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# THE WEIBULL GENERALIZED FLEXIBLE WEIBULL EXTENSION DISTRIBUTION

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*Abstract:* This paper introduces a new four parameters model called the Weibull Generalized Flexible Weibull extension (WGFWE) distribution which exhibits bathtub-shaped hazard rate. Some of it's statistical properties are obtained including ordinary and incomplete moments, quantile and generating functions, reliability and order statistics. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is derived. We illustrate the usefulness of the proposed model by applications to real data.

*Key words*: Weibull-G class, Flexible Weibull Extension distribution, Generalized Weibull, Reliability, Hazard function, Moments, Maximum likelihood estimation.

## 1. Introduction

The Weibull distribution is a highly known distribution due to its utility in modelling lifetime data where the hazard rate function is monotone Weibull (1951). In recent years new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub hazard failure rate Xie and Lai (1995). Exponentiated modified Weibull extension distribution by Sarhan and Apaloo (2013) are few among others.

Exponentiated Weibull family, Mudholkar and Srivastava (1993), Modified Weibull distribution, Lai et al. (2003) and Sarhan and Zaindin (2009), Beta Weibull distribution, Famoye et al. (2005), A flexible Weibull extension, Beb bington et al. (2007), Extended flexible Weibull, Bebbington et al. (2007), Generalized modified Weibull distribution, Carrasco et al. (2008), Kumaraswamy Weibull distribution, Cordeiro et al. (2010), Beta modified Weibull distribution,

Silva et al. (2010) and Nadarajah et al. (2011), Beta generalized Weibull distribution, Singla et al. (2012) and a new modified Weibull distribution, Almalki and Yuan (2013). A good review of these models is presented in Pham and Lai (2007) and Murthy et al. (2003).

The Flexible Weibull Extension (FWE) distribution has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies, Bebbington et al. (2007) and Singh et al. (2013, 2015). The origin and other aspects of this distribution can be found in Bebbington et al. (2007). A random variable X is said to have the Flexible Weibull Extension (FWE) distribution with parameters $\alpha, \beta > 0$ if it's probability density function (pdf) is given by

$$g(x) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\}, \quad x > 0, \tag{1}$$

while the cumulative distribution function (cdf) is given by

$$G(x) = 1 - \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\}, \quad x > 0. \tag{2}$$

The survival function is given by the equation

$$S(x) = 1 - G(x) = \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\}, \quad x > 0, \tag{3}$$

and the hazard function is

$$h(x) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}}. \tag{4}$$

Weibull distribution introduced by Weibull (1951), is a popular distribution for modeling phenomenon with monotonic failure rates. But this distribution does not provide a good fit to data sets with bathtub shaped or upside-down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering and biological studies. Hence a number of new distributions modeling the data in a better way have been constructed in literature as ramifications of Weibull distribution. Marcelo et al. (2014) introduced and studied generality a family of univariate distributions with two additional parameters, similarly as the extended Weibull, Gurvich et al. (1998) and Gamma-families, Zografos and Balakrshnan (2009), using the Weibull generator applied to the odds ratio $\frac{G(x)}{1-G(x)}$. If G(x) is the baseline cumulative distribution function (cdf) of a random variable, with probability density function (pdf) g(x) and the Weibull cumulative distribution function is

$$F(x; a, b) = 1 - e^{-ax^b}, \quad x \geq 0, \tag{5}$$

with parameters a and b are positive. Based on this density, by replacing x with ratio $\frac{G(x)}{1-G(x)}$.The cdf of Weibull- generalized distribution, say Weibull-G distribution with two extra parameters a and b, is defined by Marcelo et al. (2014)

$$\begin{aligned} F(x; a, b, \theta) &= \int_0^{\frac{G(x;\theta)}{1-G(x;\theta)}} abt^{b-1} e^{-at^b} dt \\ &= 1 - \exp\left\{-a\left[\frac{G(x;\theta)}{1-G(x;\theta)}\right]^b\right\}, \quad x \geq 0, \quad a, b \geq 0, \end{aligned} \tag{6}$$

where G(x; θ) is a baseline cdf, which depends on a parameters vector θ. The corresponding family pdf becomes

$$f(x;a,b,\theta) = ab\, g(x;\theta) \frac{[G(x;\theta)]^{b-1}}{[1-G(x;\theta)]^{b+1}} \exp\left\{-a\left[\frac{G(x;\theta)}{1-G(x;\theta)}\right]^b\right\}. \quad (7)$$

A random variable X with pdf (7) is denoted by X distributed Weibll-G(a, b, θ), x ∈ R, a, b > 0. The additional parameters induced by the Weibull generator are sought as a manner to furnish a more flexible distribution. If b = 1, it corresponds to the exponential- generator. An interpretation of the Weibull-G family of distributions can by given as follows (Corollary, Cooray (2006)) is a similar context. Let Y be a lifetime random variable having a certain continuous G distribution. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $\frac{G(x)}{1-G(x)}$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Weibull model with scale a and shape b. We can write

$$Pr(Y \le x) = Pr\left(X \le \frac{G(x)}{1-G(x)}\right) = F(x;a,b,\theta).$$

Which is given by Eq. (6). The survival function of the Weibull-G family is given by

$$S(x;a,b,\theta) = 1 - F(x;a,b,\theta) = \exp\left\{-a\left[\frac{G(x)}{1-G(x)}\right]^b\right\}, \quad (8)$$

and hazard rate function of the Weibull-G family is given by

$$\begin{aligned} h(x;a,b,\theta) &= \frac{f(x;a,b,\theta)}{S(x;a,b,\theta)} = \frac{ab\, g(x;\theta)\,[G(x;\theta)]^{b-1}}{[1-G(x;\theta)]^{b+1}} \\ &= ab\, h(x;\theta) \frac{[G(x;\theta)]^{b-1}}{[1-G(x;\theta)]^{b}}, \quad (9) \end{aligned}$$

where h(x; θ) = $\frac{G(x;\theta)}{1-G(x;\theta)}$ . The multiplying quantity $\frac{ab\cdot g(x;\theta)[G(x;\theta)]^{b-1}}{[1-G(x;\theta)]^{b}}$ works as a corrected factor for the hazard rate function of the baseline model (6) can deal with general situation in modeling survival data with various shapes of the hazard rate function. By using the power series for the exponential function, we obtain

$$\exp\left\{-a\left[\frac{G(x)}{1-G(x)}\right]^b\right\} = \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \left[\frac{G(x;\theta)}{1-G(x;\theta)}\right]^{ib}, \quad (10)$$

substituting from Eq.(10) into Eq. (7), we get

$$f(x;a,b,\theta) = ab\cdot g(x;\theta) \sum_{i=0}^{\infty} \frac{(-1)^i a^i}{i!} \frac{[G(x;\theta)]^{b(i+1)-1}}{[1-G(x;\theta)]^{b(i+1)+1}}. \quad (11)$$

Using the generalized binomial theorem we have

$$[1-G(x;\theta)]^{-(b(i+1)+1)}=\sum_{j=0}^{\infty}\frac{\Gamma\big(b(i+1)+j+1\big)}{j!\Gamma\big(b(i+1)+1\big)}\,[G(x;\theta)]^{j}\,. \qquad (12)$$

Inserting Eq. (12) in Eq. (11), the Weibull-G family density function is

$$f(x;a,b,\theta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i}a^{i+1}b\,\Gamma\big(b(i+1)+j+1\big)}{i!j!\Gamma\big(b(i+1)+1\big)}\,g(x;\theta)\,[G(x;\theta)]^{b(i+1)+j-1}\,. \qquad (13)$$

This paper is organized as follows, we define the cumulative, density and hazard functions of the Weibull-G Flexible Weibull Extension (WGFWE) distribution in Section 2. In Sections 3 and 4, we introduced the statistical properties include, quantile function skewness and kurtosis, rth moments and moment generating function. The distribution of the order statistics is expressed in Section 5. The maximum likelihood estimation of the parameters is determined in Section 6. Real data sets are analyzed in Section 7 and the results are compared with existing distributions. Finally, Section 8 concludes.

