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RATIO AND INVERSE MOMENTS OF MARSHALL-OLKIN EXTENDED BURR TYPE XII DISTRIBUTION BASED ON LOWER GENERALIZED ORDER STATISTICS

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Abstract: In this small note we have established some new explicit expressions for ratio and inverse moments of lower generalized order statistics for the Marshall-Olkin extended Burr type XII distribution. These explicit expressions can be used to develop the relationship for moments of ordinary order statistics, record statistics and other ordered random variable techniques. Further, a characterization result of this distribution has been considered on using the conditional moment of the lower generalized order statistics.

Keywords: Lower generalized order statistics, order statistics, record values, Marshall-Olkin extended Burr type XII distribution, ratio and inverse moments, characterization.

1. Introduction

The concept of generalized order statistics (gos) was introduced by Kamps (1995) as a general framework for models of ordered random variables. Moreover, many other models of ordered random variables, such as, order statistics, k-th upper record values, upper record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics are seen to be particular cases of gos. These models can be effectively applied, e.g., in reliability theory. However, random variables that are decreasingly ordered cannot be integrated into this framework. Consequently, this model is inappropriate to study, e.g. reversed ordered order statistic and lower record values models. Using the concept of gos, Pawlas and Szynal (2001) introduced the concept of lower generalized order statistics (lgos) to enable a common approach to descending order statistics, which was further studied by Burkschat et al. (2003) with the name dual generalized order statistics. The lgos models enable us to study decreasingly ordered random variables like reversed order statistics, lower record values and lower Pfeirfer records, through a common approach below:

Suppose $X(1,n,m,k),\ldots,X(n,n,m,k)$, $(k \geq 1,\ m$ is a real number), are $n\,lgos$ from an absolutely continuous cumulative distribution function (cdf) $F(x)$ with probability density function (pdf) $f(x)$, if their joint pdf is of the form

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$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$,

where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.

If $m = 0$ and $k = 1$, then this model reduces to the $(n-r+1)-$th order statistic, from the sample $X_1, X_2, \ldots, X_n$ and (1.1) will be the joint *pdf* of n order statistics. If $k = 1$ and $m = -1$, then (1.1) will be the joint *pdf* of the first n record values of the identically and independently distributed (iid) random variables with *cdf* $F(x)$ and corresponding *pdf* $f(x)$.

In view of (1.1), the marginal *pdf* of the $r-$th *lgos*, $X^*(r,n,m,k)$, $1 \leq r \leq n$, is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)), \tag{1.2}$$

and the joint *pdf* of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$, $1 \leq r < s \leq n$, is

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad h_m(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

Moments of order statistics play an important role in quality control testing and reliability to predict the failure of future items based on the times of few early failures. An application of the first moments of order statistics can be considered in calculating the L-moments which are in fact the linear combinations of the expected order statistics. See Hosking (1990) for details.The concept of recurrence relations has its own importance. They are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. Furthermore, they are used in characterizing the distributions, which is an important area, permitting the identification of population distribution from the properties of the sample. Recurrence relations and identities have attained importance as it reduces the amount of direct computation, time and labour.

Various developments on *lgos* and related topics have been studied by Pawlas and Szynal (2001), Ahsanullah (2004, 2005), Mbah and Ahsanullah (2007). Recurrence relations for single and product moment of *lgos* from the exponentiated Kumarswamy, exponentiated Pareto, exponentiated gamma and generalized exponential distributions are derived by Kumar (2011) and Khan and Kumar (2010, 2011a, 2011b) respectively. Kumar (2013a, b) established recurrence relations for marginal and joint moments generating function of *lgos* from extended type I generalized logistic distribution and generalized logistic distribution respectively. Keseling (1999) characterized some continuous distributions based on conditional distributions of gos. Bieniek and Szynal (2003) characterized some distributions via linearity of regression of gos. Cramer *et al.* (2004) gave a unifying approach on characterization via linear regression of ordered random variables.

