3424	0.3533	0.3541	0.2690	0.2481	0.2355
18	$0*6, 1*5, 0*7$	0.3142	0.3201	0.3170	0.3274	0.3133	0.3084
16	$7*1, 0*15$	0.3532	0.3586	0.3512	0.3882	0.3702	0.3660
16	$0*15, 7*1$	0.2374	0.2802	0.3410	0.6264	0.5406	0.3225
16	$1*7, 0*9$	0.3323	0.3383	0.3340	0.3562	0.3405	0.3343
16	$0*9, 1*7$	0.3463	0.3570	0.3593	0.2490	0.2288	0.2126
16	$0*6, 2*3, 1*1, 0*6$	0.2961	0.3038	0.3008	0.3734	0.3550	0.3495

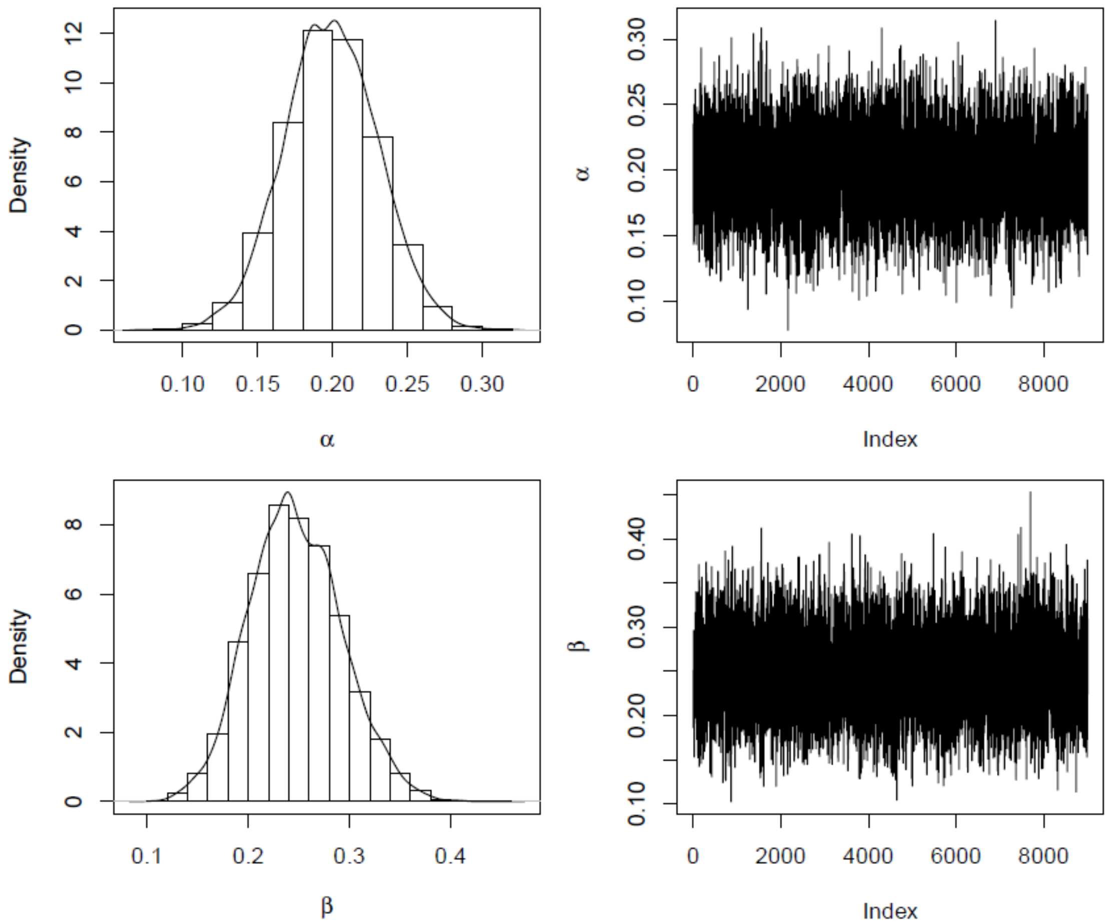

Figure 1: The density and trace plots of simulated α and β based on complete data.

