Journal of Data Science15(2017), 723-740

The Transmuted Topp-Leone G Family of Distributions: Theory, Characterizations and Applications

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Abstract: In this paper, we introduce a new family of continuous distributions called the transmuted Topp-Leone G family which extends the transmuted class pioneered by Shaw and Buckley (2007). Some of its mathematical properties including probability weighted moments, mo- ments, generating functions, order statistics, incomplete moments, mean deviations, stress- strength model, moment of residual and reversed residual life are studied. Some useful char- acterizations results based on two truncated moments as well as based on hazard function are presented. The maximum likelihood method is used to estimate its parameters. The Monte Carlo simulation is used for assessing the performance of the maximum likelihood estimators. The usefulness of the new model is illustrated by means of two real data set.

1 Introduction

In the literature, several continuous univariate distributions have been extensively used for modeling data in many areas such as economics, engineering, biological studies and environmental sciences. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. So, several classes of distributions have been constructed by extending common families of continuous distributions. These generalized distributions give more flexibility by adding one "or more" parameters to the baseline model. They were pioneered by Gupta et al. (1998) who proposed the exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin (1997), beta generalized-G family by Eugene et al. (2002), exponentiated generalized-G family by Cordeiro et al. (2013), a new method for generating families of continuous distributions by Alzaatreh et al. (2013), transmuted exponentiated generalized-G by Yousof et al. (2015), exponentiated transmuted-G by Merovc et al. (2016), Burr X-G by Yousof et al. (2016), transmuted Weibull G family by Alizadeh et al. (2016), complementary generalized transmuted Poisson-G family by Alizadeh et al. (2016b), transmuted geometric-G by Afify et al. (2016a), complementary geometric transmuted-G family Afify et al. (2016b), Kumaraswamy transmuted-G by Afify et al. (2016c), exponentiated generalized-G Poisson by Aryal and Yousof (2017), Marshall-Olkin generalized family by Yousof et al. (2017a), beta Weibull-G family of distributions by Yousof et al. (2017b), Type I general exponential class of distributions by Hamedani et al. (2017), Topp-Leone odd log-logistic family by de Brito et al. (2017), generalized odd generalized exponential family by Alizadeh et al. (2017), exponentiated Weibull-H family Cordeiro et al. (2017a), generalized transmuted-G by Nofal et al. (2017), Burr XII system of densities by Cordeiro et al. (2017b) and beta transmuted-H family by Afify et al. (2017), among others.

For an arbitrary baseline cdf $G(x)$, Shaw and Buckley (2007) defined the TG family with cdf and probability density function (pdf) given by

$$F(x) = F(x;\lambda,\psi) = H(x;\psi)\,[1+\lambda-\lambda H(x;\psi)] \quad (1)$$

and

$$f(x) = f(x;\lambda,\psi) = h(x;\psi)\,[1+\lambda-2\lambda H(x;\psi)], \quad (2)$$

respectively, where $|\lambda| \leq 1$ is a shape parameter, $x > 0$ and $\psi = (\psi_k) = (\psi_1, \psi_2, ...)$ is a parameter vector. The TG density is a mixture of the baseline density and the exponentiated-G (exp-G) density with power parameter two. For $\lambda = 0$, Equation (1) gives the baseline distribution. Due to Rezaei et al. (2016), the cdf and the pdf of the Topp Leone generated (TLG) family of distributions are specified by

$$H(x) = H(x;\alpha,\psi) = \{G(x;\psi)\,[2-G(x;\psi)]\}^{\alpha} = \left\{1-[1-G(x;\psi)]^{2}\right\}^{\alpha} \quad (3)$$

and

$$h(x) = h(x;\alpha,\psi) = 2\alpha\, g(x;\psi)\,[1-G(x;\psi)]\,\left\{1-[1-G(x;\psi)]^{2}\right\}^{\alpha-1}, \quad (4)$$