## 2. The Weibull-G Flexible Weibull Extension Distribution

In this section we studied the four parameters Weibull-G Flexible Weibull Extension (WGFWE) distribution. Using G(x) and g(x) in Eq. (13) to be the cdf and pdf of Eq. (6) and Eq. (7). The cumulative distribution function cdf of the Weibull-G Flexible Weibull Extension distribution (WGFWE) is given by

$$F(x;a,b,\alpha,\beta)=1-\exp\left\{-a\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b}\right\},\; x>0,\; a,b,\alpha,\beta>0. \qquad (14)$$

The pdf corresponding to Eq. (14) is given by

$$f(x;a,b,\alpha,\beta)=ab\left(\alpha+\frac{\beta}{x^{2}}\right)e^{\alpha x-\frac{\beta}{x}}e^{e^{\alpha x-\frac{\beta}{x}}}\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b-1}\exp\left\{-a\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b}\right\}, \qquad (15)$$

where x > 0 and , α, β > 0 are two additional shape parameters.

The survival function S(x), hazard rate function h(x), reversed- hazard rate function r(x) and cumulative hazard rate function H(x) of X ~ WGFWE(a, b, α, β)are given by

$$S(x;a,b,\alpha,\beta)=1-F(x;a,b,\alpha,\beta)=\exp\left\{-a\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b}\right\},\; x>0, \qquad (16)$$

$$h(x;a,b,\alpha,\beta)=ab\left(\alpha+\frac{\beta}{x^{2}}\right)e^{\alpha x-\frac{\beta}{x}}e^{e^{\alpha x-\frac{\beta}{x}}}\left[e^{\alpha x-\frac{\beta}{x}}-1\right]^{b-1}, \qquad (17)$$

$$r(x;a,b,\alpha,\beta)=\frac{ab\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}e^{e^{\alpha x-\frac{\beta}{x}}}\left[e^{\alpha x-\frac{\beta}{x}}-1\right]^{b-1}\exp\left\{-a\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b}\right\}}{1-\exp\left\{-a\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b}\right\}} \quad (18)$$

$$H(x;a,b,\alpha,\beta)=\int_0^x h(u)du=a\left[e^{e^{\alpha x-\frac{\beta}{x}}}-1\right]^{b}, \quad (19)$$

respectively, $x > 0$ and $a, b, \alpha, \beta > 0$.

Figures (1–5) display the cdf, pdf, survival, hazard rate and reversed hazard rate function of the WGFWE(a, b, α, β) distribution for some parameter values.

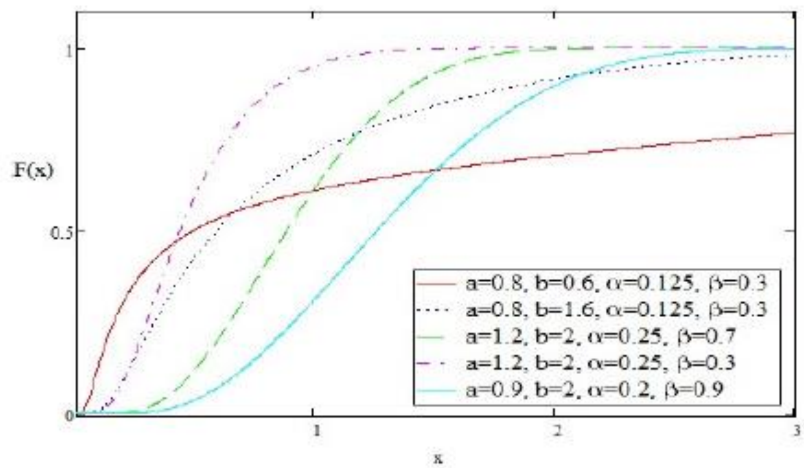


Figure 1:The cdf for different values of parameters.

