Adapting The Extended Neyman's Smooth Test to Be Used in Accelerated Failure Time Models

Abdalla Abdel-Ghaly*, Hanan Aly and Elham Abdel-Rahman

Cairo University

Abstract: Accelerated life testing (ALT) has gained greater importance because of dealing with high reliability units. As a result, there is a big need to use a goodness of fit (GOF) technique for testing the underlying lifetime distribution. But there is a difficulty due to the existence of several stress levels with different samples of units at each level. Then, the choice of a certain GOF technique is based on its capability to combine the failure times from all stress levels to reach a conclusion about the adequacy of a certain lifetime distribution at each stress level.

In this paper, the extended Neyman's smooth test (ENST) is chosen. It is then modified in order to be used in validating the distributional assumption of accelerated failure time (AFT) model. This modified method is called; the adapted extended Neyman's smooth test (AENST). It is applied to test for both Weibull and exponential distributions in case of constant stress under complete sampling. To check the performance of the AENST, a comparison is made with the conditional probability integral transformation test (CPITT) via a simulation study. Moreover, a real data set is provided to illustrate the application of the introduced AENST. The results revealed that the AENST is a powerful test comparing with the CPITT. Thus, the AENST is recommended for testing the AFT models.

Key words: Accelerated life testing, Accelerated failure time model, Constant stress, Goodness of fit tests, Extended Neyman's smooth test, Partial scores, Conditional probability integral transformation test.

1. Introduction

Global competition in companion with using higher manufacturing technologies resulted in producing high reliability products. Testing whether these products are sufficiently reliable to be utilized may require long time. This led to the use of accelerated life testing (ALT) in today's industry in order to obtain failures more quickly. Under ALT, items are subjected to stresses at several levels higher than those experienced under normal conditions so as to obtain failures under normal levels. This can be achieved via using accelerated life models (ALM) that relate

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* Corresponding author.

the lifetime distribution to the stress. The difference between the ALM proposed in the literature is in the influence of the applied stress on the reliability (for more details, see Bagdonavičius and Nikulin (2002)).

The commonly used model in reliability analysis is the accelerated failure time (AFT) model. Under this model, increasing the stress level has the effect of shrinking time through a scale factor (For more details, see Meeker and Escobar (1993)); that is the stress affects the scale parameter of the lifetime distribution through a certain relationship. The commonly used relationships are the Arrhenius and the inverse power law relationships.

There is a great amount of literature on estimating the parameters of the lifetime distribution in the case of constant stress. This literature treats a variety of models (distributions and relationships), approaches of estimation, and different types of censoring. For example, see Abdel Ghaly *et al.* (1998), Drop and Mazzuchi (2004), Escobar and Meeker (2006), Liu and Tang (2009), and Abdel Ghaly *et al.* (2015).

Most of work in ALT assumes that the lifetime distribution is known in advance and the problem is to estimate the parameters of this distribution under the normal operating conditions. Actually, if the underlying distribution of the lifetime data does not coincide with the hypothesized one, the results drawn from ALT are nonsense and misleading. Thus, it is essential to check for the underlying lifetime distribution of the lifetime data before extracting any results from the test. This problem is known as goodness of fit (GOF) testing problem and there are some techniques proposed in the literature concerning solving this problem.

GOF tests proposed in the field of ALT can be divided into two types; the first one for ALM and the other to validate the assumptions of the AFT model. Since ALM are proposed to model the effect of the stress on the lifetime distribution, they can be thought of as some hypotheses about the influence of the applied stress on the reliability. Accordingly, some GOF techniques are proposed to assess these hypotheses. (For more details, see Bagdonavičius and Nikulin (2002), Bagadonavičius *et al.* (2004), Bagadonavičius *et al.* (2011), and Balakrishnan *et al.* (2013)).

When applying GOF tests for the assumptions of an AFT model, it is assumed that the AFT model holds and GOF techniques are proposed to validate its underlying assumptions; which are

1. The life-stress relationship.
2. The family of distributions to which the failure time at each stress level belongs.

In the case of testing for the life-stress relationship, Nelson (1990) used the F statistic to test the linear life-stress relationship for log-failure time variable (log T), where T was assumed to have exponential, Weibull and lognormal distributions. Lawless (2003) dealt with the same case but using likelihood ratio test (LRT). Eguchi (1992) investigated the validity of the inverse power law relationship assuming a bivariate exponential distribution using a test statistic based on a projection method. Teng and Yeon (2002) proposed D-statistic based on the transformed least square (LS) estimation approach to assess the validity of the log-linear life-stress relationship

against the log-quadratic one in case of step stress ALT experiment under exponential type II censored data.

Regarding testing for the underlying distribution at each stress level, there is no much work in the literature. This little work was based on some traditional GOF tests. For example, Sethuurman and Singpurwalla (1982) used Kolmogrov-Smirnov statistic to test that the unknown distributions at different stress levels belong to a common parametric location-scale family. Nelson (1990) used the LRT for the same case and the same distributions. Xiong and Ji (2004) used the LRT to assess the validity of the cumulative exposure model used for analyzing step stress ALT. Wang (2009) proposed a procedure based on sample spacings to test the exponentiality of the lifetime distribution for each stress level based on type II censored k-stages step- stress ALT in the existence of the log linear life-stress relationship. Galanova *et al.* (2012) and Bagadonaviˇcius *et al.* (2013) discussed some theoretical problems inherent in using nonparametric GOF tests (Kolmogorov, Cramer-von Mises-Smirnov, and Anderson-Darling statistics) to validate parametric AFT models based on analyzing a sample of residuals.

Thus, there is much need to introduce other GOF tests to be used in examining the fit of certain family of distributions in AFT models. The contribution of this paper is the adaptation of the extended Neyman's smooth test (ENST) in order to validate it to be used as a GOF technique in AFT models. This test is then called; the adapted extended Neyman's smooth test (AENST). Our interest here, is on the problem of GOF pertaining to the distribution to which the lifetime at each stress level belongs. Hence, assuming that the AFT model is appropriate, we are interested in testing if the underlying distribution at each stress level belongs to a certain family. We focus on the log-location-scale family of distributions, specifically the exponential and Weibull distributions. The AENST is applied in case of constant stress and complete sampling.

Abdel-Ghaly *et al.* (2016) applied the conditional probability integral transfor- mation test (CPITT) in the case of AFT models. In order to check the superiority of the proposed AENST as a goodness of fit technique to be used in the case of AFT models, a numerical comparison is made between the AENST and CPITT under the same assumptions.

