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An Extended Fr echet Distribution: Properties and Applications

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Abstract: In this paper, we introduce an extended four-parameter Fr´echet model called the exponentiated exponential Fr´echet distribution, which arises from the quantile function of the standard exponential distribution. Various of its mathematical properties are derived including the quantile function, ordinary and incomplete moments, Bonferroni and Lorenz curves, mean deviations, mean residual life, mean waiting time, generating function, Shannon entropy and order statistics. The model parameters are estimated by the method of maximum likelihood and the observed information matrix is determined. The usefulness of the new distribution is illustrated by means of three real lifetime data sets. In fact, the new model provides a better fit to these data than the Marshall-Olkin Fr´echet, exponentiated-Fr´echet and Fr´echet models.

Key words: Exponentiated-exponential, Fr´echet distribution, hazard function, reliability function, Shannon entropy.

1. Introduction

The Fr´echet distribution is a well-defined limiting distribution for the maximum of random variables with non-negative real support. It is a popular and widely used model for characterizing variables having extreme phenomenons like floods, rains, cash flow (finance), etc. A random variable Z has the two-parameter Fr´echet (Fr) distribution with scale parameter $\delta > 0$ and shape parameter $\theta > 0$, if its cumulative distribution function (cdf) is given by

$$G_R(x) = \mathrm{e}^{-(\delta/x)^{\theta}}, \quad x > 0. \tag{1}$$

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The probability density function (pdf) corresponding to (1) is

$$g_R(x) = \theta\, \delta^\theta\, x^{-(\theta+1)}\; \mathrm{e}^{-(\delta/x)^\theta}. \tag{2}$$

Henceforth, we denote by Z ~Fr(δ, θ) the random variable having density (2) with parameters δ and θ.

The rth ordinary moment of Z (for $r < \theta$) is given by

$$\mu'_r = \delta^r\, \Gamma\Big(1 - \tfrac{r}{\theta}\Big), \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. The mean and variance of Z are $E(Z) = \delta\, \Gamma(1 - \frac{1}{\theta})$ and $Var(Z) = \delta^2\,.[\Gamma(1 - \frac{2}{\theta}) - \Gamma^2(1 - \frac{1}{\theta})]$, respectively.

The rth incomplete moment of Z can be expressed as

$$\mu'_{(r,Z)}(z) = \int_0^z x^r\, g_R(x)dx = \delta^r\, \Gamma\Big(1 - r/\theta, (\delta/z)^\theta\Big), \quad r < \theta, \tag{4}$$

where $\Gamma(p, x) = \int_x^\infty w^{p-1}\, \mathrm{e}^{-w} dw$ (for $p > 0$) is the upper incomplete gamma function.

Let T , R and Y be random variables with cdfs $F_T(x) = P(T \le x)$, $G_R(x) = P(R \le x)$ and $D_Y(x) = P\,(Y \le x)$. The corresponding quantile functions (qfs) are $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, where the qf is defined by $Q_Z(p) = \inf\{z : F_Z(z) \ge p\}$, $0 < p < 1$. If the densities exist, we denote them by $f_T(x)$, $g_R(x)$ and $d_Y(x)$. We assume that the random variables $T \in (a, b)$ and $Y \in (c, d)$, for $-\infty \le a < b \le \infty$ and $-\infty \le c < d \le \infty$. Alzaatreh *et al.* (2014) defined the cdf of the T-R{Y} family of distributions by

$$F_X(x) = \int_a^{Q_Y(G_R(x))} f_T(t)\, dt = F_T\Big(Q_Y\big(G_R(x)\big)\Big). \tag{5}$$

The pdf and hazard rate function (hrf) corresponding to (5) are given by (Alzaatreh *et al.*, 2014)

$$f_X(x) = G_R(x) \times \frac{F_T\Big(Q_Y\big(G_R(x)\big)\Big)}{d_Y\Big(Q_Y\big(G_R(x)\big)\Big)} \tag{6}$$

and

$$h_X(x) = h_R(x) \times \frac{h_T\Big(Q_Y\big(G_R(x)\big)\Big)}{h_Y\Big(Q_Y\big(G_R(x)\big)\Big)}, \tag{7}$$

respectively.

Let T have the exponentiated exponential (EE) distribution with pdf

$$f_T(t) = \alpha\,\beta\,\mathrm{e}^{-\alpha t}\,\left(1-\mathrm{e}^{-\alpha t}\right)^{\beta-1}.$$

Then, by using the qf of the standard exponential distribution, $Q_Y(\mathrm{p}) = -\log(1-p)$, the cdf of X follows from (5) is

$$F_X(x) = \{1-[1-G_R(x;\boldsymbol{\xi})]^{\alpha}\}^{\beta}. \tag{8}$$

The pdf of X is given by

$$f_X(x) = \alpha\,\beta\,g_R(x;\boldsymbol{\xi})\,\{1-G_R(x;\boldsymbol{\xi})\}^{\alpha-1}\,\{1-[1-G_R(x;\boldsymbol{\xi})]^{\alpha}\}^{\beta-1}. \tag{9}$$

For simplicity, the family (9) will be called the exponentiated exponential-R ("EE-R") family. Cordeiro et al. (2013) and Ghosh and Alzaatreh (2015) studied some general properties of (9). Some of its special models have been proposed in the literature such as the EE-logistic by Ghosh and Alzaatreh (2015).

The paper is organized as follows. In Section 2, we define an extended Fr´echet model, named the exponentiated-exponential Fr´echet ("EEFr") distribution, and discuss the shapes of its pdf and hrf. In Section 3, some of its mathematical properties are obtained. The density of its order statistics is derived in Section 4. In Section 5, the model parameters are estimated by the method of maximum likelihood and the observed information matrix is determined. In Section 6, we explore the usefulness of the proposed distribution by means of three real data sets. Finally, Section 7 offers some concluding remarks.

M. Mansoor, M. H. Tahir, Ayman Alzaatreh, Gauss M. Cordeiro,M. Zubair and Syed S.A. Ghazali

2. The Exponentiated-Exponential Fr´echet Distribution

If $\boldsymbol{R}$ follows the Fréchet distribution, then from equations (8) and (9), the cdf and pdf of the EEFr distribution are given by

$$F(x) = F(x;\alpha,\beta,\delta,\theta) = \left\{1-\left[1-\mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta} \tag{10}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\delta,\theta) &= \alpha\,\beta\,\theta\,\delta^{\theta}\,x^{-(\theta+1)}\,\mathrm{e}^{-(\delta/x)^{\theta}}\,\left[1-\mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha-1} \\ &\quad\times\left\{1-\left[1-\mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta-1}, \end{aligned} \tag{11}$$

respectively. Henceforth, a random variable having pdf (11) is denoted by $X \sim \mathrm{EEFr}(\alpha, \beta, \delta, \theta)$.

Some special sub-models of the EEFr distribution are now cited:

(i) If $\beta = 1$ in (11), the EEFr distribution is equal to the exponentiated-Fr´echet distribution.