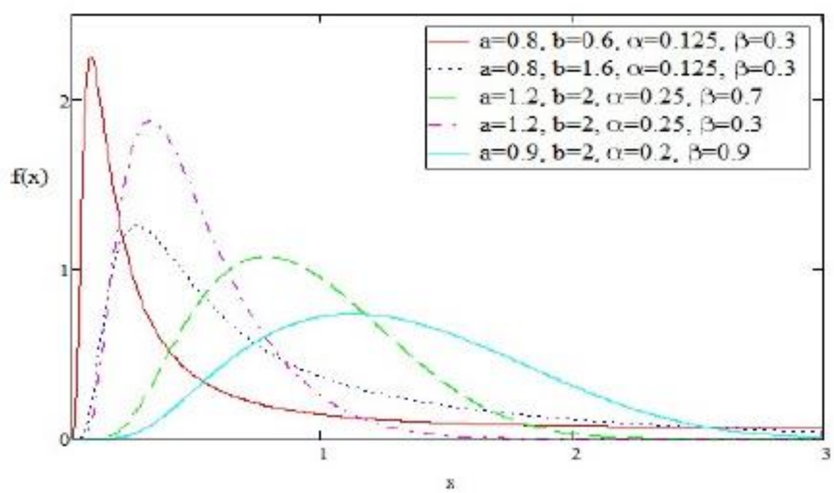


Figure 2:The pdf for different values of parameters

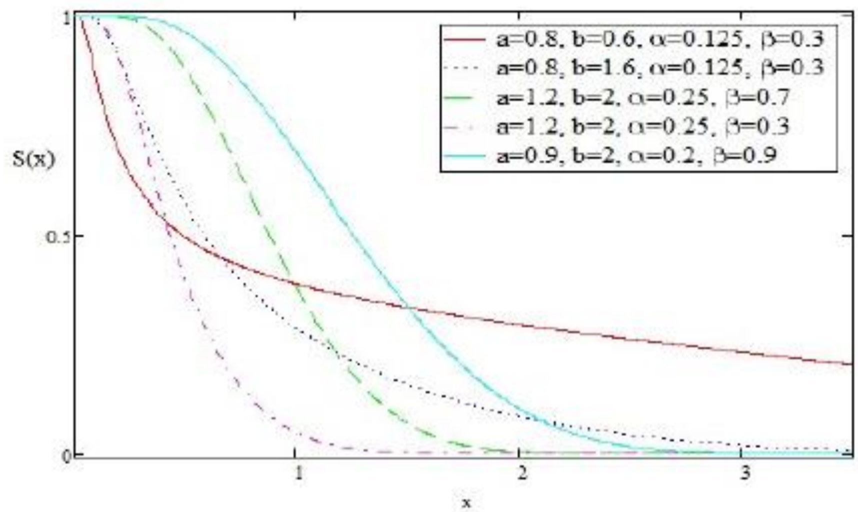


Figure 3:The survival function for different values of parameters

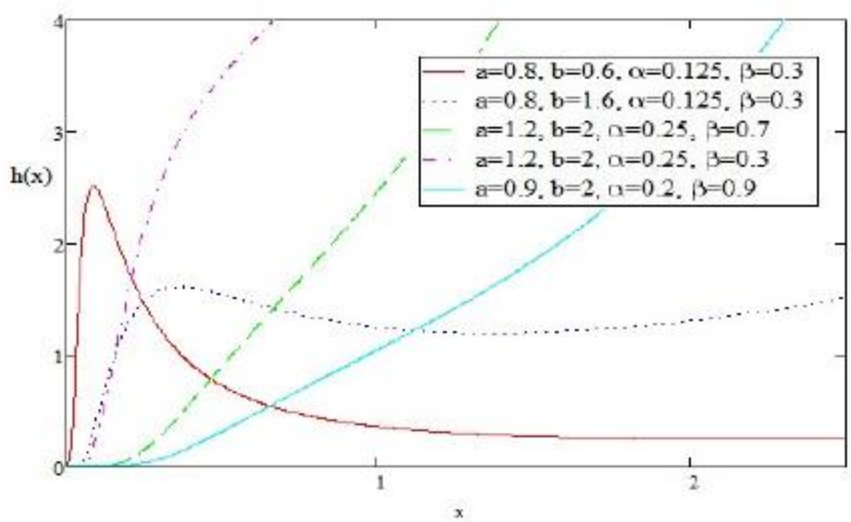


Figure 4:The hazard rate function for different values of parameters

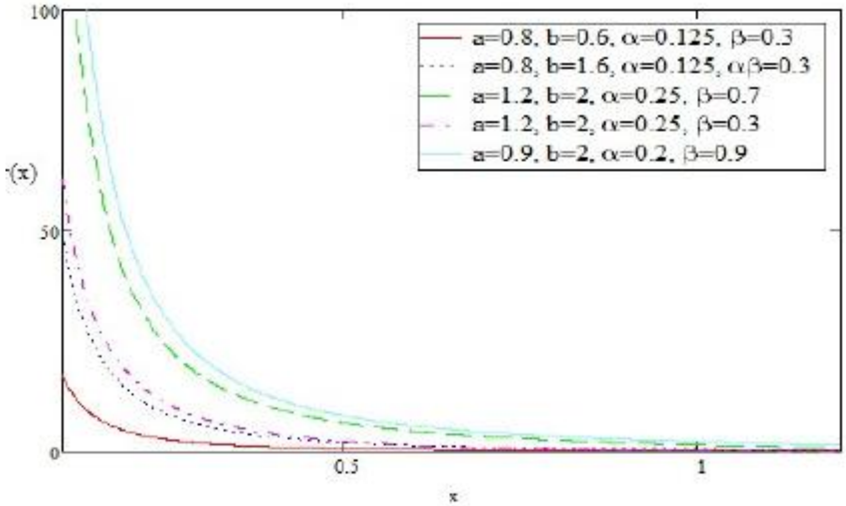


Figure 5:The reversed hazard rate function for different values of parameters.

From Figures 1–5, the WGFWE distribution is unimodal distribution, the hazard rate is decreasing, increasing and constant, decreasing reversed hazard rate and survival function.

## 3. Statistical Properties

In this section, we study the statistical properties for the WGFWE distribution, specially quantile function and simulation median, skewness, kurtosis and moments.

### 3.1 Quantile and simulation

The quantile $x_q$ of the WGFWE(a, b, α, β) random variable is given by.

$$F(x_q) = q, \quad 0 < q < 1. \tag{20}$$

Using the distribution function of WGFWE, from (14), we have

$$\alpha x_q^2 - k(q)x_q - \beta = 0, \tag{21}$$

Where

$$k(q) = \ln\left\{\ln\left[1 + \left(-\frac{\ln(1-q)}{a}\right)^{\frac{1}{b}}\right]\right\}. \tag{22}$$

So, the simulation of the WGFWE random variable is straightforward. Let U be a uniform variate on the unit interval (0, 1). Thus, by means of the inverse transformation method, we consider the random variable X given by

$$X = \frac{k(u) \pm \sqrt{k(u)^2 + 4\alpha\beta}}{2\alpha}. \tag{23}$$

Since the median is 50% quantile, we obtain the median M of X by setting q = 0.5 in Eq. (21).

### 3.2 The Skewness and Kurtosis

The analysis of the variability Skewness and Kurtosis on the shape parameters α, β can be investigated based on quantile measures. The short comings of the classical Kurtosis measure are well-known. The Bowely's skewness based on quartiles is given by, Kenney and Keeping (1962).

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}}, \tag{24}$$

and the Moors Kurtosis based on quantiles, Moors (1998)

$$K_u = \frac{q_{(0.875)} - q_{(0.625)} - q_{(0.375)} + q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}}, \tag{25}$$

where q(.) represents quantile function.

### 3.3 The Moment

In this subsection we discuss the *rth* moment for WGFWE distribution. Moments are important in any statistical analysis, especially in applications. It can. be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

**Theorem 1.** If X has WGFWE (a, b, α, β) distribution, then the *rth* moments of random variable X, is given by the following

$$\begin{aligned} \mu_r^{'} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+k+n+m} a^{i+1} \beta^m (k+1)^n \Gamma(b(i+1)+j+1)}{i!j!n!m!\alpha^{r-m-1}(n+1)^{r-2m-1}\Gamma((b+i)+1)} \\ &\quad \times \binom{b(i+1)+j+1}{k} \left[\frac{\Gamma(r-m+1)}{\alpha(n+1)^2} + \beta\Gamma(r-m-1)\right]. \end{aligned} \tag{26}$$

**Proof.** We start with the well known distribution of the *rth* moment of the random variable X with probability density function f (x) given by

$$\mu_r^{'} = \int_0^{\infty} x^r f(x; a, b, \alpha, \beta) dx. \tag{27}$$

Substituting from Eq. (1) and Eq. (2) into Eq. (13) we get

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i a^{i+1} b\,\Gamma\left(b(i+1)+j+1\right)}{i!j!\Gamma\left((b+i)+1\right)} \times \int_0^{\infty} x^r\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}e^{-e^{\alpha x-\frac{\beta}{x}}}\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{b(i+1)+j-1}dx,$$

since $0 < 1\text{-}e^{-e^{\alpha x-\frac{\beta}{x}}} < 1$ ,for x > 0, the binomial series expansion of $[1-e^{-e^{\alpha x-\frac{\beta}{x}}}]^{b(i+1)+j-1}$ yields

$$\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{b(i+1)+j-1} = \sum_{k=0}^{\infty}(-1)^k\binom{b(i+1)+j-1}{k}e^{-ke^{\alpha x-\frac{\beta}{x}}},$$

Then we get

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}a^{i+1}\Gamma\left(b(i+1)+j+1\right)}{i!j!\Gamma\left((b+i)+1\right)}\binom{b(i+1)+j-1}{k} \times \int_0^{\infty} x^r\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}e^{-(k+1)e^{\alpha x-\frac{\beta}{x}}}dx,$$