This paper is organized as follows. In Section 2, we present the idea on which Neyman's smooth test (NST) was built. Moreover, the extension of this test in case of composite hypothesis, named; the extended Neyman's smooth test (ENST) is also presented. In Section 3, we clarify how to adapt the ENST in order to be applicable in AFT models. This test is then called; the adapted extended Neyman's smooth test (AENST). Section 4 is devoted for summarizing the work of Abdel-Ghaly (2016) in applying the CPITT in the case of AFT models. In Section 5, a simulation study is conducted to investigate the performance of the AENST comparing with that of the CPITT. A real data set is considered in Section 6. Finally, the conclusions reached from this paper are summarized in Section 7.

2. Neyman's Smooth and Extended Neyman's Smooth Tests

In this Section, Neyman's smooth test (NST) and the extended Neyman's smooth test (ENST) are explained.

2.1 Neyman's Smooth Test (NST)

Neyman (1937) was the pioneer to introduce the GOF smooth tests. His target was to propose a test that does not depend on the form of the null density $f_0(t; \beta)$. To achieve his target, the test was based on the probability integral transformation (PIT) which implies that; if T is a real-valued random variable with continuous CDF $F(t; \beta)$, then $U = F(T; \beta)$ has a uniform distribution on the interval $(0, 1)$. Thus, under NST, testing for

$$H_0 : f(t; \beta) = f_0(t; \beta) \tag{2.1}$$

is equivalent to testing whether the random variable U has a uniform distribution on the interval $(0, 1)$. Neyman (1937) considered an alternative density to the uniform one of the form

$$g(u; \theta) = \exp\{\sum_{s=1}^{m} \theta_s \pi_s(u) - D(\theta)\}, \tag{2.2}$$

where $u = F(t; \beta)$, $D(\theta)$ is a normalizing constant, $\underline{\theta}` = (\theta 1, \theta 2, ..., \theta m)$, m is a fixed known positive integer recommended not to exceed four, and $\pi_s(u)$ is the sth degree of Legendre polynomials.

Neyman (1937) called (2.2) the order m smooth alternative density. Testing for (2.1) is equivalent to test for $H_0 : \underline{\theta} = \underline{0}$ in the smooth alternative density and this is accomplished using a quadratic statistic in the form

$$\Psi_m^2 = \sum_{s=1}^{m} \frac{1}{n} \{\sum_{i=1}^{n} \pi_s(u_i)\}^2. \tag{2.3}$$

The limiting null distribution of Ψ_m^2 is χ^2 distribution with m degrees of freedom. Based on the p-value, one can judge the fit of $f_0(t; \beta)$ for the data in hand. NST was proposed for fully specified distributions in the case of independent identically distributed (i.i.d) uncensored data. For more details, see D'Agostino and Stephens (1986).

2.2 Extended Neyman's Smooth Test (ENST)

Thomas and Pierce (1979) suggested using power functions instead of using the Legendre polynomials in (2.2) in order to extend Neyman's smooth test to the case of composite hypotheses. In this case, the smooth alternative density (2.2) became of the form

$$g(u; \theta) = \exp\{\sum_{s=1}^{m} \theta_s u^s - D(\theta)\}, \tag{2.4}$$

where m, $D(\theta)$, and u are as defined in equation (2.2).

The smooth alternative density (2.4) could be written in terms of the original variable T and its pdf $f(t; \beta)$, as follows

$$g(t; \beta, \theta) = f(t; \beta) \exp\{\sum_{s=1}^{m} \theta_s F^s(t; \beta) - D(\theta)\}, \qquad (2.5)$$

where m and $D(\theta)$ are as defined in equation (2.2), and $F(t; \beta)$ is the CDF corresponding to $f(t; \beta)$.

Thomas and Pierce (1979) used partial scores to test for $\theta = 0$ which is equivalent to test for (2.1). The testing procedure could be presented as:

Let $T_1, T_2, ..., T_n$ be independent continuous random variables. The CDF of T_i is given by $F(t_i; \beta)$, where $\underline{\beta}$ is a vector of p unknown parameters, with corresponding pdf denoted by $f(t_i; \beta)$. The smooth alternative proposed by Thomas and Pierce (1979) is given by (2.5). They entioned that for some distributions as the exponential and Weibull ones, it is convenient to replace $F(t; \beta)$ by $\bar{F}(t; \beta) = 1 - F(t; \beta)$, in (2.5) without changing the ultimate test. They used partial scores to test the null hypothesis $H_0 : \theta = 0$ which is equivalent to test the null density $f_0(t; \beta)$ in (2.1). This can be performed by writing the log-likelihood function of the data under (2.5) in the form

$$\ell(\theta, \beta) = \sum_{i=1}^{n} \{\ln f(t_i, \beta) + \sum_{s=1}^{m} \theta_s F^s(t_i; \beta) - D(\theta)\}, \qquad (2.6)$$

and then getting the partial scores from (2.6) under $H_0 : \theta = 0$ as

$$\frac{\partial \ell(\theta, \beta)}{\partial \theta_s} = \sum_{i=1}^{n} \{F^s(t_i; \beta) - \frac{\partial D(\theta_0)}{\partial \theta_s}\} = \sum_{i=1}^{n} \{F^s(t_i; \beta) - (s+1)^{-1}\}, s = 1, 2, ..., m, \quad (2.7)$$

and

$$\frac{\partial \ell(\theta, \beta)}{\partial \beta_w} = \sum_{i=1}^{n} \{\frac{\partial \ln f(t_i, \beta)}{\partial \beta_w}\}, \quad w = 1, 2, ..., p, \qquad (2.8)$$

where $\theta 0$ denotes the evaluation at $\theta = 0$, and $\dfrac{\partial D(\theta_0)}{\partial \theta_s} = E[F^s(t_i; \beta)]$