(ii) If $\beta = 1$ and $\alpha = 1$ in (11), the EEFr distribution reduces to the Fr´echet distribution.

Note that the distribution in (11) is constructed using the qf of the standard exponential distribution in (5). By using different qf's of other distributions with support $(0, \infty)$, we can propose different versions of the EEFr distributions. Table 1 lists some EEFr distributions using the qfs of the Weibull, log-logistic, Rayleigh, Dagum and Lomax distributions.

Table 1: Special EEFr distributions.

S.No. Y	$Q_Y(p)$	$F_T(x)$ for the EEFr{Y} distribution
(a) Weibull	$\gamma\left\{-\log(1-p)\right\}^{1/c},\ \gamma, c > 0$	$\left[1-\exp\left\{-\lambda\ \ -log\ \ 1-e^{-(\delta/x)^{\theta}}\ \ ^{1/c}\right\}\right]^{\beta},\ \lambda = \alpha\gamma.$
(b) Log-logistic	$a\left\{\frac{p}{1-p}\right\}^{1/b},\ a, b > 0$	$\left[1-\exp\left\{-\lambda\left[\frac{e^{-(\delta/x)^{\theta}}}{1-e^{-(\delta/x)^{\theta}}}\right]^{1/b}\right\}\right]^{\beta},\ \lambda = a\,\alpha.$
(c) Rayleigh	$\left\{-2b^2\log(1-p)\right\}^{1/2},\ b > 0$	$\left[1-\exp\left\{-\lambda\ \ -\log\ \ 1-e^{-(\delta/x)^{\theta}}\ \ ^{1/2}\right\}\right]^{\beta},\ \lambda = \sqrt{2}\,\alpha\, b.$
(d) Dagum	$a\left\{\frac{p^{1/c}}{1-p^{1/c}}\right\}^{1/b},\ a, b, c > 0$	$\left[1-\exp\left\{-\lambda\left[\frac{e^{-\frac{1}{c}(\delta/x)^{\theta}}}{1-e^{-\frac{1}{c}(\delta/x)^{\theta}}}\right]^{1/b}\right\}\right]^{\beta},\ \lambda = a\,\alpha.$
(e) Lomax	$\frac{1}{a}\left\{\frac{1-(1-p)^{1/k}}{(1-p)^{1/k}}\right\}^{1/b},\ a, k > 0$	$\left[1-\exp\left\{-\lambda\left[\frac{1-\ \ 1-e^{-(\delta/x)^{\theta}}\ \ ^{1/k}}{1-e^{-(\delta/x)^{\theta}}\ \ ^{1/k}}\right]^{1/b}\right\}\right]^{\beta},\ \lambda = \alpha/a.$

The survival function (sf) ($S(x)$), hrf ($h(x)$) and cumulative hazard rate function (chrf) ($H(x)$) of X are given by

$$S(x;\alpha,\beta,\delta,\theta) = 1 - \left\{1 - \left[1 - e^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta}, \qquad (12)$$

$$h(x;\alpha,\beta,\delta,\theta) = \frac{\alpha\,\beta\,\theta\,\delta^{\theta}\,x^{-(\theta+1)}\ e^{-(\delta/x)^{\theta}}\ \left[1-e^{-(\delta/x)^{\theta}}\right]^{\alpha-1}\left\{1-\left[1-e^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta-1}}{1-\left\{1-\left[1-e^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta}}$$

and

$$H(x;\alpha,\beta,\delta,\theta) = -\ \log\left[1-\left\{1-\left[1-e^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta}\right],$$

respectively.

Figures 1 and 2 display some plots of the pdf and hrf of X for selected parameter values. Figure 1 indicates that the EEFr distribution is well-suited for right-skewed data. Further, Figure 2 shows that the EEFr hrf can produce shapes such as increasing, decreasing and reversed-J.

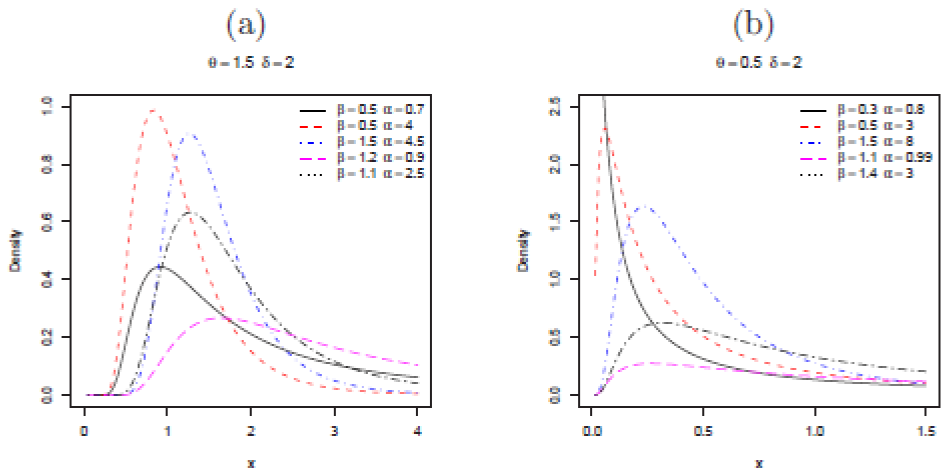

Figure 1: Plots of the EEFr densities.

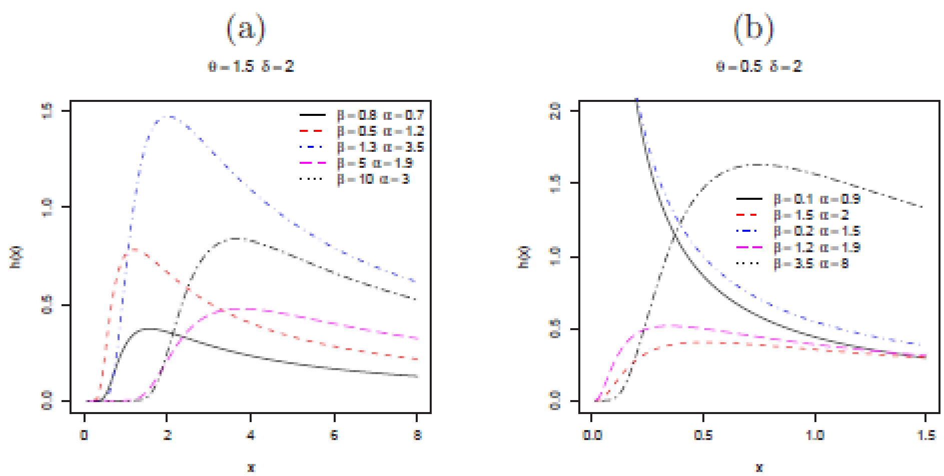

Figure 2: Plots of EEFr hazard rates.