Using series expansion of $e^{-(k+1)e^{\alpha x-\frac{\beta}{x}}}$ ,

$$e^{-(k+1)e^{\alpha x-\frac{\beta}{x}}} = \sum_{n=0}^{\infty}\frac{(-1)^n(k+1)^n}{n!}e^{n(\alpha x-\frac{\beta}{x})},$$

We obtain

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{i+k+n}a^{i+1}(k+1)^n\Gamma\left(b(i+1)+j+1\right)}{i!j!n!\Gamma\left((b+i)+1\right)} \times \binom{b(i+1)+j-1}{k}\int_0^{\infty} x^r\left(\alpha+\beta x^{-2}\right)e^{\alpha(n+1)x}e^{-(n+1)\frac{\beta}{x}}dx,$$

Using series expansion of $e^{-(n+1)\frac{\beta}{x}}$ ,

$$e^{-(n+1)\frac{\beta}{x}} = \sum_{m=0}^{\infty}\frac{(-1)^m[(n+1)\beta]^m}{m!}x^{-m},$$

We obtain

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+k+n+m} a^{i+1} (k+1)^n [(n+1)\beta]^m \Gamma\left(b(i+1)+j+1\right)}{i!j!n!m!\Gamma\left((b+i)+1\right)} \times \binom{b(i+1)+j-1}{k} \left[\int_0^{\infty} \alpha x^{r-m} e^{\alpha(n+1)x} dx + \int_0^{\infty} \beta x^{r-m-2} e^{\alpha(n+1)x} dx\right],$$

By using the the definition of gamma function in the form, Zwillinger (2014),

$$\Gamma(z) = x^z \int_0^{\infty} e^{tx} t^{z-1} dt, \quad z, x > 0.$$

Finally, we obtain the *rth* moment of WGFWE distribution in the form

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+k+n+m} a^{i+1} \Gamma\left(b(i+1)+j+1\right) [(n+1)\beta]^m (k+1)^n}{i!j!n!m!\Gamma\left((b+i)+1\right)} \times \binom{b(i+1)+j+1}{k} \left[\frac{\Gamma(r-m+1)}{\alpha^{r-m}(n+1)^{r-m+1}} + \frac{\beta\Gamma(r-m-1)}{\alpha^{r-m-1}(n+1)^{r-m-1}}\right].$$

This completes the proof.

## 4. The Moment Generating Function

The moment generating function (mgf) MX (t) of a random variable X pro- vides the basis of an alternative route to analytic results compared with working directly with the pdf and cdf of X.

**Theorem 2**. The moment generating function (mgf) of WGFWE distribution is given by

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{i+k+n+m} a^{i+1} \beta^m (k+1)^n t^r \Gamma(b(i+1)+j+1)}{i!j!n!m!r!\alpha^{r-m-1}(n+1)^{r-2m-1}\Gamma((b+i)+1)} \times \binom{b(i+1)+j+1}{k} \left[\frac{\Gamma(r-m+1)}{\alpha(n+1)^2} + \beta\Gamma(r-m-1)\right].$$

**Proof.** We start with the well known distribution of the moment generating function of the random variable X with probability density function f (x) given by

$$M_X(t) = \int_0^{\infty} e^{tx} f(x) dx. \tag{28}$$

Substituting from Eq. (1) and Eq. (2) into Eq. (13) we get

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i a^{i+1} b\Gamma(b(i+1)+j+1)}{i!j!\Gamma((b+i)+1)} \times \int_0^{\infty} e^{tx}(\alpha+\frac{\beta}{x^2})e^{\alpha x-\frac{\beta}{x}}e^{-e^{\alpha x-\frac{\beta}{x}}}\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{b(i+1)+j-1}dx,$$

since $0 < 1\text{-}e^{-e^{\alpha x-\frac{\beta}{x}}}<1$ for $x > 0$, the binomial series expansion of $[1-e^{-e^{\propto x-\frac{\beta}{x}}}]^{b(i+1)+j-1}$ yields

$$\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{b(i+1)+j-1} = \sum_{k=0}^{\infty}(-1)^k\binom{b(i+1)+j-1}{k}e^{-ke^{\alpha x-\frac{\beta}{x}}},$$

Then we get

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}a^{i+1}\Gamma(b(i+1)+j+1)}{i!j!\Gamma((b+i)+1)}\binom{b(i+1)+j-1}{k} \times \int_0^{\infty} e^{tx}\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}e^{-(k+1)e^{\alpha x-\frac{\beta}{x}}}\left[1-e^{-e^{\alpha x-\frac{\beta}{x}}}\right]^{b(i+1)+j-1}dx,$$

Using series expansion of $e^{-(k+1)e^{\alpha x-\frac{\beta}{x}}}$,

$$e^{-(k+1)e^{\alpha x-\frac{\beta}{x}}} = \sum_{n=0}^{\infty}\frac{(-1)^n(k+1)^n}{n!}e^{n(\alpha x-\frac{\beta}{x})},$$

We obtain

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{i+k+n}a^{i+1}(k+1)^n\Gamma(b(i+1)+j+1)}{i!j!n!\Gamma((b+i)+1)} \times \binom{b(i+1)+j-1}{k}\int_0^{\infty} e^{tx}\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha(n+1)x}e^{-(n+1)\frac{\beta}{x}}dx.$$

Using series expansion of $e^{-(n+1)\frac{\beta}{x}}$ and $e^{tx}$,

$$e^{-(n+1)\frac{\beta}{x}} = \sum_{m=0}^{\infty} \frac{(-1)^m[(n+1)\beta]^m}{m!} x^{-m},$$
$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r,$$

We obtain

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{i+k+n+m} a^{i+1}(k+1)^n t^r \Gamma(b(i+1)+j+1)}{i!j!n!m!r!\Gamma((b+i)+1)} \times [(n+1)\beta]^m \binom{b(i+1)+j-1}{k} \int_0^{\infty} \left[\alpha x^{r-m} + \beta x^{r-m-2}\right] e^{\alpha(n+1)x} dx,$$

By using the the definition of gamma function in the form, Zwillinger (2014),

$$\Gamma(Z) = x^z \int_0^{\infty} e^{tx} t^{z-1} dt, \quad z, x > 0.$$

Finally, we obtain the moment generating function of WGFWE distribution in the form

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{i+k+n+m} a^{i+1} \beta^m (k+1)^n t^r \Gamma(b(i+1)+j+1)}{i!j!n!m!r!\alpha^{r-m-1}(n+1)^{r-2m-1}\Gamma((b+i)+1)} \times \binom{b(i+1)+j+1}{k} \left[\frac{\Gamma(r-m+1)}{\alpha(n+1)^2} + \beta\Gamma(r-m-1)\right].$$

This completes the proof.