Under $H0 : \theta = 0$, the Fisher information matrix denoted by

$$I(\theta, \beta) = Cov(\frac{\partial \ell(\theta, \beta)}{\partial \theta}, \frac{\partial \ell(\theta, \beta)}{\partial \beta})|_{\hat{\theta}, \hat{\beta}} \qquad (2.9)$$

partitioned in the form

$$I(\theta, \beta)| = \begin{pmatrix} I_{\theta\theta}(\theta, \beta) & I_{\theta\beta}(\theta, \beta) \\ I_{\beta\theta}(\theta, \beta) & I_{\beta\beta}(\theta, \beta) \end{pmatrix}, \qquad (2.10)$$

where

$$I_{\theta\theta}(\theta,\beta) = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ & a_{22} & . & . & . & a_{2m} \\ & & . & . & . & . \\ & & & . & . & . \\ & & & & . & . \\ & & & & & a_{mm} \end{pmatrix}, \quad I_{\beta\beta}(\theta,\beta) = \begin{pmatrix} i_{11} & i_{12} & . & . & . & i_{1p} \\ & i_{22} & . & . & . & i_{2p} \\ & & . & . & . & . \\ & & & . & . & . \\ & & & & & i_{pp} \end{pmatrix},$$

$$I_{\theta\beta}(\theta,\beta) = \begin{pmatrix} O_{11} & O_{12} & . & . & . & O_{1p} \\ O_{21} & O_{22} & . & . & . & O_{2p} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ O_{m1} & O_{m2} & . & . & . & O_{mp} \end{pmatrix} = I_{\beta\theta}(\theta,\beta),$$

$$a_{sh} = -E\{\frac{\partial^2 \ell(\theta,\beta)}{\partial\theta_s \partial\theta_h}\} = \frac{nsh}{(s+h+1)(s+1)(h+1)}, \quad s,h = 1,2,...,m,$$

$$\frac{\partial^2 D(\theta_0)}{\partial\theta_s \partial\theta_h} = Cov(F^s(t_i;\beta), F^h(t_i;\beta)),$$

$$i_{wr} = -E\{\frac{\partial^2 \ell(\theta,\beta)}{\partial\beta_w \partial\beta_r}\} = -nE\{\frac{\partial^2 \ln f(t_i;\beta)}{\partial\beta_w \partial\beta_r}\}, \quad w,r = 1,2,...,p, \text{ and}$$

$$O_{sw} = -E\{\frac{\partial^2 \ell(\theta,\beta)}{\partial\theta_s \partial\beta_w}\} = -nE\{\frac{\partial F^s(t_i;\beta)}{\partial\beta_w}\}, \quad s = 1,2,...,m, w = 1,2,...,p.$$

Under $H_0 : \theta = 0$, the quadratic partial score test statistic considered by Thomas and Pierce (1979) takes the form

$$Q_m = \frac{1}{n}(\frac{\partial\ell(\theta,\hat{\beta}_0)}{\partial\theta})`(n^{-1}I_{\theta\theta|\beta})^{-1}(\frac{\partial\ell(\theta,\hat{\beta}_0)}{\partial\theta}) = (\frac{\partial\ell(\theta,\hat{\beta}_0)}{\partial\theta})`(I_{\theta\theta|\beta})^{-1}(\frac{\partial\ell(\theta,\hat{\beta}_0)}{\partial\theta}), (2.11)$$

where $\frac{\partial\ell(\theta,\hat{\beta}_0)}{\partial\theta}$ is an ($m \times 1$) vector of the partial scores, $\hat{\beta}_0$ is the MLE for β when $\theta = 0$, and $I_{\theta\theta|\beta}$ is an ($m \times m$) matrix which represents $Cov(\frac{\partial\ell(\theta,\hat{\beta}_0)}{\partial\theta})$ and is defined as

$$I_{\theta\theta|\beta} = I_{\theta\theta} - I_{\theta\beta}I_{\beta\beta}^{-1}I_{\beta\theta}. \quad (2.12)$$

The asymptotic null distribution of Q_m is χ_m^2 distribution. Thomas and Pierce(1979) derived $n(I_{\theta\theta|\beta})^{-1}$ for m = 4 in the case of exponential, Weibull and normal distributions. They concluded that when the distribution under test is of a location-scale type, $n(I_{\theta\theta|\beta})^{-1}$ is β free, that is; it does not depend on the true values of $\underline{\beta}` = (\beta1, \beta2, ..., \beta p)$. For other distributions, if $n(I_{\theta\theta|\beta})^{-1}$ depends on β, one should replace β with $\hat{\beta}_0$ or use the observed information matrix, without changing the limiting distribution of Q_m. Thomas and Pierce (1979) used the extended Neyman's smooth test to examine the validity of the exponential, Weibull, and lognormal distributions.

3.   Adapted Extended Neyman's Smooth Test (AENST) and its use in AFT models

In this Section, the ENST explained in sub-section 2.2 is adapted such that it be- comes valid to be used as a GOF test in AFT models. We call this test; Adapted Extended Neyman's Smooth Test (AENST). The partial scores and the elements of Fisher information matrix are derived for Weibull AFT model in sub-section 3.1, while exponential AFT model is considered as a special case in sub-section 3.2.

3.1 The case of Weibull AFT model

Under Weibull inverse power law AFT model, the experiment is conducted according to the following conditions

1. A random sample of size n units are put on test, and all run to failure.
2. There are k test stress levels and nj units are tested at stress level Vj, $j = 1, 2, ..., k$.
3. The total number of test units is $n = n1 + n2 + ... + nk$.
4. The shape parameter η of Weibull distribution is independent of the stress which means that this parameter is the same at each stress level.
5. The stress Vj affects the scale parameter, αj, through the inverse power law relationship

$$\varpi_j = \frac{C}{V_j^P}, \ C > 0. \tag{3.1}$$

6. The failure times at stress level Vj, Tij, $i = 1, 2, ..., nj$, $j = 1, 2, ..., k$, are assumed to have Weibull distribution with pdf given by

$$f(t_{ij}; \varpi_j, \eta) = \frac{\eta}{\varpi_j} (\frac{t_{ij}}{\varpi_j})^{\eta-1} \exp[-(\frac{t_{ij}}{\varpi_j})^{\eta}], \ t_{ij} > 0, \varpi_j, \eta > 0. \tag{3.2}$$

By substituting (3.1) in (3.2), the pdf of the failure times, T_{ij}, $i = 1, 2, ..., n_j$, $j = 1, 2, ..., k$, takes the following form

$$f(t_{ij}; C, P, \eta) = \frac{\eta v_j^P}{C} (\frac{v_j^P t_{ij}}{C})^{\eta-1} \exp[-(\frac{v_j^P t_{ij}}{C})^{\eta}], \ t_{ij} > 0, C, \eta > 0. \tag{3.3}$$

For convenience, we use the notations $\beta 1 = C$, $\beta 2 = P$, and $\beta 3 = \eta$. In this case, the pdf in (3.3) and the corresponding reliability function have the following forms, respectively

$$f(t_{ij}; \beta) = (\frac{\beta_3 v_j^{\beta_2}}{\beta_1})(\frac{v_j^{\beta_2} t_{ij}}{\beta_1})^{\beta_3-1} \exp\{-(\frac{v_j^{\beta_2} t_{ij}}{\beta_1})^{\beta_3}\}, t_{ij} > 0, \beta_1, \beta_3 > 0, \tag{3.4}$$

$$\bar{F}(t_{ij}; \beta) = \exp[-(\frac{v_j^{\beta_2} t_{ij}}{\beta_1})^{\beta_3}], \ t_{ij} > 0, \beta_1, \beta_3 > 0. \tag{3.5}$$