2.1. Shapes of the Density and Hazard Rate Functions

The shapes of the density and hazard rate functions can be described analytically. The critical points of the EEFr density are the roots of the equation:

$$\left\{1+\left(1+\frac{1}{\theta}\right)(1-z)\log(1-z)\right\}-\left\{\frac{\alpha-1}{z}+\frac{\alpha(\beta-1)^{\alpha-1}}{1-z^{\alpha}}\right\}=0, \qquad (13)$$

where $z = [1 - e^{-(\delta/x)^{\theta}}]$. There may be more than one root to (13).

The critical point of $h(x) = h(x; \alpha, \beta, \delta, \theta)$ are obtained from the equation

$$\left[\left(\frac{1+\theta}{\theta}\right)\left(\frac{x}{\delta}\right)^{\theta} - 1\right] - (1 - z)\left[\frac{\alpha(\beta-1)\, z^{\alpha-1}}{1-z^{\alpha}} + \frac{\alpha\,\beta\, z^{\alpha}(1-z^{\alpha})^{\beta-1}}{1-(1-z^{\alpha})^{\beta}} - \frac{\alpha-1}{z}\right] = 0. \quad (14)$$

There may be more than one root to (14).

3. Mathematical Properties

Established algebraic expansions to determine some structural properties of the EEFr distribution can be more efficient than computing those directly by numerical integration of its density function. We provide some of its mathematical properties in the next sections.

3.1 Expansion of the EEFr Density

In order to obtain a simplified form for the EEFr pdf, we expand (11) in power series. Let

$$A = \left\{1 - \left[1 - \mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha}\right\}^{\beta-1}.$$

By using the generalized binomial expansion, the quantity A reduces to

$$A = \sum_{k=0}^{\infty} \frac{(-1)^k\,(\beta-1)_k}{k!}\,\left[1 - \mathrm{e}^{-(\delta/x)^{\theta}}\right]^{k\,\alpha},$$

where $\beta_k = \beta(\beta - 1) \cdot \cdot \cdot (\beta - k + 1)$ is the descending factorial.

Inserting this result in (11), we have

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,(\beta-1)_k}{k!}\,\alpha\,\beta\,\theta\,\delta^{\theta}\,x^{-(\theta+1)}\,\mathrm{e}^{-(\delta/x)^{\theta}}\,\underbrace{\left[1 - \mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha(k+1)-1}}_{B}.$$

Letting $B = \{1 - \mathrm{e}{-}(\delta/x)\}$ and using the generalized binomial expansion again,

$$B = \sum_{j=0}^{\infty} \frac{(-1)^j\,(\alpha(k+1)-1)_j}{j!}\,\mathrm{e}^{-j(\delta/x)^{\theta}}.$$

Let Z_{j+1} be a Fr random variable with scale parameter $(j + 1)^{1/\theta}\,\delta > 0$ and shape parameter $\theta > 0$ and pdf $h_{j+1}(x) = h_{j+1}(x; \delta, \theta)$. Combining the last two results, we obtain

$$f(x) = \underbrace{\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+j}\,(\beta-1)_k\,(\alpha(k+1)-1)_j}{k!\,(j+1)!}\,\alpha\,\beta}_{v_j} \times \underbrace{(j+1)\,\theta\,\delta^{\theta}\,x^{-(\theta+1)}\;\mathrm{e}^{-(j+1)(\delta/x)^{\theta}}}_{h_{j+1}(x;\delta,\theta)}.$$

In a more simplified form, the last equation becomes

$$f(x) = f(x;\alpha,\beta,\delta,\theta) = \sum_{j=0}^{\infty} v_j\; h_{j+1}(x), \qquad (15)$$

where

$$v_j = \alpha\,\beta \sum_{k=0}^{\infty} \frac{(-1)^{k+j}\,(\beta-1)_k\,(\alpha(k+1)-1)_j}{k!\,(j+1)!}.$$

Equation (15) reveals that the density function of *X* is a mixture of Fr´echet densities. So, several mathematical properties of the EEFr distribution can be derived from those of the Fr distribution. This equation is the main result of this section.

3.1 Quantile Function and Simulation

The qf of a distribution has many uses in both the theory and applications of statistics. The qf of *X* is obtained by inverting (10). We have

$$x = Q(u) = \delta\left\{\log\left[1-\left(1-u^{1/\beta}\right)^{1/\alpha}\right]^{-1/\theta}\right\}^{-1}. \qquad (16)$$

If *U* has a uniform distribution in (0, 1), then X = *Q*(*U*) has the EEFr(*α, β, δ, θ*) distribution.

The analysis of the variability of the skewness and kurtosis on the shape parameters α and *β* can be investigated based on quantile measures. The Bowley skewness (Kenney and Keeping, 1962) based on quartiles is given by

$$B = \frac{Q(3/4)+Q(1/4)-2\,Q(2/4)}{Q(3/4)-Q(1/4)}.$$

The Moors kurtosis (Moors, 1998) based on octiles is given by

$$M = \frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}.$$

These measures are less sensitive to outliers and they exist even for distributions without moments. In Figure 3, we plot the measures *B* and *M* of *X*. They indicate the variability of these measures on the shape parameters. Further, it is clear from Figure 3 that the EEFr is a right-skewed distribution.

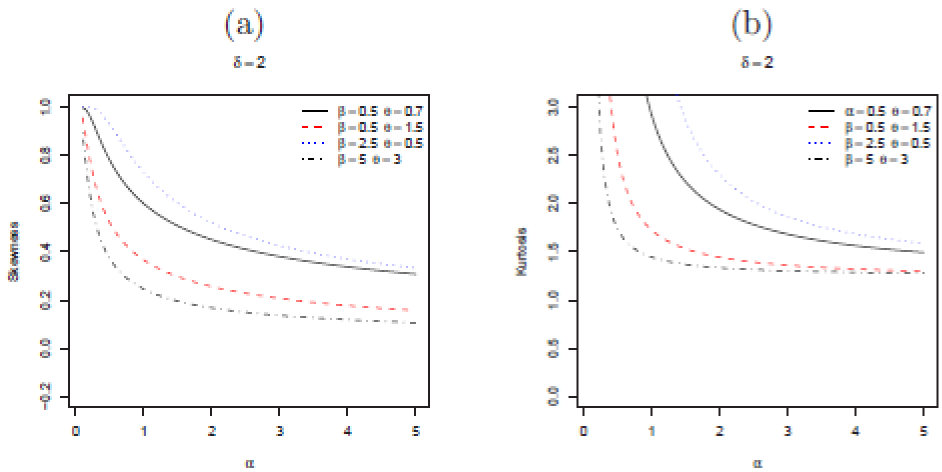

Figure 3: Skewness (a) and Kurtosis (b) plots of the EEFr distribution.

3.2 Moments

The *r*th moment of *X* can be expressed from (15) as

$$\mu'_r = E(X^r) = \sum_{j=0}^{\infty} v_j \int_0^{\infty} x^r \, h_{j+1}(x)\, dx.$$

Using (3), we obtain

$$\mu'_r = E(X^r) = \delta^r \, \Gamma(1 - r\theta^{-1}) \sum_{j=0}^{\infty} v_j \; (j+1)^{r/\theta}. \qquad (17)$$

Setting $r = 1$ in (17), it follows the mean $\mu'_1 = E(X)$.