1.1. Marshall-Olkin extended Burr type XII distribution

A set of family of distributions which might be useful for fitting data has been proposed by Burr (1942) as a modeling lifetime data or survival data, which has twelve different forms of cumulative distribution functions. Among those twelve distribution functions, Burr Type X and Burr Type XII received the maximum attention. These distributions has been extensively used in various fields such as, simulation, quantal response, approximation of distributions, and development of non-normal control charts, in reliability theory and survival analysis. A number of standard theoretical distributions are limiting forms of Burr distributions. Burr type XII distribution plays major role in the analyses of lifetime and survival data. Shao *et al.* (2004) studied the models for extended three parameter of Burr type XII distribution and used this distribution to model extreme event with application to flood frequency. The flexibility of Burr type XII distribution has been studied by Rodriguez (1977). The Burr type XII has been widely used in various fields of sciences, such as in actuarial science, forestry, ecoloxicology, reliability and survival analysis.

The construction of the Marshall-Olkin extended Burr type XII distribution is rather simple and was first proposed by Marshall and Olkin (1997). As a new family of distributions by adding a parameter to obtain new families of distributions in connection with more flexible and represent a wide range of behavior than the original distributions. This distribution relates to a number of distributions such as Burr type XII, Marshall - Olkin extended Lomax distribution and hence its applicability in real life situations is significant. Since then, the Marshall-Olkin distributions have been widely studied in statistics and many authors have developed various Marshall-Olkin type distributions based on some well known distributions. See, for example the Marshall-Olkin extended Pareto distribution with application by Ghitany (2005), the Marshall-Olkin extended Weibull distribution by Ghitany *et al.* (2005), the Marshall-Olkin extended Lomax distribution by Ghitany *et al.* (2007), Marshall-Olkin Extended Uniform Distribution by Josea and Krishna (2011), Marshall-Olkin Extended Log-Logistic Distribution by Gui (2013), and so on.

The uniqueness of this study comes from the fact that we provide a comprehensive description ratio and inverse moments of lower generalized order statistics of this distribution with hope that they will attract wider applications in life time analysis.

The Marshall-Olkin extended Burr type XII distribution was introduced by Al-Saiari *et al.* (2014). He studied some properties and maximum likelihood estimation of the unknown parameters.

A random variable X is said to have Marshall-Olkin extended Burr type XII distribution if its pdf is of the form

$$f(x)=\frac{\alpha\,\beta\,\lambda x^{\beta-1}(1+x^{\beta})^{-(\lambda+1)}}{[1-(1-\alpha)(1+x^{\beta})^{-\lambda}]^{2}}\,, \quad x>0,\ \alpha\,,\ \beta,\lambda>0 \tag{1.4}$$

and the corresponding cdf is

$$F(x)=\frac{1-(1+x^{\beta})^{-\lambda}}{1-(1-\alpha)(1+x^{\beta})^{-\lambda}}\,, \quad x>0,\ \alpha\,,\ \beta,\lambda>0 \tag{1.5}$$

and the survival function is given by

$$S(x)=\frac{\alpha(1+x^{\beta})^{-\lambda}}{1-(1-\alpha)(1+x^{\beta})^{-\lambda}}\,, \quad x>0,\ \alpha\,,\ \beta,\lambda>0\,. \tag{1.6}$$

Burr XII and Marshall-Olkin extended Lomax distributions are consider as a special case of Marshall-Olkin extended Burr type XII distribution when $\alpha=1$ and $\beta=1$ respectively.

The paper is organized as follows. Some new expressions of the inverse moments of *lgos* from Marshall-Olkin extended Burr type XII distribution are derived in Section 2. In Section 3, some explicit expressions of ratio moments of *lgos* are derived. In Section 4, we obtained a characterization result of this distribution by using conditional expectation of *lgos*. Section 5 end with the some applications.

2. Relations for inverse moments

In this Section, the explicit expressions for inverse moments of *lgos* are considered. For the Marshall-Olkin extended Burr type XII distribution as given in (1.5), the $j-$th moments of $X^{*}(r,n,m,k)$ is given as

$$E[X^{*j-\beta}(r,n,m,k)]=\int_0^{\infty}x^{j-\beta}f_{X^{*}(r,n,m,k)}(x)\,dx$$

$$=\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^{j-\beta}[F(x)]^{\gamma_r-1}f(x)\,g_m^{r-1}(F(x))\,dx \tag{2.1}$$