Testing for (3.4) is equivalent to test for $\theta = 0$ in the smooth alternative given by (2.5) and this is accomplished using the score test. Considering (3.4) and (3.5), the log-likelihood function of the data under (2.5) becomes

$$\ell(\theta, \beta) = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\{\ln f(t_{ij}, \beta) + \sum_{s=1}^{m}\theta_s \bar{F}^s(t_{ij}; \beta) - D(\theta)\}$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}\{\ln(\frac{\beta_3 v_j^{\beta_2}}{\beta_1}) + (\beta_3 - 1)\ln(\frac{v_j^{\beta_2}t_{ij}}{\beta_1}) - (\frac{v_j^{\beta_2}t_{ij}}{\beta_1})^{\beta_3}$$

$$+ \sum_{s=1}^{m}\theta_s \exp[-s(\frac{v_j^{\beta_2}}{\beta_1}t_{ij})^{\beta_3}] - D(\theta)\}. \tag{3.6}$$

The partial scores from (3.6) under $H_0 : \theta = 0$ are given by

$$\frac{\partial\ell(\theta, \beta)}{\partial\theta_s} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\{\bar{F}^s(t_{ij}; \beta) - \frac{\partial D(\theta_0)}{\partial\theta_s}\}$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}\{\exp[-s(\frac{v_j^{\beta_2}}{\beta_1}t_{ij})^{\beta_3}] - (s+1)^{-1}\}, \ s = 1, 2, ..., m, and \tag{3.7}$$

$$\frac{\partial\ell(\theta, \beta)}{\partial\beta_w} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\{\frac{\partial\ln f(t_{ij}; \beta)}{\partial\beta_w}\}$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}\{\frac{\partial\{\ln(\frac{\beta_3 v_j^{\beta_2}}{\beta_1}) + (\beta_3 - 1)\ln(\frac{v_j^{\beta_2}t_{ij}}{\beta_1}) - (\frac{v_j^{\beta_2}t_{ij}}{\beta_1})^{\beta_3}\}}{\partial\beta_w}\}, \tag{3.8}$$

for s=1,2,...,m, and w=1,2,3. The elements of the information matrix $I(\theta, \beta)$ under H_0 are given by

$$a_{sh} = nsh/(s+h+1)(s+1)(h+1), \ s, h = 1, 2, ..., m,$$

$$i_{11} = n(\beta_3/\beta_1)^2, \ i_{22} = \beta_3^2 \sum_{j=1}^{k} n_j(\ln v_j)^2, \ i_{33} = [n/\beta_3^2][1 + (1/6)\pi^2 - 2\gamma + \gamma^2],$$

$$i_{12} = (-\beta_3^2/\beta_1)\sum_{j=1}^{k} n_j \ln v_j, \ i_{13} = (-n/\beta_1)(1-\gamma), \ i_{23} = (1-\gamma)\sum_{j=1}^{k} n_j \ln v_j,$$

$$O_{s1} = [-ns\beta_3][\beta_1(s+1)^2], \ O_{s2} = [s\beta_3/(s+1)^2]\sum_{j=1}^{k} n_j \ln v_j, \ and$$

$$O_{s3} = [ns/\beta_3(s+1)^2][1-\gamma-\ln(s+1)],$$

where $\gamma = 0.5772...$ is the Euler's constant.

By getting the $I_{\theta\theta|\beta}$, it was found that it is β free which means that it does not depend on the true values of $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$. Thus, under $H_0 : \theta = 0$, the quadratic partial score test statistic is given by

$$Q_m = \left(\frac{\partial \ell(\theta, \hat{\beta}_0)}{\partial \theta}\right)`(I_{\theta\theta|\beta})^{-1}\left(\frac{\partial \ell(\theta, \hat{\beta}_0)}{\partial \theta}\right), \tag{3.9}$$

where the s_{th} element of $\frac{\partial \ell(\theta, \hat{\beta}_0)}{\partial \theta}$ can be computed as

$$\frac{\partial \ell(\theta, \hat{\beta}_0)}{\partial \theta_s} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left\{ \exp\left[-s\left(\frac{v_j^{\hat{\beta}_2}t_{ij}}{\hat{\beta}_1}\right)^{\hat{\beta}_3}\right] - (s+1)^{-1}\right\}. \tag{3.10}$$

In this case, the limiting null distribution of Q_m is χ^2 distribution with m degrees of freedom. By comparing the p-value of Q_m computed from χ_m^2 distribution with the significance level, the adequacy of Weibull distribution as the parametric model for the lifetime variable at each stress level can be judged.

3.2 The case of exponential AFT model

Since, the exponential distribution is a special case of Weibull distribution, then by putting $\eta = 1$ in (3.2), the pdf of T_{ij} , $i = 1, 2, ..., n_j$, $j = 1, 2, ..., k$, becomes

$$f(t_{ij}; \varpi_j) = \frac{1}{\varpi_j}exp\left[-\left(\frac{t_{ij}}{\varpi_j}\right)\right], \; t_{ij} > 0, \varpi_j > 0. \tag{3.11}$$

Using inverse power law relationship, the pdf in (3.11) takes the following form

$$f(t_{ij}; \beta) = \frac{v_j^{\beta_2}}{\beta_1} \exp\left(-\frac{v_j^{\beta_2}}{\beta_1}t_{ij}\right), \; t_{ij} > 0, \beta_1 > 0. \tag{3.12}$$

Similarly, under $H_0 : \theta = 0$, the quadratic partial score test statistic is given by (3.9).

4.   Conditional Probability Integral Transformation Test   (CPITT)

O'Reilly and Quesenberry (1973) introduced the CPITT that was based on transforming the original set of n random variables into a smaller set of $(n - p)$ - where p is the number of estimated parameters - (i.i.d) Uniform, U (0, 1) random variables. Then, test of uniformity can be applied. Abdel-Ghaly et al. (2016) applied this method in the case of AFT model. They assumed that the failure times at each stress level could be transformed separately into i.i.d. U (0, 1) random variables. Then the transformed random variables obtained using the CPITT at each stress level are pooled together as one sample hypothesized to be drawn from U (0, 1) distribution. Finally, the unifor- mity of the pooled sample and accordingly the adequacy of the hypothesized family of distributions can be judged.

Applying the CPITT using both Weibull and exponential distributions are briefly explained (for more details, see Abdel-Ghaly et al. (2016)) in the following subsections.