The central moments (μ_n) and cumulants (κ_n) of *X* are obtained from equation (17) as

$$\mu_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu_1'^k \, \mu'_{n-k} \qquad \text{and} \qquad \kappa_n = \mu'_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k \, \mu'_{n-k},$$

respectively, where $\kappa_1 = \mu'_1$. The skewness and kurtosis can be calculated from the third and fourth standardized cumulants as $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and $\gamma_2 = \kappa_4/\kappa_2^2$.

They are also important to derive Edgeworth expansions for the cdf and pdf of the standardized sum and mean of independent and identically distributed random variables having the EEFr distribution.

3.3 Incomplete Moments

Using (4), the rth incomplete moment of X follows from (15) as

$$m_{r,X}(z) = \delta^r \sum_{j=0}^{\infty} v_j \, (j+1)^{r/\theta} \, \Gamma\Big(1 - r\theta^{-1}, (j+1)(\delta/z)^{\theta}\Big). \qquad (18)$$

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in several fields. For a given probability π, they are defined by $B(\pi) = m_1(q)/(\pi\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $m_1(q)$ comes from (18) with $r = 1$ and $q = Q(\pi)$ is evaluated from (16).

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median defined by $\delta_1 = \int_0^{\infty} |x - \mu_1'| f(x)dx$ and $\delta_2(x) = \int_0^{\infty} |x - M| f(x)dx$, respectively, where $\mu_1' = E(X)$ is the mean and $M = Q(0.5)$ is the median. These measures can be expressed as $\delta_1 = 2\mu_1' F(\mu_1') - 2m_1(\mu_1')$ and $\delta_2 = \mu_1' - 2m_1(M)$, where $F(\mu_1')$ comes from (10).

Further applications of the first incomplete moment are related to the mean residual life and mean waiting time given by $m(t;\alpha,\beta,\delta,\theta) = [1 - m_1(t)]/S(t) - t$ and $\mu(t;\alpha,\beta,\delta,\theta) = t - [m_1(t)/F(t)]$, respectively, where $F(t;\alpha,\beta,\delta,\theta)$ and $S(t;\alpha,\beta,\delta,\theta) = 1 - F(t;\alpha,\beta,\delta,\theta)$ are obtained from (10).

3.3 Generating Function

The moment generating function (mgf) of a random variable X provides the basis of an alternative route to analytical results compared with working directly with the pdf and cdf of X. The mgf of the Fréchet distribution (for $t < \theta$) is given by

$$M_R(t) = \theta\,\delta^{\theta} \int_0^{\infty} x^{-\theta-1} \, e^{-\delta^{\theta} x^{-\theta}} \, e^{tx} \, dx = \theta\,\delta^{\theta} \int_0^{\infty} x^{-\theta-1} \, e^{-\delta^{\theta} x^{-\theta} + tx} \, dx. \quad (19)$$

The calculations of the integral in (19) involve the generalized hypergeometric function defined in equation (2.3.1.14) (Prudnikov et al., 1986, p. 322).

For $a > 0$ and $s > 0$, if $b = p/q$ ($p \geq 1$ and $q \geq 1$ co-primes integers), we can write

$$\begin{aligned} J(a,\gamma,b,s) &= \int_0^\infty x^{a-1}\, \mathrm{e}^{-\gamma\, x^{-b}-sx}\, dx \\ &= \sum_{j=0}^{q-1} \frac{(-\gamma)^j}{j!}\, \Gamma\Big(1+\frac{p(1+j)}{q}\Big)\ s^{-\left(1+\frac{p(1+j)}{q}\right)} \\ &\qquad \times\ {}_1F_{p+q}\left(1; \triangle(q,\, 1+\tfrac{p(1+j)}{q}), \triangle(q,\, 1+j); z\right) \\ &+ \sum_{j=0}^{p-1} \frac{q\,(-s)^j}{p\, j!}\, \Gamma\Big(1+\frac{q(1+j)}{p}\Big)\ \gamma^{-\left(1+\frac{q(1+j)}{p}\right)} \\ &\qquad \times\ {}_1F_{p+q}\left(1, \triangle(q,\, 1+\tfrac{q(1+j)}{p}), \triangle(p,\, 1+j); z\right), \qquad (20) \end{aligned}$$

where $z = (-1)^{p+q}\, s^p\, a^q/(p^p\, q^q)$, $\triangle(k, a)$ represents the sequence

$$\triangle(k,a) = (a/k, (a+1)/k, \ldots, (a+k-1)/k),$$

${}_mF_n$ denotes the generalized hypergeometric function defined by

$${}_mF_n\left(\alpha_1, \alpha_2 \ldots, \alpha_m; \beta_1, \beta_2 \ldots, \beta_n;\ z\right) = \sum_{k=0}^\infty \frac{(\alpha_1)_k\,, (\alpha_2)_k\ \ldots, (\alpha_m)_k}{(\beta_1)_k\,, (\beta_2)_k\ \ldots, (\beta_n)_k}\, \frac{z^k}{k!}$$

and $(c)_k = c(c+1)\ldots(c+k-1)$ denotes the ascending factorial.

Numerical routines for computing the generalized hypergeometric function are available in most mathematical packages, e.g., MAPLE, MATLAB, MATHE-MATICA and Ox. Nadarajah and Kotz (2005) and Nadarajah (2007) also used this result to obtain the properties of the distribution of the difference between two independent Gumbel variates, and to Iacobellis and Fiorentino (2000), and Fiorentino et al. (2006)'s model for peak streamflow.

By using (20), the mgf of the Fr distribution follows as

$$M_R(t) \quad = \quad \theta\, \delta^\theta\ J(-\theta, \delta^\theta, \theta, -t). \qquad (21)$$

Based on the representation (15), the mgf of X can be expressed as

$$M(t) = \sum_{j=0}^\infty\ v_j\ \int_0^\infty\ \mathrm{e}^{tx}\ h_{j+1}(x)\, dx$$

and then using (21), $M(t)$ reduces to

$$M(t) = \theta\, \delta^\theta \sum_{j=0}^\infty (j+1)\, v_j\, J(-\theta, (j+1)\delta^\theta, \theta, -t). \qquad (22)$$

Equation (22) is the main result of this section.

3.4 Shannon Entropy

Entropy has been used in various situations in science as a measure of variation of the uncertainty. Numerous measures of entropy have been studied and compared in the literature. The Shannon's entropy (Shannon, 1948) is used as a measure of uncertainty and plays an important role in many fields such as engineering and information theory. Shannon's entropy of a random variable X with pdf f (x) is defined as $\eta X = -E[\log(f(X))]$. According to Ghosh and Alzaatreh (2015), the Shannon's entropy for the EE-*X* family in (8) is given by

$$-E\{\log[g_R(G_R^{-1}(1-\mathrm{e}^{-\mathrm{T}}))]\} - \log(\alpha\beta) + [\psi(\beta)-\psi(1)](1-\alpha^{-1}) + 1 - \beta^{-1}, \quad (23)$$

where $T \sim \mathrm{EE}(\alpha, \beta)$ and $\psi(\cdot)$ is the digamma function.