Further, on using the binomial expansion, we can rewrite (2.1) as

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}$$

$$\times\int_0^{\infty}x^{j-\beta}[F(x)]^{\gamma_{r-u}-1}f(x)\,dx. \tag{2.2}$$

Now letting $t=F(x)$ in (2.2), we get

$$E[X^{*j-\beta}(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^r} \sum_{u=0}^{r-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{u+l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \binom{r-1}{u}$$

$$\times \left(\frac{p}{\lambda}\right) \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} B\left(\frac{k}{m+1}+n-r+u+\frac{l+q}{m+1},\ 1\right). \quad (2.3)$$

Since

$$\sum_{a=0}^{b}(-1)^a \binom{b}{a} B(a+k,\ c) = B(k,\ c+b) \quad (2.4)$$

where $B(a,\ b)$ is the complete beta function. Therefore,

$$E[X^{*j-\beta}(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^r} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \left(\frac{p}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} B\left(\frac{k}{m+1}+n-r+\frac{l+q}{m+1},\ r\right)$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \left(\frac{p}{\lambda}\right) \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)}$$

$$\times \frac{1}{\prod_{i=1}^{r}\left(1+\frac{q+l}{\gamma_i}\right)}. \quad (2.5)$$

Special cases

i) Putting $m=0$, $k=1$ in (2.5), we get moments of order statistics from Marshall-Olkin extended Burr type XII distribution as;

$$E[X_{n-r+1:n}^{j-\beta}] = \frac{n!}{(n-r)!} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \left(\frac{p}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \frac{\Gamma[n-r+1+q+l]}{\Gamma[n+1+q+l]}.$$

If we replace $n-r+1$ by r we get

$$E[X_{r:n}^{j-\beta}] = \frac{n!}{(r-1)!} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \left(\frac{p}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \frac{\Gamma[r+q+l]}{\Gamma[n+1+q+l]}$$

and at $r=n$

$$E[X_{n:n}^{j-\beta}] = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \left(\frac{p}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \frac{n}{(n+q+l)}.$$

ii) Putting $m=-1$ in (2.5), to get moments of $k-$th record value from Marshall-Olkin extended Burr type XII distribution as;

$$E[X^{*j-\beta}(r,n,-1,k)] = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l}{p!q!l!} \left(\frac{p}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \frac{1}{\left(1+\frac{q+l}{k}\right)^r}.$$

3. Relations for ratio moments

In this Section, the explicit expressions for ratio moments of $lgos$ are considered. For Marshall-Olkin Burr type XII distribution, the ratio moments of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is given as

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)] = \int_0^{\infty}\int_0^{x} x^i y^{j-\beta} f_{X^*(r,n,m,k)\ X^*(s,n,m,k)}(x,y)\,dxdy.$$

On using (1.3) and binomial expansion, we have

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]$$

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times\int_0^\infty x^i[F(x)]^{(s-r+u-v)(m+1)-1}f(x)I(x)\,dx,\quad x>y, \tag{3.1}$$

where

$$I(x)=\int_0^x y^{j-\beta}[F(y)]^{\gamma_{s-v}-1}f(y)\,dy. \tag{3.2}$$

By setting $t=F(y)$ in (3.2), we obtain

$$I(x)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l+(j/\beta)-1}(1-\alpha)^l(p/\lambda)}{p!q!l!}\frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)}\frac{[F(x)]^{\gamma_{s-v}+q+l}}{[\gamma_{s-v}+q+l]}.$$

On substituting the above expression of $I(x)$ in (3.1), we find that

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{u+v+l+(j/\beta)-1}(1-\alpha)^l(p/\lambda)}{p!q!l!}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times\frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)(\gamma_{s-v}+q+l)}\int_0^\infty x^i[F(x)]^{\gamma_{r-u}+q+l-1}f(x)\,dx. \tag{3.3}$$

Again by setting $z=F(x)$ in (3.3) and simplifying the resulting equation, we get

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{w=0}^{\infty}\frac{(-1)^{l+w+t+((1+j)/\beta)-1}(1-\alpha)^{l+t}}{p!q!l!w!t!a!}\left(\frac{p+w}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} B\left(\frac{k}{m+1}+n-r+u+\frac{q+l+a+t}{m+1}, 1\right)$$

$$\times \frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\frac{w}{\lambda}+1-t\right)} \sum_{v=0}^{s-r-1}(-1)^v \binom{s-r-1}{v} B\left(\frac{k}{m+1}+n-s+v+\frac{q+l}{m+1},\ 1\right). \tag{3.4}$$