4.1 The case of exponential AFT model

The failure times, T_{ij} , $i = 1, 2, ..., n_j$, $j = 1, 2, ..., k$ that follow the Weibul distribution in the form (3.4), are transformed into a new set of random variables, X_{ij} , where

$$X_{ij} = T_{ij}^{\beta_3}, \ i = 1, 2, ..., n_j, \ j = 1, 2, ..., k. \tag{4.1}$$

In this case, the transformation to $U\ (0,\ 1)$ is obtained by

$$U_{ij} = 1 - \{\frac{1 - (n_j - i + 1)Z_{ij}/(Z_{ij} + ... + Z_{n_j j})}{1 - (n_j - i + 1)Z_{(i-1)j}/(Z_{ij} + ... + Z_{n_j j})}\}^{n_j - i}, i \ = \ 1, 2, ..., n_j - 1,$$
$$j \ = \ 1, 2, ..., k, \tag{4.2}$$

where Z_{ij} is the ith order statistic from the jth stress level, and $Z_{0j} = 0$.

If the shape parameter β_3 is unknown, it can be replaced by the MLE computed from each stress level separately. This will result in omitting an extra observation when using the CPITT, which in turn leads to replacing $i = 1, 2, ..., n_j - 1$ by $i = 1, 2, ..., n_j - 2$ in (4.2). The transformed $(n - 2k)$ U values obtained from all stress levels are pooled together as one sample. The uniformity of the transformed values are tested by using the modified Watson statistic, named U_{Mod}^2 (called U_{Mod-W}^2 in the case of Weibull distribution). Thus, the assumption of Weibull distribution can be validated.

4.2 The case of exponential AFT model

Using $T_{ij}, i = 1, 2, ..., n_j, j = 1, 2, ..., k$ with *pdf* of exponential distribution given by (3.12), the transformed U values from each sub-sample at each stress level are given by (4.2). By testing the uniformity of these $(n - k)$ U values, using the same statistic, U_{Mod}^2 (called U_{Mod-E}^2 in the case of exponential distribution), the assumption of exponentiality can be validated.

5. Simulation Study

In this Section, the performance of the AENST in validating the underlying distri- butional assumption in AFT models is evaluated. The performance is measured by the size and the power of the test. At each size, the power of both the AENST and CPITT are compared. Testing for both Weibull and exponential distributions are explained in the following subsections.

5.1 Testing for the Weibull distribution

In this subsection, the assumption of Weiull distribution in AFT model is to be examined. Firstly, the power of the AENST, Q_m given by (3.9) with $m = 1$ (called Q_{1-W} in this case) is found. Secondly, the power of the CPITT, U_{Mod}^2 (called U_{Mod-W}^2 is also given.

The simulation study is conducted under the following   experiment
- There are $k = 4$ stress levels with different four values: $V_1 = 24$, $V_2 = 26$, $V_3 = 28$ and $V_4 = 30$.

- Four different sample sizes, n, and their division on stress levels, n_j, $j = 1, 2, 3, 4$, are arbitrary chosen as shown in Table 1.
- The true values of the parameters of the inverse power law relationship defined in (3.1), C and P are arbitrary chosen to be (0.5, 1.5, 3.5) and (0.1, 0.5, 0.9), respectively. Then, nine different combinations of these values are considered.
- The computed values of the variance of the partial score, $I_{\theta\theta|\beta}$, needed to calculate the statistic Q_{1-W} are given in Table 2.
- For each combination of (C, P, n), 1000 samples are generated using Mathcad program from
  - Exponential distribution with pdf given by (3.11).
  - Weibull distribution with pdf given by (3.3), with $\eta = 0.5$. The case of $\eta = 2$ is also considered, but it is found that there is no much difference between both cases. So, the case of $\eta = 0.5$ is the only presented in this paper. When $\eta = 1$ -which is the case of the exponential distribution- is discussed in subsection 5.2.
  - Lognormal distribution with pdf givenby

$$f(t_{ij}; C, P, \sigma) = \frac{1}{\sigma t_{ij}\sqrt{2\pi}} \exp\{-\frac{[\ln t_{ij} - \ln C + P \ln V_j]^2}{2\sigma^2}\},$$
$$t_{ij} > 0, C > 0, \ \sigma > 0, \qquad (5.1)$$

with $\sigma = 2$ and the scale parameter, $\exp(\mu_j)$, $j = 1, 2, 3, 4$, is expressed by the inverse power law relationship defined by

$$\exp(\mu_j) = \frac{C}{V_j^P}, \quad \text{or} \quad \mu_j = \ln C - P \ln V_j, \ C > 0. \qquad (5.2)$$

- For each sample, the MLE of the parameters of these distributions are obtained using Mathcad program with tolerance value $\epsilon = 0.00001$. Then, the statistics Q_{1-W} and U^2_{Mod-W} are calculated and the power for each one of them is estimated by

Power = Number of times rejecting H_0/1000, where
H_0: T_{ij} follows Weibull distribution with pdf given by (3.2).
H_1: Not H_0.

The estimated power values of Q_{1-W} and U^2_{Mod-W} statistics are given in Table 3.
From this Table, it is seen that

- In general, The estimated power for Q_{1-W} statistic increases as the sample sizes increase. For large sample sizes, the power of the this statistic becomes very high and is close to 1.
- Under the lognormal alternative and for all sample sizes, Q_{1-W} has observable higher power than U^2_{Mod-W} statistic.
- When testing for the Weibull distribution against Weibull alternative, the estimated power of Q_{1-W} is very small, whatever the sample size. This means that there is a very

small probability to reject H_0 when the failure times have already follow Weibull distribution.
- Under the exponential alternative, the estimated power of Q_{1-W} is not so high. This can be justified by the fact that when the shape parameter of Weibull distribution equals 1, the distribution is reduced to the exponential one.
- The case when n = 33 is not considered when calculating the power of the U_{Mod-W}^2 statistic because some of the sub-samples at each stress level are very small and using this statistic requires obtaining the MLE of the shape parameter of Weibull distribution at each stress level separately. This problem does not appear when using the Q_{1-W} statistic.

5.2 Testing for the exponential distribution

For the exponential case, to examine the power of the AENST (called Q_{1-E}), the simulation study is conducted under the same experiment of Weibull case but with different values of the variance of the partial score, $I_{\theta\theta|\beta}$, also given in Table 2.

In this case, the hypotheses are given by

H_0: T_{ij} follows exponential distribution with pdf given by (3.11).

H_1: Not H_0.