## 5. Order Statistics

In this section, we derive closed form expressions for the probability density function of the *rth* order statistic of the WGFWE distribution. Let $X_{1:n}, X_{2:n} \ldots, X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, \cdots, X_n$ which taken from a continuous population with cumulative distribution function cdf $F(x;\varphi)$ and probability density function pdf $f(x;\varphi)$,then the probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x;\varphi) = \frac{1}{B(r, n-r+1)} [F(x;\varphi)]^{r-1} [1 - F(x;\varphi)]^{n-r} f(x;\varphi), \tag{29}$$

where $f(x;\varphi)$, $F(x;\varphi)$are the pdf and cdf of WGFWE($\varphi$) distribution given by Eq. (15) and Eq. (14) respectively, $\varphi = (a, b, \alpha, \beta)$ and B(., .) is the Beta function, also we define first order statistics $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$, and the last order statistics as $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$. Since $0 < F(x;\varphi) < 1$ for x > 0, we can use the binomial expansion of $[1 - F(x;\varphi)]^{n-r}$ given as follows

$$[1 - F(x;\varphi)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x;\varphi)]^i. \qquad (30)$$

Substituting from Eq. (30) into Eq. (29), we obtain

$$f_{r:n}(x;\varphi) = \frac{1}{B(r, n-r+1)} f(x;\varphi) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left[F(x;\varphi)\right]^{i+r-1}. \qquad (31)$$

Substituting from Eq. (14) and Eq. (15) into Eq. (31), we obtain

$$f_{r:n}(x; a, b, \alpha, \beta) = \sum_{i=0}^{n-r} \frac{(-1)^i n!}{i!(r-1)!(n-r-i)!} \left[F(x,\varphi)\right]^{i+r-1} f(x;\varphi). \qquad (32)$$

Relation (32) shows that fr:n(x; ϕ) is the weighted average of the Weibull-G Flexible Weibull Extension distribution withe different shape parameters.

## 6. Parameters Estimation

In this section, point and interval estimation of the unknown parameters of the WGFWE distribution are derived by using the method of maximum likelihood based on a complete sample.

### 6.1 Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ denote a random sample of complete data from the WGFWE distribution. The Likelihood function is given as

$$L = \prod_{i=1}^{n} f(x_i; a, b, \alpha, \beta), \qquad (33)$$

substituting from (15) into (33), we have

$$L = \prod_{i=1}^{n} ab\left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}} e^{e^{\alpha x_i - \frac{\beta}{x_i}}} \left[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right]^{b-1} e^{-a[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1]^b}.$$

The log-likelihood function is

$$\begin{aligned} \mathcal{L} \;=\; & n\ln(ab) + \sum_{i=1}^{n} \ln(\alpha + \frac{\beta}{x_i^2}) + \sum_{i=1}^{n} \left(\alpha x_i - \frac{\beta}{x_i}\right) + \\ & \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} + (b-1)\sum_{i=1}^{n} \ln(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1) - a\sum_{i=1}^{n} \left[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right]^b. \quad (34) \end{aligned}$$

The maximum likelihood estimation of the parameters are obtained by differentiating the log-likelihood function L with respect to the parameters a, b, α and β and setting the result to zero, we have the following normal equations.

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} [e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1]^b = 0 \tag{35}$$

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1) - a \sum_{i=1}^{n} [e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1]^b \ln(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1) = 0 \tag{36}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \alpha} &= \sum_{i=1}^{n} \frac{x_i^2}{\beta + \alpha x_i^2} + \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} + (b-1) \sum_{i=1}^{n} \frac{x_i e^{\alpha x_i - \frac{\beta}{x_i}}}{1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}} \\ &\quad - ab \sum_{i=1}^{n} x_i D_i \left[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right]^{b-1} = 0\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \beta} &= \sum_{i=1}^{n} \frac{1}{\beta + \alpha x_i^2} - \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}} - (b-1) \sum_{i=1}^{n} \frac{e^{\alpha x_i - \frac{\beta}{x_i}}}{x_i \left(1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}\right)} \\ &\quad + ab \sum_{i=1}^{n} \frac{D_i}{x_i} \left[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right]^{b-1} = 0,\end{aligned} \tag{38}$$

where $D_i = \exp\{\alpha x_i - \frac{\beta}{x_i} + e^{\alpha x_i - \frac{\beta}{x_i}}\}$. The MLEs can be obtained by solving the nonlinear equations previous, (35)-(38), numerically for a, b, α and β.

## 6.2 Asymptotic confidence bounds

In this section, we derive the asymptotic confidence intervals when a, b, α > 0 and β > 0 as the MLEs of the unknown parameters a, b, α > 0 and β > 0 can not be obtained in closed forms, by using variance covariance matrix $I^{-1}$see Lawless (2003), where $I^{-1}$ is the inverse of the observed information matrix which defined as follows

$$\mathbf{I^{-1}} = \begin{pmatrix} -\frac{\partial^2\mathcal{L}}{\partial a^2} & -\frac{\partial^2\mathcal{L}}{\partial a\partial b} & -\frac{\partial^2\mathcal{L}}{\partial a\partial\alpha} & -\frac{\partial^2\mathcal{L}}{\partial a\partial\beta} \\ -\frac{\partial^2\mathcal{L}}{\partial b\partial a} & -\frac{\partial^2\mathcal{L}}{\partial b^2} & -\frac{\partial^2\mathcal{L}}{\partial b\partial\alpha} & -\frac{\partial^2\mathcal{L}}{\partial b\partial\beta} \\ -\frac{\partial^2\mathcal{L}}{\partial\alpha\partial a} & -\frac{\partial^2\mathcal{L}}{\partial\alpha\partial b} & -\frac{\partial^2\mathcal{L}}{\partial\alpha^2} & -\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\mathcal{L}}{\partial\beta\partial a} & -\frac{\partial^2\mathcal{L}}{\partial\beta\partial b} & -\frac{\partial^2\mathcal{L}}{\partial\beta\partial\alpha} & -\frac{\partial^2\mathcal{L}}{\partial\beta^2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} var(\hat{a}) & cov(\hat{a},\hat{b}) & cov(\hat{a},\hat{\alpha}) & cov(\hat{a},\hat{\beta}) \\ cov(\hat{b},\hat{a}) & var(\hat{b}) & cov(\hat{b},\hat{\alpha}) & cov(\hat{b},\hat{\beta}) \\ cov(\hat{\alpha},\hat{a}) & cov(\hat{\alpha},\hat{b}) & var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) \\ cov(\hat{\beta},\hat{a}) & cov(\hat{\beta},\hat{b}) & cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) \end{pmatrix}. \quad (39)$$

The second partial derivatives included in *I* are given as follows.

$$\frac{\partial^2\mathcal{L}}{\partial a^2} = -\frac{n}{a^2} \quad (40)$$

$$\frac{\partial^2\mathcal{L}}{\partial a\partial b} = -\sum_{i=1}^{n}[e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1]^b \ln(e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1) \quad (41)$$

$$\frac{\partial^2\mathcal{L}}{\partial a\partial\alpha} = -b\sum_{i=1}^{n} x_i D_i[e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1]^{b-1} \quad (42)$$

$$\frac{\partial^2\mathcal{L}}{\partial a\partial\beta} = b\sum_{i=1}^{n} \frac{D_i}{x_i}[e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1]^{b-1} \quad (43)$$

$$\frac{\partial^2\mathcal{L}}{\partial b^2} = -\frac{n}{b^2} - a\sum_{i=1}^{n}[e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1]^b\left[\ln(e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1)\right]^2 \quad (44)$$

$$\frac{\partial^2\mathcal{L}}{\partial b\partial\alpha} = \sum_{i=1}^{n}\frac{D_i x_i}{e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1} - a\sum_{i=1}^{n} D_i x_i E_i \quad (45)$$