7. One sample Bayes prediction

Under the progressive Type-II censoring scheme, the experiment is terminated as soon as the desired numbers of units obtained and so some surviving units are removed from the test. In practice, the experimenter may be interested to know the life-times of the $(n - m)$ removed surviving units at *m*th stage on the basis of observed sample. Let $Y_s = X_{m+s}, m < s \leq n$ represents the failure life time of the remaining units, then conditional distribution of $Y_{(s)}$th order statistics given progressive Type-II censored sample $\underset{\sim}{x}$ is given by

$$f\left(y_{(s)} \middle| x_{(m)}, \alpha, \beta\right) = \frac{(n-m)!\left[1-F\left(y_{(s)}\right)\right]^{n-m-s}}{(s-1)!(n-m-s)!\left[1-F\left(x_{(m)}\right)\right]^{n-m}} \left[F(y_s) - F(x_m)\right]^{s-1} f(y_{(s)}) \tag{26}$$

Substituting (1) and (2) in (26), we will have

$$f\left(y_{(s)}\middle|x_{(m)},\alpha,\beta\right)=\frac{(n-m)!}{(s-1)!(n-m-s)!}\left(\alpha y_{(s)}+\frac{\beta}{y_s^2}\right)\phi(y_{(s)})\sum_{k=1}^{s-1}(-1)^k\binom{s-1}{k} \quad (27)$$

where, $\varphi(z)=exp\ (\alpha z-\frac{\beta}{z})$. One sample Bayes predictive density of $y_{(s)}$ th ordered future sample can be obtained as follows

$$f\left(y_{(s)}\middle|\underset{\sim}{x}\right)=\int_0^\infty\int_0^\infty f\left(y_{(s)}\middle|\underset{\sim}{x},\alpha,\beta\right)\pi\left(\alpha,\beta\middle|\underset{\sim}{x}\right)d\alpha d\beta \quad (28)$$

Putting (13) in (28), we get

$$f\left(y_{(s)}\middle|\underset{\sim}{x}\right)=\frac{(n-m)!}{(s-1)!(n-m-s)!}\int_0^\infty\int_0^\infty\alpha^{b-1}\beta^{d-1}\exp\left[\alpha(S-a)-\beta(S_1+c)\left(\alpha y_{(s)}-\frac{\beta}{y_{(s)}}\right)\right]$$
$$\phi(y_{(s)})\exp(-(r_i+1)\sum_{i=1}^m\phi(x_i))\exp\left(\sum_{i=1}^m\log\left(\alpha+\frac{\beta}{x_i^2}\right)\right)\sum_{k=1}^{s-1}(-1)^k\binom{s-1}{k}\exp[(n-$$
$$-m-s+k+1)\phi(x_{(m)})-(k+n-m-s+1)\phi(y_{(s)})d\alpha d\beta \quad (29)$$

where, $S_1=\sum_{i=1}^m x_i^{-1},\sum_{i=1}^m x_i$. The above equation for $f(y_{(s)}\,|\underset{\sim}{x})$ cannot be expressed in closed form and hence it cannot be evaluated analytically. Therefore, the MCMC techniques have been used for obtaining the approximate solution of the above predictive density. If $(\alpha_i,\beta_i);\ i=1,2,\dots,M$ obtained from $\pi(\alpha,\beta|\underset{\sim}{x})$ using Gibbs sampling can be utilized to obtain the consistent estimate of $f(y_{(s)}\,|\underset{\sim}{x})$. It can be obtained by

$$f\left(y_{(s)}\middle|\underset{\sim}{x}\right)=\frac{1}{M}\sum_{i=1}^M f(y_{(s)}|\alpha_i,\beta_i) \quad (30)$$

Thus, we can obtain the two sided $100(1-\psi)\%$ prediction interval (L,U) for future sample by solving the following two equations:

$$P\left(Y_{(s)}>U\middle|\underset{\sim}{x}\right)=\frac{\psi}{2}\text{ and }P\left(Y_{(s)}>L\middle|\underset{\sim}{x}\right)=1-\frac{\psi}{2}.$$

It is not possible to obtain the solutions analytically. We need to apply suitable numerical techniques for solving these non-linear equations. Alternatively, we can also use the MCMC approach by [5], in the following way: Let $(y_{(i:s)})$; $i=1,2,\cdots,M$ be the corresponding ordered MCMC sample of $(y_{i;s})$; $i=1,2,\cdots,M$ from (29). Then, the $100(1-\psi)\%$ HPD intervals for $y_{(s)}$ is $y_{(j*:s)}$, $y_{j^*}+[(1-\psi)M]:s$, where j^* is chosen so that

$$y_{j^*}+[(1-\psi)M]:s\}-y_{(j^*:s)}=\min_{1\le j\le M-[(1-\psi)M]}[y_{j^*}+[(1-\psi)M:s]-y_{(j^*:s)}]$$

8. Two sample Bayes prediction

Let $y_{(1)},y_{(2)},\dots,y_{(m)}$ be a ordered future sample of size m from flexible Weibull distribution, indepen- dent of the informative sample $x_{(1)},x_{(2)},\dots,x_{(n)}$. The density function of $y_{(k)}$th ordered future sample is

$$p(y_{(k)}\mid\alpha,\beta)=\frac{m!}{(k-1)!(m-k)!}\left[F(y_{(k)}\mid\alpha,\beta)\right]^{k-1}\left[1-F(y_{(k)}\mid\alpha,\beta)\right]^{m-k}f(y_{(k)}\mid\alpha,\beta) \quad (31)$$