The estimated power values of Q_{1-E} statistic in testing the exponential distribution in AFT model are given in Table 4 under different values of C, P, and n. From this Table, it is seen that
- The estimated power increases as the sample size increases.
- The Q_{1-E} statistic is powerful for testing the exponential distribution since the estimated power of this statistic is very high for all sample sizes versus each one of the lognormal and Weibull alternatives. In some cases the power equals 1.
- Comparing with the CPITT test, U_{Mod-E}^2, Q_{1-E} statistic is more powerful for testing H0 versus both the lognormal and Weibull alternatives.
- In the case of Exponential alternative, Q_{1-E} statistic has a very small power to reject H0. In the majority of cases, the estimated power of Q_{1-E} is smaller than that of U_{Mod-E}^2.

6. A Real Data Set

In this Section, we apply the AENST to investigate the distribution of times to breakdown of an insulating fluid. The data is taken from Nair (1982). This data represents the times to breakdown of an insulating fluid under three elevated voltage stresses.

Nelson (1990) suggested using the exponential inverse power law model given by (3.12) to fit the data. To validate this suggestion, the AENST statistic with $m = 1$, Q_{1-E} statistic is computed. The results of applying the test are summarized in Table 5. The MLE of the parameters are obtained using Mathcad program. From Table 5, it is clear that the test gave strong evidence that the exponential inverse power law model is a suitable model to represent the times to breakdown of the insulating fluid given by Nair (1982).

7. Conclusions

In accelerated life testing (ALT), the failure times obtained under high stress con- ditions are used to obtain the lifetime distribution of products under the normal conditions. The incorrect specification of the lifetime distributions of products may result in expensive costs and bad decisions.

The tools used to assess the validity of certain distribution in representing the data are called goodness of fit (GOF) techniques. These techniques are based on measuring the discrepancy of the sample data from the null or the hypothesized distribution.

When dealing with accelerated failure time (AFT) models, it is assumed that a certain model holds. Then, it is desired to test the validity of the underlying distribution. Under constant stress ALT experiment, there are different stress levels. Thus, the failure times at all stress levels should be combined to reach a conclusion about the adequacy of a certain lifetime distribution at each stress level. Thus, there is a need to modify some GOF techniques to be used in this case.

The Neyman's smooth test (NST) is a GOF test that dose not depend on the form of the null density. This test was extended to be used in the case of composite hypotheses. It is called the extended Neyman's smooth test (ENST).

In this paper, the process of adapting the ENST is clarified. The resulting test; which we called the adapted extended Neyman's smooth test (AENST) is then ap- plied to examine the fit of a certain family of distributions under AFT models. We assumed that the AFT model holds and showed how to use the AENST to assess the underlying lifetime distribution. The case of constant stress under complete sam- pling is considered and the method is applied to test for both Weibull and exponential distributions.

A simulation study is conducted to examine the estimated power of the AENST in validating both the Weibull (the test is called Q_{1-W} in this case) and exponential (it is called Q_{1-E}) distributions when using a certain AFT model. Another GOF test; which is the conditional probability integral transformation test (CPITT) is also applied on the same experiments to the purpose for comparison. The results showed that the estimated powers for both Q_{1-W} and Q_{1-E} statistics increase as the sample sizes increase and for large sample sizes, the power of these statistics become very high and sometimes equal 1. It is also seen that the power of the AENST is much better than that of the CPITT in the majority of cases. Thus, it is recommended to use the AENST as a GOF test when dealing with AFT models. To summarize, we can say that adapting the ENST will help the development in this field.

Finally, the AENST is applied on a real data set that includes the times to break- down of an insulating fluid to investigate the adequacy of the exponential inverse power law model suggested by Nelson (1990). The results give an evidence that the exponential inverse power law model is suitable to describe the data.

The future work of this study is to modify the AENST in order to be valid to test the underlying distributions in AFT model under step stress ALT. Under step stress, the stress on each unit is increased at pre-specified periods of time, or upon the occurrence of a fixed number of failures.

Table 1. Total sample sizes (n) and sub-samples (n_j)

n	n_1	n_2	n_3	n_4
33	3	5	10	15
63	13	15	17	18
103	18	20	30	35
203	35	45	55	68

Table 2. Computed values of $I_{\theta\theta|\beta}$ for Weibull and exponential distributions at different n

Distribution	33	63	103	203
Weibul	0.085	0.163017	0.266520	0.525278
Exponentiall	0.687499	1.312500	2.145833	4.229166

Table 3: Estimated values of the power for testing Weibull distribution

| | | | | U^2_{Mod-W} | | | Q_{1-W} | | |
| | | | | α | | | α | | |
C	P	Alternative	n	0.1	0.05	0.01	0.1	0.05	0.01
0.5	0.1	Exponential	33	-	-	-	0.095	0.049	0.012
			63	0.097	0.059	0.017	0.108	0.060	0.013
			103	0.095	0.048	0.020	0.091	0.045	0.010
			203	0.095	0.053	0.011	0.112	0.052	0.012
		Weibull	33	-	-	-	0.105	0.050	0.014
			63	0.102	0.052	0.008	0.093	0.046	0.008
			103	0.090	0.048	0.014	0.109	0.056	0.016
			203	0.100	0.058	0.014	0.117	0.061	0.009
		Lognormal	33	-	-	-	0.498	0.322	0.038
			63	0.186	0.101	0.024	0.708	0.549	0.213
			103	0.295	0.173	0.051	0.873	0.787	0.474
			203	0.601	0.448	0.233	0.988	0.969	0.886
0.5	0.5	Exponential	33	-	-	-	0.095	0.048	0.016
			63	0.092	0.043	0.010	0.107	0.060	0.016
			103	0.088	0.049	0.020	0.099	0.046	0.014
			203	0.077	0.039	0.009	0.119	0.056	0.010
		Weibull	33	-	-	-	0.079	0.033	0.006
			63	0.101	0.052	0.009	0.109	0.061	0.009
			103	0.090	0.046	0.013	0.104	0.052	0.012
			203	0.082	0.038	0.007	0.115	0.069	0.013
		Lognormal	33	-	-	-	0.444	0.240	0.018
			63	0.195	0.114	0.034	0.653	0.500	0.182
			103	0.303	0.197	0.078	0.891	0.796	0.500
			203	0.609	0.472	0.253	0.986	0.972	0.894
0.5	0.9	Exponential	33	-	-	-	0.107	0.054	0.020
			63	0.087	0.051	0.016	0.087	0.053	0.010
			103	0.072	0.033	0.004	0.105	0.052	0.008
			203	0.072	0.036	0.008	0.122	0.054	0.008
		Weibull	33	-	-	-	0.119	0.064	0.011
			63	0.093	0.041	0.013	0.106	0.054	0.011
			103	0.078	0.039	0.010	0.108	0.050	0.017
			203	0.085	0.039	0.013	0.116	0.071	0.016
		Lognormal	33	-	-	-	0.500	0.310	0.0460
			63	0.172	0.100	0.021	0.712	0.563	0.247
			103	0.281	0.170	0.057	0.901	0.829	0.550
			203	0.617	0.491	0.263	0.980	0.967	0.883

Table 3: Estimated values of the power for testing Weibull distribution (Cont.)