On using relation (2.4) in (3.4), we get

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{w=0}^{\infty} \frac{(-1)^{l+w+t-1+(i+j)/\beta}(1-\alpha)^{l+t}}{p!q!l!w!t!a!}\left(\frac{p+w}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\frac{w}{\lambda}+1-t\right)}$$

$$\times B\left(\frac{k}{m+1}+n-r+\frac{q+l+a+t}{m+1},\ r\right) B\left(\frac{k}{m+1}+n-s+\frac{q+l}{m+1},\ s-r\right).$$

Which after simplification yields

$$E[X^{*i}(r,n,m,k)\ X^{*j-\beta}(s,n,m,k)]$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{w=0}^{\infty} \frac{(-1)^{l+w+t-1+(1+j)/\beta-1}(1-\alpha)^{l+t}}{p!q!l!w!t!s!}\left(\frac{p+w}{\lambda}\right)$$

$$\times \frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)} \frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\frac{w}{\lambda}+1-t\right)}$$

$$\times \frac{1}{\prod_{\phi=1}^{r}\left(1+\frac{q+l+a+t}{\gamma_{\phi}}\right)\prod_{\psi=r+1}^{s}\left(1+\frac{q+l}{\gamma_{\psi}}\right)}. \tag{3.5}$$

Special cases

i) Setting $m=0$, $k=1$ in (3.5), we get product moments of order statistics from Marshall-Olkin Burr type XII distribution as

$$E[X_{n-r+1:n}^{i}\, X_{n-s+1:n}^{j}] = \frac{n!}{(n-s)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{l+t+w-1+(i+j)/\beta}$$

$$\times \frac{\binom{\frac{p+w}{\lambda}}{}(1-\alpha)^{l+t}}{p!q!l!t!w!a!}\frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)}\frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\frac{w}{\lambda}+1-t\right)}$$

$$\times \frac{\Gamma(n-r+1+l+q+a+t)\,\Gamma(n-s+1+q+l)}{\Gamma(n+1+q+l+a+t)\,\Gamma(n-r+1+q+l)}.$$

Replace $n-s+1$ by r and $n-r+1$ by s

$$E[X_{r:n}^{i}\, X_{s:n}^{j}] = \frac{n!}{(r-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{l+t+w-1+(i+j)/\beta}$$

$$\times \frac{\binom{\frac{p+w}{\lambda}}{}(1-\alpha)^{l+t}}{p!q!l!t!w!a!}\frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)}\frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\frac{w}{\lambda}+1-t\right)}$$

$$\times \frac{\Gamma(s+l+q+a+t)\,\Gamma(r+q+l)}{\Gamma(n+1+q+l+a+t)\,\Gamma(s+q+l)}.$$

ii) If $m=-1$ in (3.5), we get product moments of $k-$th record values from Marshall-Olkin Burr type XII distribution as

$$E[(Z_{r}^{(k)})^{i}(Z_{s}^{(k)})^{j}] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{l+t+w-1+(i+j)/\beta}$$

$$\times \frac{\binom{\frac{p+w}{\lambda}}{}(1-\alpha)^{l+t}}{p!q!l!t!w!a!}\frac{\Gamma\left(1-\frac{j}{\beta}+p\right)\Gamma\left(\frac{p}{\lambda}+q\right)}{\Gamma\left(1-\frac{j}{\beta}\right)\Gamma\left(\frac{p}{\lambda}+1-l\right)}\frac{\Gamma\left(\frac{i}{\beta}+1\right)\Gamma\left(\frac{w}{\lambda}+a\right)}{\Gamma\left(\frac{i}{\beta}+1-w\right)\Gamma\left(\frac{w}{\lambda}+1-t\right)}$$

$$\times \frac{1}{\left(1+\frac{q+l+a+t}{k}\right)^{r}\left(1+\frac{q+l}{k}\right)^{s-r}}.$$