respectively. The various properties of the Topp-Leone's distribution have been studied by several authors. For example: moments by (Nadarajah and Kotz, 2003); reliability measures and stochastic orderings by (Ghitany et al., 2005); distributions of sums, products and ratios by (Zhou et al., 2006); behavior of kurtosis by (Kotz and Seier, 2007); record values by (Zghoul, 2011); moments of order statistics by (Genc, 2012); stress-strength modeling by (Genc, 2013); Bayesian estimation under trimmed samples by (Sindhu et al., 2013) and Censored maximum likelihood estimation by (Rezaei et al., 2016). The objective of this study is to define a new class of distributions called the transmuted Topp-Leone G (TTL-G for short) family of distributions and study its mathematical properties. Based on the TG and TLG families, we construct a new generator by inserting (3) into (1), to have

$$F(x) = (1+\lambda)\left\{1-[1-G(x;\psi)]^{2}\right\}^{\alpha} - \lambda\left\{1-[1-G(x;\psi)]^{2}\right\}^{2\alpha}, x \geq 0, \quad (5)$$

where $G(x;\psi) = G(x)$ is the baseline cdf and $\alpha > 0$ and $|\lambda| \leq 1$ are two additional shape parameters. The TTL-G is a wider class of continuous distributions. It includes the TG and TLG families of distributions.

The rest of the paper is outlined as follows. In Section 2, we define the univariate extensions of the TTL-G family. A useful mixture representation for the new pdf are derived in the same Section. In Section 3, we derive some of its mathematical properties including probability weighted moments (PWMs), residual life and reversed residual life functions, stress-strength model, ordinary and incomplete moments, generating functions and finally order statistics and their moments are introduced at the end of the section. Some characterization results are provided in Section 4. Maximum likelihood estimation of the model parameters is addressed in Section 5. In section 6, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. In Section 7, we define two special models and provide the plots of their pdf's and hazard rate functions (hrf's). In Section 8, we provide the applications to real data to illustrate the importance of the new family. Finally, some concluding remarks are presented in Section 9.

2 The new family

The pdf corresponding to (5) is

$$\begin{aligned} f(x) &= 2\alpha\, g(x;\psi)\,[1-G(x;\psi)]\,\left\{1-[1-G(x;\psi)]^{2}\right\}^{\alpha-1} \\ &\quad \times\left\{1+\lambda-2\lambda\left\{1-[1-G(x;\psi)]^{2}\right\}^{\alpha}\right\}, x > 0. \end{aligned} \quad (6)$$

The hrf for the new family can be expressed as

$$\tau(x) = 2\alpha\, g(x;\boldsymbol{\psi})\,[1-G(x;\boldsymbol{\psi})]\,\frac{1+\lambda-2\lambda\left\{1-[1-G(x;\boldsymbol{\psi})]^2\right\}^{\alpha}}{\left\{1-[1-G(x;\boldsymbol{\psi})]^2\right\}^{1-\alpha}} \times\left[1-(1+\lambda)\left\{1-[1-G(x;\boldsymbol{\psi})]^2\right\}^{\alpha}+\lambda\left\{1-[1-G(x;\boldsymbol{\psi})]^2\right\}^{2\alpha}\right]^{-1}. \quad (7)$$

For simulation of this new family if $u \sim u(0,1)$ then for $\lambda \neq 0$ the solution of non-linear equation

$$x_u = G^{-1}\left(1-\left\{1-\left[\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda\, u}}{2\lambda}\right]^{1/\alpha}\right\}^{0.5}\right),$$