C	P	Alternative	n	U^2_{Mod-W} α 0.1	0.05	0.01	Q_{1-W} α 0.1	0.05	0.01
1.5	0.1	Exponential	33	-	-	-	0.096	0.050	0.013
			63	0.094	0.042	0.014	0.095	0.044	0.008
			103	0.084	0.041	0.008	0.124	0.065	0.014
			203	0.09	0.046	0.015	0.102	0.056	0.016
		Weibull	33	-	-	-	0.094	0.045	0.011
			63	0.095	0.046	0.003	0.091	0.050	0.010
			103	0.083	0.038	0.006	0.108	0.054	0.010
			203	0.069	0.035	0.010	0.120	0.062	0.010
		Lognormal	33	-	-	-	0.464	0.275	0.043
			63	0.155	0.086	0.026	0.749	0.597	0.298
			103	0.275	0.181	0.069	0.878	0.784	0.479
			203	0.606	0.462	0.249	0.987	0.978	0.919
1.5	0.5	Exponential	33	-	-	-	0.097	0.05	0.014
			63	0.089	0.038	0.010	0.101	0.051	0.014
			103	0.084	0.047	0.012	0.078	0.049	0.009
			203	0.089	0.050	0.009	0.112	0.063	0.014
		Weibull	33	-	-	-	0.076	0.031	0.007
			63	0.089	0.038	0.010	0.112	0.065	0.014
			103	0.074	0.040	0.011	0.101	0.059	0.013
			203	0.079	0.033	0.009	0.124	0.068	0.013
		Lognormal	33	-	-	-	0.485	0.293	0.047
			63	0.184	0.100	0.023	0.724	0.565	0.052
			103	0.307	0.184	0.071	0.880	0.785	0.469
			203	0.615	0.473	0.246	0.984	0.964	0.887
1.5	0.9	Exponential	33	-	-	-	0.090	0.042	0.016
			63	0.087	0.039	0.012	0.110	0.054	0.012
			103	0.083	0.038	0.009	0.115	0.060	0.018
			203	0.084	0.041	0.014	0.091	0.053	0.009
		Weibull	33	-	-	-	0.107	0.048	0.014
			63	0.090	0.052	0.010	0.122	0.072	0.013
			103	0.106	0.049	0.013	0.113	0.064	0.013
			203	0.073	0.035	0.007	0.113	0.072	0.012
		Lognormal	33	-	-	-	0.485	0.288	0.040
			63	0.188	0.101	0.024	0.648	0.484	0.194
			103	0.298	0.189	0.068	0.870	0.758	0.484
			203	0.616	0.490	0.282	0.979	0.959	0.890

Table 3: Estimated values of the power for testing Weibull distribution (Cont.)

C	P	Alternative	n	U^2_{Mod-W} α			Q_{1-W} α		
				0.1	0.05	0.01	0.1	0.05	0.01
3.5	0.1	Exponential	33	-	-	-	0.087	0.037	0.008
			63	0.084	0.045	0.005	0.117	0.062	0.014
			103	0.082	0.036	0.006	0.107	0.057	0.015
			203	0.075	0.038	0.011	0.114	0.072	0.016
		Weibull	33	-	-	-	0.099	0.048	0.012
			63	0.085	0.041	0.010	0.087	0.044	0.006
			103	0.084	0.041	0.006	0.116	0.062	0.010
			203	0.085	0.032	0.006	0.126	0.062	0.016
		Lognormal	33	-	-	-	0.479	0.301	0.041
			63	0.163	0.079	0.027	0.661	0.516	0.222
			103	0.283	0.173	0.067	0.871	0.753	0.456
			203	0.621	0.480	0.249	0.979	0.970	0.889
3.5	0.5	Exponential	33	-	-	-	0.094	0.048	0.007
			63	0.092	0.045	0.010	0.099	0.045	0.018
			103	0.087	0.051	0.015	0.106	0.054	0.010
			203	0.117	0.064	0.017	0.096	0.049	0.012
		Weibull	33	-	-	-	0.094	0.048	0.010
			63	0.099	0.048	0.012	0.111	0.070	0.013
			103	0.084	0.044	0.017	0.100	0.059	0.016
			203	0.080	0.038	0.008	0.123	0.062	0.015
		Lognormal	33	-	-	-	0.481	0.279	0.051
			63	0.169	0.086	0.021	0.701	0.547	0.259
			103	0.310	0.181	0.059	0.883	0.790	0.495
			203	0.627	0.480	0.231	0.980	0.972	0.878
3.5	0.9	Exponential	33	-	-	-	0.090	0.055	0.010
			63	0.108	0.047	0.010	0.101	0.056	0.015
			103	0.070	0.028	0.007	0.124	0.062	0.020
			203	0.093	0.052	0.015	0.106	0.060	0.012
		Weibull	33	-	-	-	0.109	0.059	0.014
			63	0.084	0.053	0.019	0.117	0.059	0.007
			103	0.087	0.040	0.005	0.120	0.059	0.012
			203	0.088	0.033	0.007	0.126	0.066	0.016
		Lognormal	33	-	-	-	0.464	0.275	0.043
			63	0.187	0.093	0.029	0.610	0.443	0.140
			103	0.289	0.175	0.052	0.855	0.742	0.443
			203	0.641	0.469	0.246	0.981	0.969	0.889