For the EEFr distribution, we have

$$\begin{aligned} E\left\{\log[g_R(G_R^{-1}(1-\mathrm{e}^{-T}))]\right\} &= \log(\theta/\delta) + E(\log(1-\mathrm{e}^{-T})) \\ &\quad +(1+\theta^{-1})\, E\left\{\log[-\log(1-\mathrm{e}^{-T})]\right\}. \quad (24) \end{aligned}$$

First, we consider $E[\log(1-\mathrm{e}^{-T})]$. By expanding $\log(1-\mathrm{e}^{-T})$ in Taylor series, one can get

$$E[\log(1-\mathrm{e}^{-\mathrm{T}})] = -\beta \sum_{\mathrm{j}=1}^{\infty} \mathrm{j}^{-1}\, \mathrm{B}(\beta, 1+\mathrm{j}/\alpha), \quad (25)$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. Setting $u = -\log(1-\mathrm{e}^{-t})$ gives

$$\begin{aligned} E\left\{\log[-\log(1-\mathrm{e}^{-T})]\right\} &= \sum_{j=0}^{\infty}(j+1)\, v_j \int_0^{\infty} \mathrm{e}^{-(j+1)u} \log(u)\, du \\ &= -\sum_{j=0}^{\infty} v_j \left\{(\log(j+1) - \psi(1)\right\}. \quad (26) \end{aligned}$$

Using (23)-(26) and the fact that $v_0 = 0$, the Shannon's entropy of X reduces to

$$\eta_X = C + \sum_{j=1}^{\infty} \left\{\beta\, j^{-1} B(\beta, 1+j/\alpha) + (1+\theta^{-1})\, v_j \left[\log(j+1) - \psi(1)\right]\right\},$$

where $C = \log(\delta/\theta) - \log(\alpha\beta) + [\psi(\beta) - \psi(1)](1-\alpha^{-1}) + 1 - \beta^{-1}$.

4. Order Statis

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from the EEFr distribution. Let $X_{i:n}$ denote the ith order statistic. Then, the pdf of $X_{i:n}$ can be expressed as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x)\, F(x)^{i-1} \, \{1-F(x)\}^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1}. \end{aligned}$$

Inserting (10) and (11) in the last equation and after some algebra, we obtain

$$\begin{aligned} f_{i:n}(x) &= \sum_{j=0}^{n-i} \frac{(-1)^j\, \Gamma(n+1)\, (i+j)^{-1}}{\Gamma(i)\, \Gamma(j+1)\Gamma(n-i-j+1)} \Big\{ \alpha\, \beta(i+j)\, \theta\, \delta^{\theta}\, x^{-(\theta+1)}\, \mathrm{e}^{-(\delta/x)^{\theta}} \\ & \left[1-\mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha-1} \left[1-\left(1-\mathrm{e}^{-(\delta/x)^{\theta}}\right)^{\alpha}\right]^{\beta(i+j)-1} \Big\}. \end{aligned}$$

Hence,

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \eta_j \; f_{\alpha,(i+j)\beta,\delta,\theta}(x), \tag{27}$$

where

$$\eta_j = \frac{(-1)^j\, \Gamma(n+1)}{(i+j)\, \Gamma(i)\, \Gamma(j+1)\, \Gamma(n-i-j+1)}$$

and $f_{\alpha,(i+j)\beta,\delta,\theta}$ (x) is the EEFr density with parameters (α, $(i+j)\beta$, δ, θ). Equation (27) is the main result of this section. It reveals that the pdf of the EEFr order statistics is a linear combination of EEFr densities. So, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, mgf, mean deviations and several others can be derived from those quantities of the EEFr distribution.

5. Estimation and Information Matrix

Here, we consider the estimation of the unknown parameters of the new distribution by the maximum likelihood method. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and regions and deliver simple approximations that work well in finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically. Let $x_1, \ldots, x_n$ be a sample of size n from the EEFr distribution given by (11). The log-likelihood function for the vector of parameters $\Theta = (\alpha, \beta, \delta, \theta)^T$ can be expressed as

$$\begin{aligned}\ell &= n\log(\alpha\,\beta\,\theta) + n\,\theta\,\log(\delta) - (\theta+1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}(\delta/x_i)^{\theta} \\ &\quad +(\alpha-1)\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-(\delta/x_i)^{\theta}}\right] + (\beta-1)\sum_{i=1}^{n}\log\left\{1-\left[1-\mathrm{e}^{-(\delta/x_i)^{\theta}}\right]^{\alpha}\right\}.\end{aligned}$$

Let $z_i = [1 - \mathrm{e}^{-(\delta/xi)^{\theta}}]$. Then, we can write A as

$$\begin{aligned}\ell &= n\log(\alpha\,\beta\,\theta) + n\,\theta\,\log(\delta) - (\theta+1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}(\delta/x_i)^{\theta} \\ &\quad +(\alpha-1)\sum_{i=1}^{n}\log z_i + (\beta-1)\sum_{i=1}^{n}\log\,(1-z_i^{\alpha}).\end{aligned}$$

The components of the score vector U (Θ) are given by

$$\begin{aligned}U_{\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log\,z_i - (\beta-1)\sum_{i=1}^{n}\left[\frac{z_i^{\alpha}\log\,\alpha}{1-z_i^{\alpha}}\right], \\ U_{\beta} &= \frac{n}{\beta} + \sum_{i=1}^{n}\log\,(1-z_i^{\alpha}), \\ U_{\delta} &= \frac{n\theta}{\delta} - \sum_{i=1}^{n}(\theta/x_i)\,(\delta/x_i)^{\theta-1} + (\alpha-1)\sum_{i=1}^{n}\left(\frac{z'_{i\delta}}{z_i}\right) \\ &\quad -\alpha(\beta-1)\sum_{i=1}^{n}\left[\frac{z_i^{\alpha-1}\,z'_{i\delta}}{1-z_i^{\alpha}}\right], \\ U_{\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\log\,(\delta/x_i) - \sum_{i=1}^{n}(\delta/x_i)^{\theta}\,\log\,(\delta/x_i) + (\alpha-1)\sum_{i=1}^{n}\left(\frac{z'_{i\theta}}{z_i}\right) \\ &\quad -\alpha(\beta-1)\sum_{i=1}^{n}\left[\frac{z_i^{\alpha-1}\,z'_{i\theta}}{1-z_i^{\alpha}}\right].\end{aligned}$$

Setting these equations to zero and solving them simultaneously yields the maximum likelihood estimates (MLEs) of the model parameters. There is no closed-form expression for the MLE $\widehat{\Theta}$ and its computation has to be performed numerically using a nonlinear optimization algorithm. The Newton-Raphson iterative technique could be applied to solve the likelihood equations and obtain $\widehat{\Theta}$ numerically.