$$\frac{\partial^2\mathcal{L}}{\partial b\partial\beta} = -\sum_{i=1}^{n}\frac{D_i}{x_i\left(e^{e^{\alpha x_i-\frac{\beta}{x_i}}}-1\right)} + a\sum_{i=1}^{n}\frac{D_i E_i}{x_i} \quad (46)$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2} = -\sum_{i=1}^{n}\frac{x_i^4}{(\beta+\alpha x_i^2)^2} + \sum_{i=1}^{n} x_i^2 e^{\alpha x_i-\frac{\beta}{x_i}} + (b-1)\sum_{i=1}^{n} x_i^2 F_i - ab\sum_{i=1}^{n} x_i^2 H_i \quad (47)$$

$$\frac{\partial^2 \mathcal{L}}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n}\frac{x_i^2}{\left(\beta+\alpha x_i^2\right)^2} - \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} - (b-1)\sum_{i=1}^{n} F_i + ab\sum_{i=1}^{n} H_i \quad (48)$$

$$\frac{\partial^2 \mathcal{L}}{\partial\beta^2} = -\sum_{i=1}^{n}\frac{1}{\left(\beta+\alpha x_i^2\right)^2} + \sum_{i=1}^{n}\frac{1}{x_i^2} e^{\alpha x_i - \frac{\beta}{x_i}} + (b-1)\sum_{i=1}^{n}\frac{F_i}{x_i^2} - ab\sum_{i=1}^{n}\frac{H_i}{x_i^2}, \quad (49)$$

Where

$$E_i = \left[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right]^{b-1}\left[1 + b\ln(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)\right],$$

$$F_i = \frac{e^{\alpha x_i - \frac{\beta}{x_i}}}{\left(1 - e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}\right)^2}\left[1 - \left(1 + e^{\alpha x_i - \frac{x_i}{\beta}}\right) e^{-e^{\alpha x_i - \frac{\beta}{x_i}}}\right],$$

$$H_i = \left(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right)^{b-2} D_i\left[(b-1)D_i + \left(1 + e^{\alpha x_i - \frac{\beta}{x_i}}\right)\left(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right)\right].$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters a, b, α and β, by using variance matrix as in the following forms

$$\hat{a} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{a})}, \quad \hat{b} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{b})}, \quad \hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\beta})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$-th percentile of the standard normal distribution.

## 7. Application

In this section we perform an application to two examples of real data to illustrate that the WGFWE (a, b, α, β) can be a good lifetime model, compering with many known distributions. By using some statistics, Kolmogorov Smirnov (K-S) statistic, as well as Akaike information criterion (AIC), Akaike (1974), Akaike Information Citerion with correction (AICC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Schwarz information criterion (SIC) values, Schwarz (1978).

### Exapmle 7.1:

Consider the data have been obtained from Aarset (1987), and widely reported in many literatures. It represents the lifetimes of 50 devices, and also, possess a bathtub-shaped failure rate property, Table1.

Table 1: Life time of 50 devices, see Aarset(1987).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.2 | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 6 |
| 7 | 11 | 12 | 18 | 18 | 18 | 18 | 18 | 21 | 32 |
| 36 | 40 | 45 | 46 | 47 | 50 | 55 | 60 | 63 | 63 |
| 67 | 67 | 67 | 67 | 72 | 75 | 79 | 82 | 82 | 83 |
| 84 | 84 | 84 | 85 | 85 | 85 | 85 | 85 | 86 | 86 |

Table 2 gives MLEs of parameters of the WGFWE distribution and K-S Statistics. The values of the log-likelihood functions, AIC, AICC, BIC and HQIC are in Table 3.

Table 2: MLEs and K–S of parameters for Aarset data.

| Model | MLE of the parameters | K-S |
|---|---|---|
| FW$(\alpha,\beta)$ | $\hat{\alpha}=0.0122$, $\hat{\beta}=0.7002$ | 0.4386 |
| W$(\alpha,\beta)$ | $\hat{\alpha}=44.913$, $\hat{\beta}=0.949$ | 0.2397 |
| LFR (a, b) | $\hat{a}=0.014$, $\hat{b}=2.4\times10^{-4}$ | 0.1955 |
| EW$(\alpha,\beta,\gamma)$ | $\hat{\alpha}=91.023$, $\hat{\beta}=4.69$, $\hat{\gamma}=0.164$ | 0.1841 |
| GLFR(a, b, c) | $\hat{a}=0.0038$, $\hat{b}=3.04\times10^{-4}$, $\hat{c}=0.533$ | 0.1620 |
| EFW$(\alpha,\beta,\theta)$ | $\hat{\alpha}=0.0147$, $\hat{\beta}=0.133$, $\hat{\theta}=4.22$ | 0.1433 |
| WGFWE$(a,b,\alpha,\beta)$ | $\hat{a}$ =0.204 , $\hat{b}$ =0.332 , $\hat{\alpha}$ =0.024 , $\hat{\beta}$ =1.421 | 0.10986 |

Table 3: Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted for Aarset data.

| Model | $\mathcal{L}$ | AIC | AICC | BIC | HQIC |
|---|---|---|---|---|---|
| FW$(\alpha,\beta)$ | -250.81 | 505.620 | 505.8753 | 509.4440 | 15.1718 |
| W$(\alpha,\beta)$ | -241.002 | 486.004 | 486.2593 | 489.8280 | 15.0923 |
| LFR (a, b) | -238.064 | 480.128 | 480.3833 | 483.9520 | 15.0679 |
| EW$(\alpha,\beta,\gamma)$ | -235.926 | 477.852 | 478.3737 | 483.5881 | 15.0541 |
| GLFR(a, b, c) | -233.145 | 472.290 | 472.8117 | 478.0261 | 15.0306 |
| EFW$(\alpha,\beta,\theta)$ | -226.989 | 459.978 | 460.4997 | 465.7141 | 14.9774 |
| WGFWE$(a,b,\alpha,\beta)$ | -218.639 | 445.278 | 446.1669 | 452.9261 | 14.9075 |

We find that the WGFWE distribution with four parameters provides a better fit than the previous models flexible Weibull (FW), Weibull (W), linear failure rate (LFR), exponentiated

Weibull (EW), generalized linear failure rate (GLFR) and exponentiated flexible Weibull (EFW). It has the largest likelihood, and the smallest K-S, AIC, AICC, BIC and HQIC values among those considered in this paper.

Substituting the MLE's of the unknown parameters a, b, α, β into (39), we get estimation of the variance covariance matrix as the following

$$I_0^{-1} = \begin{pmatrix} 0.018 & -0.051 & 1.471\times10^{-3} & 0.268 \\ -0.051 & 0.173 & -5.12\times10^{-3} & -0.888 \\ 1.471\times10^{-3} & -5.12\times10^{-3} & 1.535\times10^{-4} & 0.026 \\ 0.268 & -0.888 & 0.026 & 4.825 \end{pmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters a, b, α and β are [0, 0.467], [0, 1.018], [0, 0.048] and [0, 5.727], respectively.

The nonparametric estimate of the survival function using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be WGFWE, FW , W , LFR, EW , GLFR and EFW are computed and plotted in Figure 6.