Table 4: Estimated values of the power for testing the exponential distribution

C	P	Alternative	n	U^2_{Mod-E}			Q_{1-E}		
				α			α		
				0.1	0.05	0.01	0.1	0.05	0.01
0.5	0.1	Exponential	33	0.111	0.066	0.010	0.084	0.041	0.003
			63	0.112	0.059	0.014	0.085	0.041	0.010
			103	0.088	0.040	0.007	0.097	0.045	0.009
			203	0.094	0.050	0.012	0.109	0.055	0.012
		Weibull	33	0852	0.779	0.631	0.994	0.979	0.960
			63	0.993	0.983	0.946	1.000	1.000	1.000
			103	1.000	0.999	0.996	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.898	0.857	0.759	0.989	0.985	0.978
			63	0.993	0.992	0.980	1.000	1.000	0.999
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
0.5	0.5	Exponential	33	0.107	0.056	0.015	0.082	0.040	0.003
			63	0.117	0.063	0.014	0.087	0.045	0.008
			103	0.107	0.061	0.013	0.105	0.048	0.012
			203	0.100	0.049	0.015	0.097	0.053	0.012
		Weibull	33	0.844	0.781	0.624	0.993	0.989	0.967
			63	0.980	0.970	0.920	1.000	0.999	0.999
			103	0.998	0.998	0.992	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.909	0.875	0.796	0.995	0.991	0.972
			63	0.992	0.987	0.969	1.000	1.000	1.000
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
0.5	0.9	Exponential	33	0.111	0.066	0.010	0.081	0.039	0.005
			63	0.097	0.041	0.012	0.096	0.044	0.008
			103	0.093	0.035	0.008	0.105	0.045	0.007
			203	0.102	0.053	0.019	0.096	0.042	0.006
		Weibull	33	0.831	0.756	0.613	0.995	0.988	0.966
			63	0.984	0.965	0.915	1.000	0.999	0.998
			103	1.000	0.998	0.991	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.892	0.846	0.743	0.996	0.990	0.974
			63	0.995	0.992	0.979	1.000	1.000	0.996
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000

Table 4: Estimated values of the power for testing the exponential distribution (Cont.)

| | | | | U^2_{Mod-E} | | | Q_{1-E} | | |
| | | | | α | | | α | | |
C	P	Alternative	n	0.1	0.05	0.01	0.1	0.05	0.01
1.5	0.1	Exponential	33	0.091	0.048	0.010	0.107	0.054	0.010
			63	0.075	0.040	0.011	0.094	0.038	0.006
			103	0.096	0.051	0.012	0.088	0.046	0.009
			203	0.102	0.051	0.015	0.108	0.057	0.012
		Weibull	33	0.841	0.772	0.636	0.996	0.986	0.959
			63	0.981	0.969	0.920	1.000	1.000	0.999
			103	1.000	1.000	0.995	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.912	0.865	0.758	0.991	0.990	0.982
			63	0.999	0.995	0.980	1.000	1.000	1.000
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
1.5	0.5	Exponential	33	0.110	0.048	0.015	0.107	0.057	0.009
			63	0.098	0.052	0.012	0.089	0.039	0.006
			103	0.104	0.056	0.015	0.086	0.039	0.004
			203	0.103	0.046	0.015	0.109	0.057	0.012
		Weibull	33	0.863	0.781	0.662	0.992	0.990	0.975
			63	0.988	0.980	0.933	1.000	1.000	1.000
			103	1.000	1.000	0.992	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.912	0.860	0.765	0.993	0.985	0.966
			63	0.996	0.993	0.982	1.000	1.000	1.000
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
1.5	0.9	Exponential	33	0.113	0.058	0.017	0.083	0.034	0.005
			63	0.100	0.052	0.009	0.101	0.051	0.007
			103	0.109	0.052	0.012	0.093	0.035	0.003
			203	0.108	0.055	0.017	0.098	0.052	0.011
		Weibull	33	0.837	0.777	0.632	0.992	0.982	0.967
			63	0.984	0.966	0.917	1.000	0.999	0.999
			103	1.000	0.999	0.998	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.917	0.863	0.761	0.990	0.984	0.977
			63	0.996	0.991	0.971	1.000	1.000	1.000
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000

Table 4: Estimated values of the power for testing the exponential distribution (Cont.)

C	P	Alternative	n	U^2_{Mod-E}			Q_{1-E}		
				α			α		
				0.1	0.05	0.01	0.1	0.05	0.01
3.5	0.1	Exponential	33	0.100	0.054	0.012	0.080	0.039	0.009
			63	0.093	0.045	0.013	0.094	0.043	0.005
			103	0.096	0.046	0.011	0.094	0.047	0.004
			203	0.095	0.045	0.017	0.095	0.040	0.006
		Weibull	33	0.865	0.789	0.631	0.994	0.989	0.968
			63	0.981	0.968	0.926	1.000	1.000	1.000
			103	0.998	0.995	0.989	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.901	0.865	0.742	0.992	0.991	0.979
			63	0.996	0.996	0.983	1.000	1.000	1.000
			103	1.000	1.000	0.999	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
3.5	0.5	Exponential	33	0.106	0.053	0.012	0.080	0.039	0.009
			63	0.096	0.054	0.011	0.081	0.040	0.008
			103	0.102	0.047	0.009	0.119	0.067	0.018
			203	0.103	0.046	0.017	0.098	0.039	0.006
		Weibull	33	0.822	0.768	0.629	0.995	0.990	0.971
			63	0.977	0.967	0.913	1.000	1.000	1.000
			103	1.000	1.000	0.994	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.894	0.846	0.755	0.993	0.991	0.978
			63	0.992	0.987	0.977	1.000	1.000	0.997
			103	1.000	1.000	0.999	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
3.5	0.9	Exponential	33	0.100	0.061	0.020	0.102	0.050	0.008
			63	0.100	0.058	0.016	0.093	0.038	0.010
			103	0.103	0.052	0.015	0.097	0.044	0.007
			203	0.097	0.054	0.008	0.102	0.054	0.014
		Weibull	33	0.856	0.785	0.638	0.995	0.992	0.964
			63	0.987	0.978	0.936	1.000	1.000	0.999
			103	1.000	1.000	0.992	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000
		Lognormal	33	0.907	0.868	0.760	0.992	0.992	0.978
			63	0.996	0.992	0.982	1.000	1.000	1.000
			103	1.000	1.000	1.000	1.000	1.000	1.000
			203	1.000	1.000	1.000	1.000	1.000	1.000

Table 5. The results of applying Q_{1-E} statistic on the real data set

$\hat{C}$	$\hat{P}$	$I_{\theta\theta\mid\beta}$	Q_{1-E}	$p-value$	α
26.345	0.575	3.7499955	1.443	0.23	0.05

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Abdalla Abdel-Ghaly
Faculty of Economics and Political Science
Department of Statistics
Cairo University
abdelghaly@yahoo.com

Hanan Aly
Faculty of Economics and Political Science
Department of Statistics
Cairo University
han_m_hasan@yahoo.com, hananhasan@feps.edu.eg

Elham Abdel-Rahman
Faculty of Economics and Political Science
Department of Statistics
Cairo University
ea2006_feps@yahoo.com, ea2006_feps@feps.edu.eg