4. Characterization

Let $X^*(r,n,m,k)$, $r = 1,2,\ldots,n$ be gos from a continuous population with df $F(x)$ and pdf $f(x)$, then the conditional pdf of $X^*(s,n,m,k)$ given $X^*(r,n,m,k) = x$, $1 \le r < s \le n$, in view of (1.3) and (1.2), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y \mid x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}$$

$$\times \frac{[(h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1}}{[F(x)]^{\gamma_{r+1}}} f(y), \; x < y. \tag{4.1}$$

Theorem 4.1: Let X be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$E[X^*(s,n,m,k) \mid X^*(r,n,m,k) = x] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{p+q+(1/\beta)}(1-\alpha)^q (p/\lambda)}{p!q!l!}$$

$$\times \left[\frac{(1+x^\beta)^\lambda - 1}{(1+x^\beta)^\lambda - (1-\alpha)}\right]^{q+l} \frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{p}{\lambda}+l\right)}{\Gamma\left(\frac{1}{\beta}+1-p\right)\Gamma\left(\frac{p}{\lambda}+1-q\right)} \prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+q+l}\right), \tag{4.2}$$

if and only if

$$F(x) = \frac{1-(1+x^\beta)^{-\lambda}}{1-(1-\alpha)(1+x^\beta)^{-\lambda}}, \quad x > 0, \; \alpha, \; \beta, \lambda > 0.$$

Proof: From (4.1), we have

$$E[X^*(s,n,m,k) \mid X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_0^x y\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{F(x)}dy. \tag{4.3}$$

By setting $u = \dfrac{F(y)}{F(x)}$ from (1.5) in (4.3), we obtain

$$E[X^*(s,n,m,k) \mid X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{p+q+(1/\beta)}(1-\alpha)^q(p/\lambda)}{p!q!l!}\left[\frac{(1+x^\beta)^\lambda - 1}{(1+x^\beta)^\lambda-(1-\alpha)}\right]^{q+l}$$

$$\times \frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{p}{\lambda}+l\right)}{\Gamma\left(\frac{1}{\beta}+1-p\right)\Gamma\left(\frac{p}{\lambda}+1-q\right)} \int_0^1 u^{\gamma_s+q+l-1}(1-u^{m+1})^{s-r-1}du\,. \qquad (4.4)$$

Again by setting $t = u^{m+1}$ in (4.4) and simplifying the resulting expression, we derive the relation given in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_0^x y[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-1}$$

$$\times [F(y)]^{\gamma_s-1} f(y)dy = [F(x)]^{\gamma_{r+1}} H_r(x)\,, \qquad (4.7)$$

where

$$H_r(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{p+q+(1/\beta)}(1-\alpha)^q(p/\lambda)}{p!q!l!} \left[\frac{(1+x^\beta)^\lambda-1}{(1+x^\beta)^\lambda-(1-\alpha)}\right]^{q+l}$$

$$\times \frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{p}{\lambda}+l\right)}{\Gamma\left(\frac{1}{\beta}+1-p\right)\Gamma\left(\frac{p}{\lambda}+1-q\right)} \prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+q+l}\right).$$

Differentiating (4.7) both sides with respect to x, we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_0^x y[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-2}[F(y)]^{\gamma_s-1} f(y)dy$$

$$= H_r'(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1}H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x)$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x)-H_r(x)]} = \frac{\alpha\beta\lambda x^{\beta-1}(1+x^\beta)^{\lambda-1}}{[(1+x^\beta)^\lambda-(1-\alpha)][(1+x^\beta)^\lambda-1]}$$

which proves that

$$F(x) = \frac{1-(1+x^\beta)^{-\lambda}}{1-(1-\alpha)(1+x^\beta)^{-\lambda}}\,, \qquad x>0,\ \alpha,\ \beta,\lambda>0\,.$$

5. Application

Moments of order statistics play an important role in quality control testing and reliability to predict the failure of future items based on the times of few early failures. In this Section we suggest some application based on moments discussed in Section 2.

i) Estimation: The moments of order statistics and record values given in Section 2 can be used to obtain the best linear unbiased estimate of the parameters of the Marshall-Olkin Burr type XII distribution.

ii) Characterization: The Marshall-Olkin Burr type XII distribution given in (1.5) can be characterized by using conditional expectation of generalized order statistics as Theorem 4.1.

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