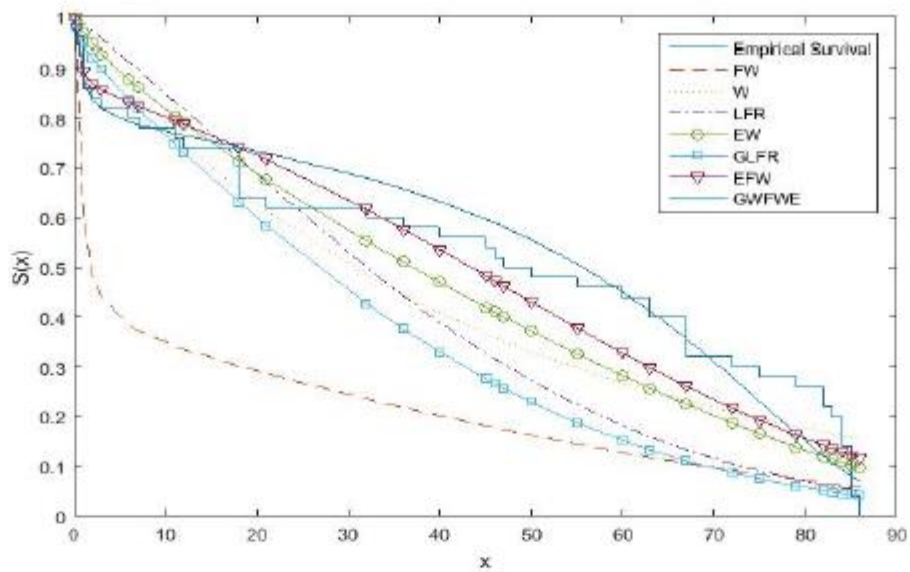


Figure 6: The Kaplan-Meier estimate of the survival function for the data

**Exapmle 7.2:**

The data have been obtained from Salman et al. (1999), it is for the time between failures (thousands of hours) of secondary reactor pumps, Table4

Table 4: Time between failures of secondary reactor pumps.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2.160 | 0.746 | 0.402 | 0.954 | 0.491 | 6.560 | 4.992 | 0.347 |
| 0.150 | 0.358 | 0.101 | 1.359 | 3.465 | 1.060 | 0.614 | 1.921 |
| 4.082 | 0.199 | 0.605 | 0.273 | 0.070 | 0.062 | 5.320 | |

Table 5 gives MLEs of parameters of the WGFWE and K-S Statistics.

Table 5: MLEs and K–S of parameters for secondary reactor pumps.

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{a}$ | $\hat{b}$ | K-S |
|---|---|---|---|---|---|---|
| WGFWE | 0.1750 | 0.4090 | – | 0.769 | 0.565 | 0.0706 |
| FW | 0.0207 | 2.5875 | – | – | – | 0.1342 |
| W | 0.8077 | 13.9148 | – | – | – | 0.1173 |
| MW | 0.1213 | 0.7924 | 0.0009 | – | – | 0.1188 |
| RAW | 0.0070 | 1.7292 | 0.0452 | – | – | 0.1619 |
| EW | 0.4189 | 1.0212 | 10.2778 | – | – | 0.1057 |

The values of the log-likelihood functions, AIC, AICC, BIC, HQIC, and SIC are in Table6.

Table 6: Log-likelihood, AIC, AICC, BIC, HQIC and SIC values of models fitted.

| Model | $\mathcal{L}$ | AIC | AICC | BIC | HQIC | SIC |
|---|---|---|---|---|---|---|
| WGFWE | -29.6650 | 67.33 | 69.5222 | 71.87198 | 10.5823 | 71.8720 |
| FW | -83.3424 | 170.6848 | 171.2848 | 172.95579 | 12.5416 | 172.9558 |
| W | -85.4734 | 174.9468 | 175.5468 | 177.21779 | 12.5915 | 177.2178 |
| MW | -85.4677 | 176.9354 | 178.1986 | 180.34188 | 12.6029 | 180.3419 |
| RAW | -86.0728 | 178.1456 | 179.4088 | 181.55208 | 12.6168 | 181.5521 |
| EW | -86.6343 | 179.2686 | 180.5318 | 182.67508 | 12.6296 | 182.6751 |

We find that the WGFWE distribution with the four-number of parameters provides a better fit than the previous models such as a Flexible Weibull (FW), Weibull (W), modified Weibull (MW), Reduced Additive Weibull (RAW) and Extended Weibull (EW) distributions, respectively. It has the largest likelihood, and the smallest K-S, AIC, AICC and BIC values among those considered in this paper.

Substituting the MLE's of the unknown parameters a, b, α, β into (39), we get estimation of the variance covariance matrix as the following

$$I_0^{-1} = \begin{pmatrix} 0.209 & -0.290 & 0.058 & 0.25 \\ -0.290 & 0.537 & -0.118 & -0.441 \\ 0.058 & -0.118 & 0.027 & 0.095 \\ 0.250 & -0.441 & 0.095 & 0.377 \end{pmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters a, b, α and β are [0, 1.666], [0, 2.205], [0, 0.498] and [0, 1.612], respectively.

To show that the likelihood equation have unique solution, we plot the profiles of the log-likelihood function of a, b, α and β in Figures 7 and 8.

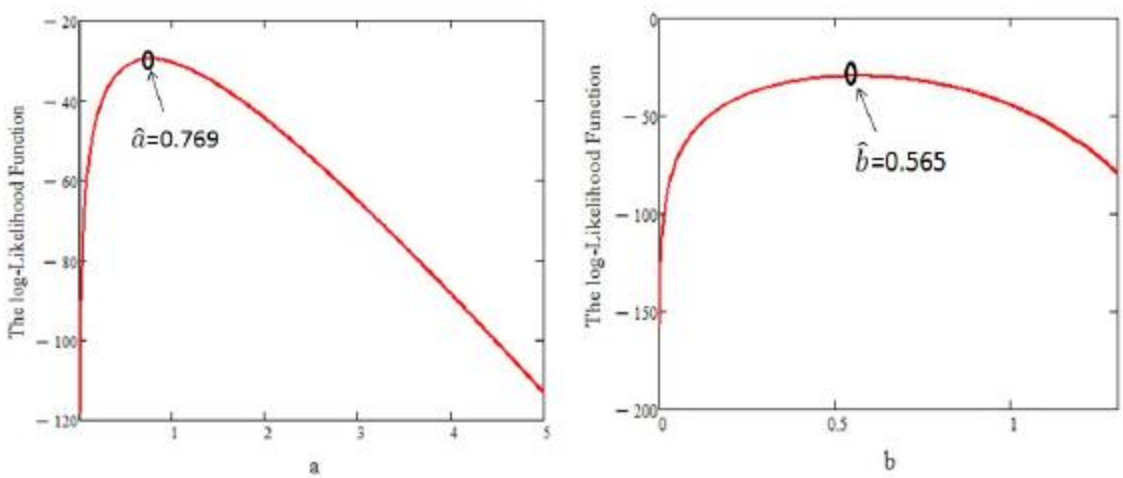


Figure 7: The profile of the log-likelihood function of a, b.