By substituting (1) and (2) in (31), we get

$$p\left(y_{(k)} \mid \alpha,\lambda \right) = \frac{m!}{(k-1)!(m-k)!}\left(\alpha y_{(k)} + \frac{\beta}{y_{(k)}^2}\right)\phi\left(y_{(k)}\right)\sum_{i=1}^{k-i}(-1)^i \binom{k-1}{i} \exp\left[-(i+m-k+1)\phi\left(y_{(k)}\right)\right] \quad (32)$$

Two sample Bayes predictive density of $y_{(k)}$th ordered future sample can be obtained as follows

$$p\left(y_{(k)} \mid \underset{\sim}{x} \right) = \int_0^\infty \int_0^\infty p\left(y_{(k)} \mid \alpha, \beta \right)\pi(\alpha,\beta|\underset{\sim}{x})\, d\alpha d\beta \quad (33)$$

The above equation for $p(y_{(k)} \,|\underset{\sim}{x})$ cannot be expressed in closed form and hence it cannot be evaluated analytically. Therefore, the MCMC techniques have been used for obtaining the approximate solution of the above predictive density. If $\{(\alpha_i\,,\beta_i\,)\,;\; i \;=\; 1,2,\cdot\cdot\cdot,M\,\}$ obtained from $\pi(\alpha,\beta|\underset{\sim}{x})$ using Gibbs sampling can be utilized to obtain the consistent estimate of $p(y_{(k)}\,|\underset{\sim}{x})$. It can be obtained by

$$p\left(y_{(k)} \middle|\underset{\sim}{x}\right) = \frac{1}{M}\sum_{i=1}^{M} p(y_{(k)}\,|\alpha_i, \beta_i) \quad (34)$$

Thus, we can obtain the two sided $100(1\;-\;\psi)\%$ prediction interval (L, U) for future sample by solving the following two equations:

$$P\left(Y_{(k)} > U\middle|\underset{\sim}{x}\right) = \frac{\psi}{2} \text{ and } P\left(Y_{(k)} > L\middle|\underset{\sim}{x}\right) = 1 - \frac{\psi}{2}.$$

It is not possible to obtain the solutions analytically. We need to apply suitable numerical techniques for solving these non-linear equations. Alternatively, we can also use the MCMC approach by [5], in the following way: Let $y_{(i:k)}\,;\; i \;=\; 1,2,\cdot\cdot\cdot,M$ be the corresponding ordered MCMC sample of $\{(y_{i:k}\,)\,;\; i \;=\; 1,2,\cdot\cdot\cdot,M\,\}$ from (33). Then, the $100(1\;-\;\psi)\%$ HPD intervals for $y_{(k)}$ is $y_{(j^*:k)}\,, y_j^* + [(1-\psi)M\,]\colon k$, where j^* is chosen so that

$$y_{j^*} +\; [(1-\psi)M]\colon k\} - y_{(j^*:k)} \;=\; \min_{1\le j\le M-[(1-\psi)M]}[y_{j^*} + [(1-\psi)M\colon k]{-}y_{(j^*:k)}]$$

For considered real data set, we calculated the mean and 95% credible intervals (predictive bounds) for future samples using one and two sample prediction techniques. The results are summarised in Table 9.

Table 9: Mean and 95 % predictive bounds for future samples based on real data for scheme 4, m = 18.

One sample prediction				Two sample prediction			
s	Mean	Bounds		k	Mean	Bounds	
		L	U			L	U
1	6.34829	5.32012	8.58725	5	0.17554	0.07270	0.30417
2	7.38246	5.33653	11.50189	10	0.49039	0.09350	1.23117
3	8.60376	5.43351	14.67601	15	1.76355	0.21747	4.46787
4	9.97251	5.73372	17.28349	20	5.86135	0.96266	11.97434
5	12.05279	6.22936	20.78702	23	10.65383	4.41187	19.48063

9. Conclusions

In this article, we have discussed classical and Bayes estimation of the parameters, reliability and hazard functions based on progressive Type-II censored sample draw from flexible Weibull distribution. From the simulation study, it was observed that Bayes estimators are superior than maximum likelihood estimators under different censoring schemes. The Lindley's approximation and MCMC techniques are utilized for Bayesian calculation. In addition, one and two sample prediction problems are also discussed under Bayesian set-up. The discussed procedures have been verified through a real data study. Finally, we can conclude that the methodology discussed in the previous sections provides the complete analysis of the progressive Type-II censored sample having flexible Weibull distribution.

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