For interval estimation of the parameters, we require the 4 × 4 observed information matrix $J(\Theta) = \{-J_{rs}\}$ (for $r, s = \alpha, \beta, \delta, \theta$) given in Appendix A.
This matrix can be evaluated numerically from standard maximization routines as part of their output; e.g., one can use the R functions optim or nlm, the Ox function MaxBFGS, the SAS procedure NLMixed, among others, to compute $J(\Theta)$ numerically.

Under standard regularity conditions, the multivariate normal $N_4(0, J(\widehat{\Theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\gamma)\%$ confidence intervals for α, β, δ and θ are given by $\widehat{\alpha} \pm z_{\gamma^*/2} \times \sqrt{var(\widehat{\alpha})}$, $\widehat{\beta} \pm z_{\gamma^*/2} \times \sqrt{var(\widehat{\beta})}$, $\widehat{\delta} \pm z_{\gamma^*/2} \times \sqrt{var(\widehat{\delta})}$, and $\widehat{\theta} \pm z_{\gamma^*/2} \times \sqrt{var(\widehat{\theta})}$, respectively, where the $var(\cdot)$'s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\gamma^*/2}$ is the quantile $(1-\gamma*/2)$ of the standard normal distribution.

The likelihood ratio (LR) statistic can be used to check if the EEFr distribution is strictly "superior" to the Fr distribution for a given data set. Then, the test of $H_0 : \alpha = \beta = 1$ versus H_1 : H_0 is not true is equivalent to compare the EEFr and Fr distributions and the LR statistic becomes $w = 2\{\ell(\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\theta}) - \ell(1, 1, \tilde{\delta}, \tilde{\theta})\}$, where $\hat{\alpha}, \hat{\beta}, \hat{\delta}$, and $\hat{\theta}$ are the MLEs under H_1 and $\tilde{\delta}$, and $\tilde{\theta}$ are the estimates under H_0.

6. Applications

In this section, we provide three applications to real data to illustrate the importance of the EEFr distribution. The model parameters are estimated by the method of maximum likelihood and five goodness-of-fit statistics are evaluated to compare the EEFr distribution with other competing models.

Data set 1: Epoxy Strands Failure at 90% Stress Level. The first data set consists of 101 observations, which represent the stress-rupture life of 49 kevlar epoxy strands. They were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact failure times. These times in hours are given by Barlow et al. (1984) and Andrews and Herzberg (1985). The data have recently been used by Cooray and Ananda (2008), Pescim et al. (2010) and Cordeiro et al. (2013).The data are: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09,0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34 ,0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

Data set 2: Rainfall data. The second data set consists of annual maximum daily precipitation (unit:mm) at Busan, Korea for the 1904-2011 period. The data were obtained from the Korean Meteorological Administration (KMA(http://www.kma.go.kr) 2013). This data set has recently been used by Jeong et al. (2014). The data are: 24.8, 140.9, 54.1, 153.5, 47.9, 165.5, 68.5, 153.1, 254.7, 175.3, 87.6, 150.6, 147.9, 354.7, 128.5, 150.4, 119.2, 69.7, 185.1,153.4, 121.7, 99.3, 126.9, 150.1, 149.1, 143, 125.2, 97.2, 179.3,125.8, 101, 89.8, 54.6, 283.9, 94.3, 165.4, 48.3, 69.2, 147.1, 114.2, 159.4, 114.9, 58.5, 76.6, 20.7, 107.1, 244.5, 126, 122.2, 219.9, 153.2, 145.3, 101.9, 135.3, 103.1, 74.7, 174, 126, 144.9, 226.3, 96.2, 149.3, 122.3, 164.8, 188.6, 273.2, 61.2, 84.3, 130.5, 96.2, 155.8, 194.6, 92, 131, 137, 106.8, 131.6, 268.2, 124.5, 147.8, 294.6, 101.6, 103.1, 247.5, 140.2, 153.3, 91.8, 79.4, 149.2,168.6, 127.7, 332.8, 261.6, 122.9, 273.4, 178, 177, 108.5, 115,241, 76, 127.5, 190, 259.5, 301.5.

Data set 3: Survival Times of 72 Guinea Pigs. The third data set consists of 72 survival times of guinea pigs injected with different amount of tubercle bacilli and was studied by Bjerkedal (1960). Guinea pigs are known to have high susceptibility of human tuberculosis, which is one of the reasons for choosing this species. The data represent the survival times of Guinea pigs in days.The data are given below: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

We fit the EEFr model to the three data sets and compared it with other models, namely: the Marshall-Olkin extended Fr´echet (MOFr) (Krishna et al., 2013a; 2013b), exponentiated Fr´echet (EFr) (Nadarajah and Kotz, 2003) and Fr distributions. The densities of the MOFr and EFr distributions are:

$$\text{MOFr}: \; f_{MOFr}(x;\alpha,\delta,\theta) = \frac{\alpha\,\theta\;(\delta/x)^{\theta+1}\;\mathrm{e}^{-(\delta/x)^{\theta}}}{\delta\;\left[\alpha+(1-\alpha)\mathrm{e}^{-(\delta/x)^{\theta}}\right]^{2}}, \quad x>0, \quad \alpha>0;\delta,\theta>0,$$

$$\text{EFr}: \; f_{EFr}(x;\alpha,\lambda) = \alpha\,\theta\,\delta^{\theta}\,x^{\theta-1}\,\mathrm{e}^{-(\delta/x)^{\theta}}\;\left[1-\mathrm{e}^{-(\delta/x)^{\theta}}\right]^{\alpha-1}, \quad x>0,\; \alpha,\delta,\theta>0.$$

Some well-known measures of goodness-of-fit statistics including the log-likelihood function evaluated at the MLEs ($\hat{\ell}$), the Akaike information criterion (AIC), the Anderson-Darling (A^{*}), the Cram´er–von Mises (W^{*}) and the Kolmogrov-Smirnov (K-S) statistics. They are evaluated to compare the fitted models. The statistics W^{*} and A^{*} are described in details in Chen and Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out using the R-software.

Table 2: MLEs and their standard errors (in parentheses) for the data sets 1, 2 and 3.