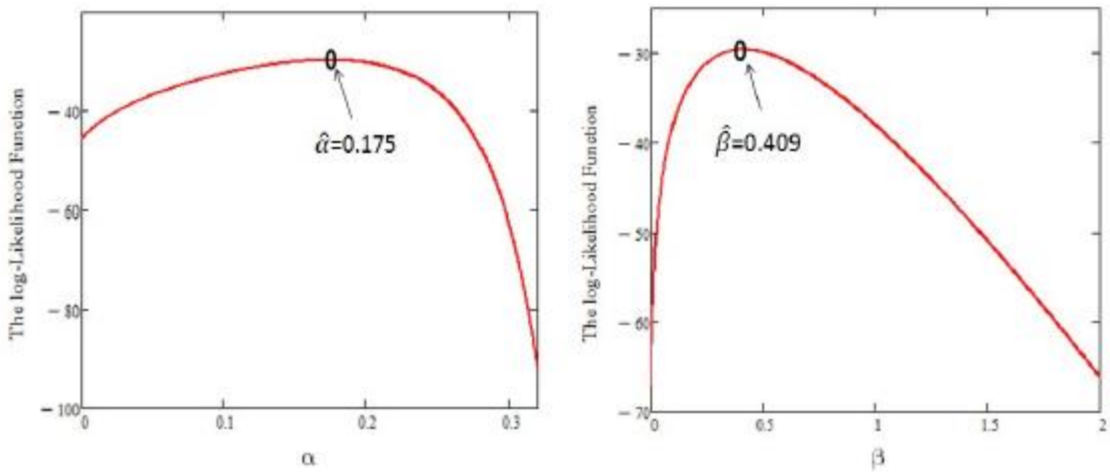


Figure 8: The profile of the log-likelihood function of α, β.

The nonparametric estimate of the survival function using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be WGFWE, FW, W, MW, RAW and EW are computed and plotted in Figure 9.

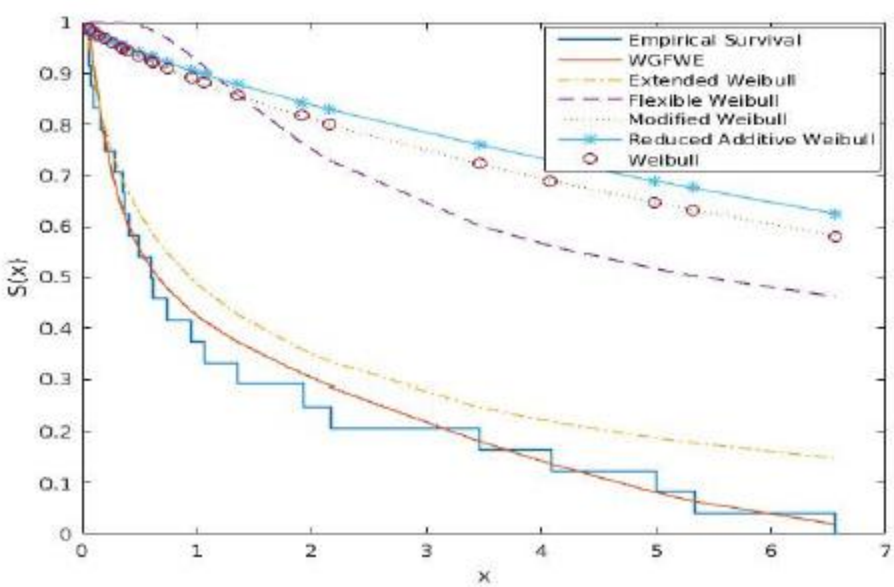


Figure 9: The Kaplan-Meier estimate of the survival function for the data.

Figures 10 and 11 give the form of the hazard rate and CDF for the WGFWE, FW, W, MW, RAW and EW which are used to fit the data after replacing the unknown parameters included in each distribution by their MLE.

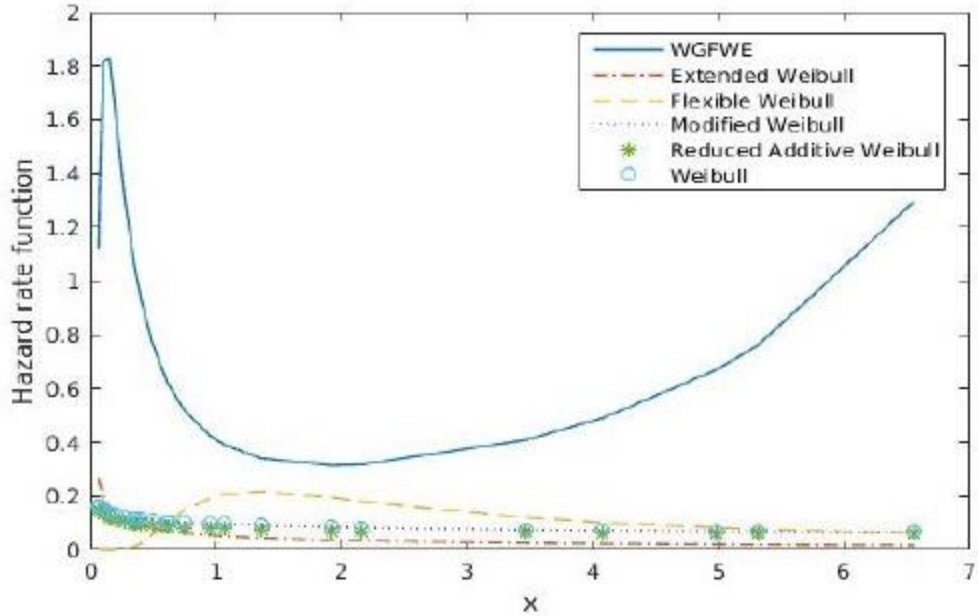


Figure 10: The Fitted hazard rate function for the data.

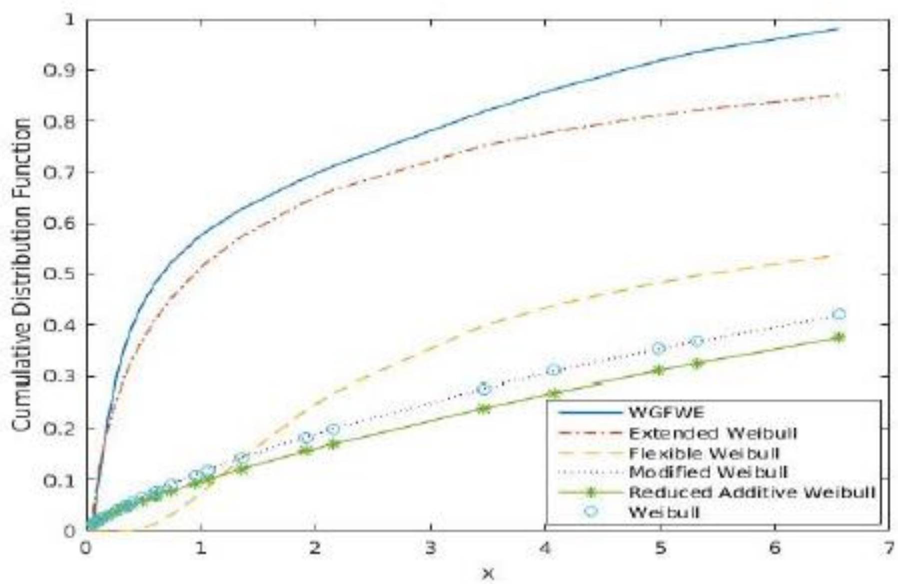


Figure 11:The Fitted cumulative distribution function for the data.

## 8. Summary

A new distribution, based on Weibull- G Family distributions, has been pro- posed and its properties are studied. The idea is to add parameter to a flexible Weibull extension distribution, so that the hazard function is either increasing or more importantly, bathtub shaped. Using Weibull generator component, the distribution has flexibility to model the second peak in a distribution. We have shown that the Weibull-G flexible Weibull extension distribution fits certain well- known data sets better than existing modifications of the Weibull-G family of probability distribution.

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