Distribution	α	β	θ	δ
Data 1				
EEFr	0.1488	12.8814	3.6329	4.4896
	(0.0195)	(2.2459)	(0.0026)	(0.0026)
MOFr	16.8914	-	8.1434	1.8443
	(20.8477)	-	(1.0696)	(0.2072)
EFr	36.5724	-	1.4822	6.6724
	(57.5848)	-	(0.5706)	(4.1862)
Fr	-	-	5.2028	2.3610
	-	-	(0.4975)	(0.0622)
Data 2				
EEFr	29.5053	0.6415	0.7419	928.9561
	(7.9135)	(0.3451)	(0.1133)	(77.8344)
MOFr	69.4851	-	3.1681	33.4437
	(28.2970)	-	(0.2199)	(3.4478)
EFr	7.2536	-	0.9080	331.1528
	(2.4567)	-	(0.0973)	(82.5980)
Fr	-	-	1.6503	99.8150
	-	-	(0.1031)	(6.2799)
Data 3				
EEFr	0.8541	1.1302E+02	0.3964	1.0163E+04
	(0.4829)	(60.4973)	(0.0528)	(207.2668)
MOFr	193.9639	-	2.4994	17.8293
	(102.1779)	-	(0.1977)	(2.9607)
EFr	3.4220	-	0.8553	251.9001
	(1.0010)	-	0.0828	(60.5894)
Fr	-	-	1.1729	105.8497
	-	-	0.0843	11.3206

Table 2 lists the MLEs and their corresponding standard errors (in parenthe- ses) of the model parameters for data sets 1, 2 and 3. The numerical values of the statistics , $\hat{\ell}$ AIC, A^* , W^* and K-S, and p-values are listed in Table 3. We note from the figures of this table that the EEFr model has the lowest values of, $\hat{\ell}$ AIC, A^*, W^*, K-S statistics and the largest p-values among the fitted MOFr, EFr and Fr models, thus suggesting that the EEFr distribution yields the best fit, and therefore could be chosen as the best model. The histogram of the data sets 1, 2 and 3 and the estimated pdfs and cdfs of the EEFr distribution and its competitive models are displayed in Figure 4. It is clear from Table 3 and Figure 4 that the EEFr model provides the best fits to the histogram of the three data sets and could be chosen as the best model.

Table 3: The statistics , $\hat{\ell}$ AIC, A^*, W^* and K-S for the data sets 1, 2 and 3.

Distribution	$\hat{\ell}$	AIC	A^*	W^*	K-S	p-value
			Data set 1			
EEFr	-43.440	94.881	0.462	0.077	0.082	0.816
MOFr	-45.974	97.948	0.783	0.132	0.099	0.602
EFr	-44.553	95.106	0.484	0.079	0.092	0.697
Fr	-49.754	103.508	1.513	0.273	0.132	0.258
			Data set 2			
EEFr	−583.317	1174.634	0.971	0.171	0.086	0.415
MOFr	-585.546	1177.094	1.002	0.169	0.108	0.189
EFr	-587.637	1181.275	1.427	0.248	0.116	0.114
Fr	-608.596	1221.194	4.121	0.712	0.145	0.023
			Data set 3			
EEFr	-426.774	861.548	0.408	0.052	0.102	0.455
MOFr	-428.545	863.090	0.599	0.078	0.125	0.211
EFr	-436.466	878.932	1.625	0.235	0.177	0.022
Fr	-449.745	903.490	3.350	0.526	0.197	0.008

7. Concluding Remarks

In this paper, we propose an extended Fr´echet distribution, called the exponentiated exponential Fr´echet, which arises from the quantile function of the exponential distribution. We study some mathematical properties of the extended Fr´echet distribution including an expansion for the density function and explicit expressions for the moments, generating function, mean deviations, quantile function, Shannon entropy and order statistics. The maximum likelihood method is employed for estimating the model parameters and the observed information matrix is determined. We fit the new distribution to three real data sets to demonstrate its flexibility. The proposed model provides consistently better fit than other competing models. We hope that the new model will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics, among others.

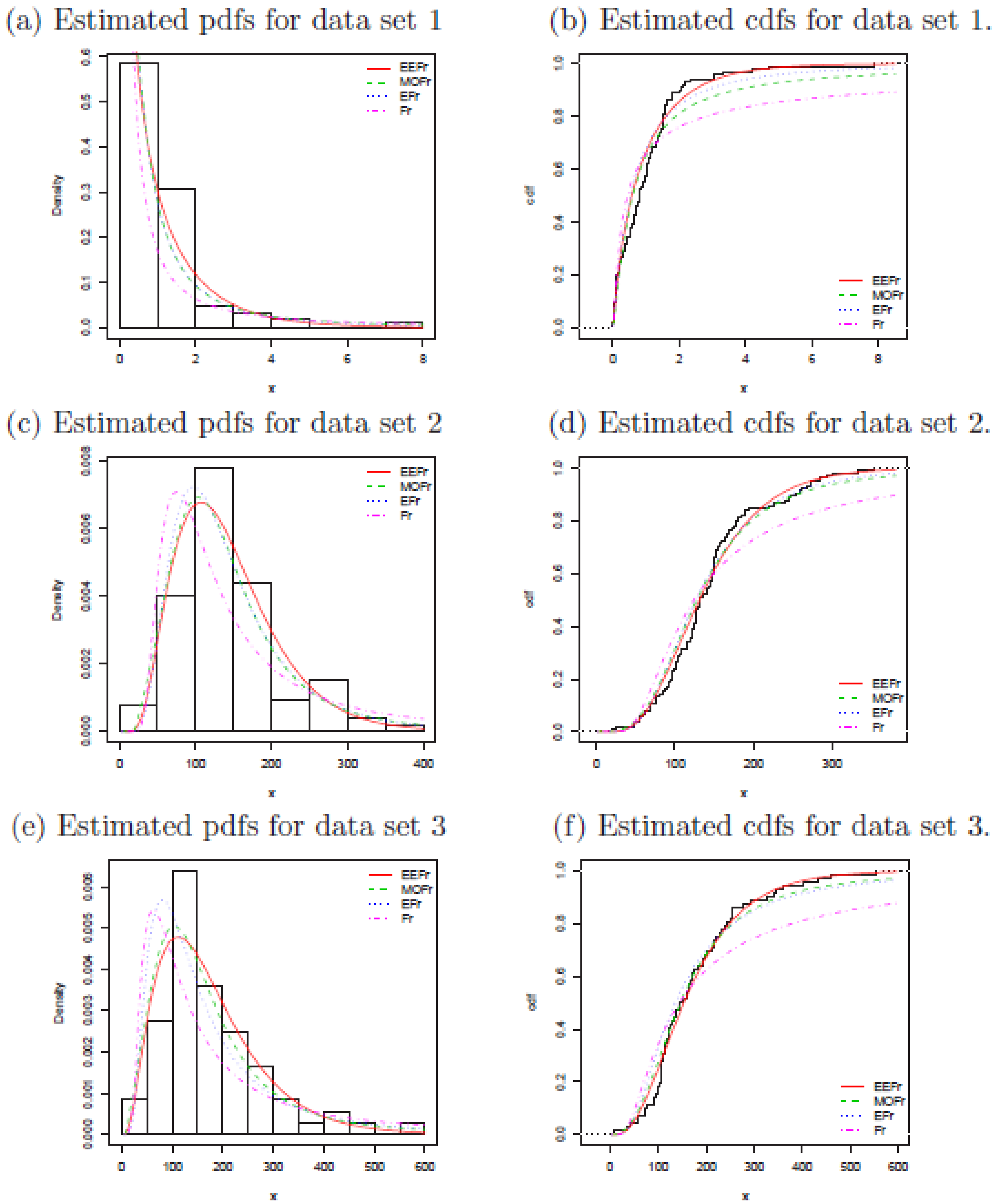

Figure 4: Plots of the estimated pdfs and cdfs of the EEFr, MOFr, EFr and Fr models for data sets 1, 2 & 3.

Appendix A

The observed information matrix for the parameter vector $\Theta = (\alpha, \beta, \delta, \theta)^T$

is given by

$$J(\Theta) = -\frac{\partial^2 \ell(\Theta)}{\partial\Theta\,\partial\Theta^\top} = -\begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\delta} & J_{\alpha\theta} \\ . & J_{\beta\beta} & J_{\beta\delta} & J_{\beta\theta} \\ . & . & J_{\delta\delta} & J_{\delta\theta} \\ . & . & . & J_{\theta\theta} \end{pmatrix},$$

whose elements are

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} - (\beta-1)\sum_{i=1}^{n}\left\{\frac{(1-z_i^\alpha)\left[\alpha^{-1}z_i^\alpha + z_i^\alpha\,(\log\alpha)^2\right] + z_i^{2\alpha}\,(\log\alpha)^2}{(1-z_i^\alpha)^2}\right\},$$

$$J_{\alpha\beta} = -\sum_{i=1}^{n}\left\{\frac{z_i^\alpha\,(\log\alpha)}{1-z_i^\alpha}\right\},$$

$$J_{\alpha\delta} = \sum_{i=1}^{n}\left\{\frac{z'_{i\delta}}{z_i}\right\} - \alpha(\beta-1)\,\log\alpha\sum_{i=1}^{n}\left\{\frac{z_i^{\alpha-1}\,z'_{i\delta}}{(1-z_i^\alpha)^2}\right\},$$

$$J_{\alpha\theta} = \sum_{i=1}^{n}\left\{\frac{z'_{i\theta}}{z_i}\right\} - \alpha(\beta-1)\,\log\alpha\sum_{i=1}^{n}\left\{\frac{z_i^{\alpha-1}\,z'_{i\theta}}{(1-z_i^\alpha)^2}\right\},$$

$$J_{\beta\beta} = -\frac{n}{\beta^2},$$

$$J_{\beta\delta} = -\alpha\sum_{i=1}^{n}\left\{\frac{z_i^{\alpha-1}\,z'_{i\delta}}{1-z_i^\alpha}\right\},$$

$$J_{\beta\theta} = -\alpha\sum_{i=1}^{n}\left\{\frac{z_i^{\alpha-1}\,z'_{i\theta}}{1-z_i^\alpha}\right\},$$

$$J_{\delta\delta} = -\frac{n\,\theta}{\delta^2} - (\theta-1)\sum_{i=1}^{n}(\theta/x_i^2)\,(\delta/x_i)^{\theta-2} + (\alpha-1)\sum_{i=1}^{n}\left\{\frac{z_i\,z''_{i\delta} - (z'_{i\delta})^2}{z_i^2}\right\}$$
$$-\alpha(\beta-1)\sum_{i=1}^{n}\left\{\frac{(1-z_i^\alpha)\left[z_i^{\alpha-1}\,z''_{i\delta} + (\alpha-1)\,z_i^{\alpha-2}\,(z'_{i\delta})^2\right] + \alpha\,z_i^{2\alpha-2}\,(z'_{i\delta})^2}{(1-z_i^\alpha)^2}\right\},$$

$$J_{\delta\theta} = \frac{n}{\delta} - \sum_{i=1}^{n}\left\{(\theta/x_i)\,(\delta/x_i)^{\theta-1}\,\log\,(\delta/x_i) + x_i^{-1}\,(\delta/x_i)^{\theta-1}\right\}$$
$$+(\alpha-1)\left\{\frac{z_i\,z''_{i\theta\delta} - z'_{i\theta}\,z'_{i\delta}}{z_i^2}\right\}$$
$$-\alpha(\beta-1)\sum_{i=1}^{n}\left\{\frac{(1-z_i^\alpha)\left[z_i^{\alpha-1}\,z''_{i\theta\delta} + (\alpha-1)\,z_i^{\alpha-2}\,z'_{i\theta}\,z'_{i\delta}\right] + \alpha\,z_i^{2\alpha-2}\,z'_{i\theta}\,z'_{i\delta}}{(1-z_i^\alpha)^2}\right\},$$

$$J_{\theta\theta} = -\frac{n}{\theta^2} - \sum_{i=1}^{n}(\delta/x_i)^\theta\,\{\log\,(\delta/x_i)\}^2 + (\alpha-1)\left\{\frac{z_i\,z''_{i\theta} - (z'_{i\theta})^2}{z_i^2}\right\}$$
$$-\alpha(\beta-1)\sum_{i=1}^{n}\left\{\frac{(1-z_i^\alpha)\left[z_i^{\alpha-1}\,z''_{i\theta} + (\alpha-1)\,z_i^{\alpha-2}\,(z'_{i\theta})^2\right] + \alpha\,z_i^{2\alpha-2}\,(z'_{i\theta})^2}{(1-z_i^\alpha)^2}\right\},$$

where $z_i = [1 - \mathbf{e}^{-(\delta/x_i)^{\theta}}]$, and

$$\begin{aligned}
z'_{i\theta} &= \frac{\partial z_i}{\partial\,\theta} = (\delta/x_i)^{\theta}\ \mathrm{e}^{-(\delta/x_i)^{\theta}}\ \log\ (\delta/x_i)\,,\\
z''_{i\theta} &= \frac{\partial^2 z_i}{(\partial\,\theta)^2} = (\delta/x_i)^{\theta}\ \mathrm{e}^{-(\delta/x_i)^{\theta}}\ \left[1-(\delta/x_i)^{\theta}\right]\ [\log\ (\delta/x_i)]^2\,,\\
z'_{i\delta} &= \frac{\partial z_i}{\partial\,\delta} = (\theta/x_i)\ (\delta/x_i)^{\theta-1}\ \mathrm{e}^{-(\delta/x_i)^{\theta}},\\
z''_{i\delta} &= \frac{\partial^2 z_i}{(\partial\,\delta)^2} = (\theta/x_i)\ (\delta/x_i)^{\theta-1}\ \mathrm{e}^{-(\delta/x_i)^{\theta}}\ \left[\frac{\theta-1}{\delta} - \frac{\theta}{x_i}\left(\frac{\delta}{x_i}\right)^{\theta-1}\right],\\
z''_{i\theta\delta} &= \frac{\partial^2 z_i}{(\partial\,\theta)\,(\partial\,\delta)} = (\delta/x_i)^{\theta}\ \mathrm{e}^{-(\delta/x_i)^{\theta}}\ \left\{\frac{1}{\delta} + \left(\frac{\theta}{\delta}\right)\left[1-(\delta/x_i)^{\theta}\right]\ \left[\log\ (\delta/x_i)\right]\right\}.
\end{aligned}$$

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