has cdf (5). For $\lambda = 0$, $x_u = G^{-1}[1-(1-u^{1/\alpha})^{0.5}]$ has cdf (5). The chief motivation of the generalized distributions for modeling lifetime data lies in the flexibility to model both monotonic and non-monotonic failure rates even though the baseline failure rate may be monotonic. The basic justifications for generating a new distribution in practice are the following: to produce a skewness for symmetrical models; to generate distributions with left-skewed, right-skewed, symmetric, or reversed-J shape; to define special models with all types of hrf; to make the kurtosis more flexible compared to that of the baseline distribution; to construct heavy-tailed distributions for modeling various real data sets; to provide consistently better fits than other generated distributions with the same underlying model. Below is a simple motivation for the development of TTL-G family of distributions. Suppose T_1 and T_1" are two independent random variables with cdf (3). Define

$$X = \begin{cases} T_{1:2} & \text{with probability } \frac{1}{2}(\lambda+1)\,; \\ T_{2:2} & \text{with probability } \frac{1}{2}(1-\lambda)\,, \end{cases}$$

where

$$T_{1:2} = min\{T_1, T_2\} \text{ and } T_{2:2} = max\{T_1, T_2\}\,.$$

Then, the cdf of X is given by (5). The TTL-G family of distributions appears to be more felxible and could be used for modeling various types of data. For illustration propose we provide pdf and hrf of some special models of this family in figures 1 and 2. It can be seen that the hazard rate can take increasing, decreasing, upside down and bathtub shapes. Therefore, this family of distribution could be used to model diverse nature of data sets. Now, we provide a useful representation for (5) as

$$F(x) = (1+\lambda)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\bar{G}(x)^{2j} - \lambda\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\bar{G}(x)^{2\,j}, \quad (8)$$

or

$$F(x) = (1+\lambda)\sum_{j=0}^{\infty}\sum_{k=0}^{2\,j}(-1)^{j+k}\binom{\alpha}{j}\binom{2\,j}{k}G(x)^k - \lambda\sum_{j=0}^{\infty}\sum_{k=0}^{2\,j}(-1)^{j+k}\binom{2\alpha}{j}\binom{2\,j}{k}G(x)^k,$$

and finally

$$F(x) = \sum_{k=0}^{2\,j} w_{j,k}\,\boldsymbol{\Pi}_k(x), \quad (9)$$

where

$$w_k = \sum_{j=0}^{\infty}(-1)^{j+k}\left[(1+\lambda)\binom{\alpha}{j} - \lambda\binom{2\alpha}{j}\right]\binom{2\,j}{k},$$

and $\Pi_\delta(x) = G(x)^\delta$ is the cdf of the exp-G distribution with power parameter δ. The corresponding TTL-G density function is obtained by differentiating (9)

$$f(x) = \sum_{k=0}^{2j} w_k\, \boldsymbol{\pi}_{k+1}(x), \tag{10}$$

where $\boldsymbol{\pi}_\delta(x) = \delta g(x)\, G(x)^{\delta-1}$ is the pdf of the exp-G distribution with power parameter δ

3 Mathematical properties

3.1 Probability weighted moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,r)th PWMs of X following the TTL-G distribution, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\left\{X^s\, F(X)^r\right\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\, dx.$$

Using equations (5) and (6), we can write

$$f(x)\; F(x)^r = \sum_{k=0}^{2j+1} b_k \boldsymbol{\pi}_{k+1}(x),$$

where

$$\begin{aligned} b_k \;=\; & \sum_{i=0}^{r}\sum_{j=0}^{\infty} \frac{2\alpha\lambda^i\,(-1)^{i+j+k}}{(k+1)\,(1+\lambda)^{-(r-i)}}\binom{r}{i}\binom{2j+1}{k} \\ & \times\left\{(1+\lambda)\binom{\alpha\,(r+i+1)-1}{j} - 2\lambda\binom{\alpha\,(r+i+2)-1}{j}\right\}. \end{aligned}$$

Then, the (s,r)th PWMs of X can be expressed as

$$\rho_{s,r} = \sum_{k=0}^{2j+1} b_k E\left(Y_{k+1}^s\right).$$

3.2 Residual life and reversed residual life functions

The nth moment of the residual life, say $m_n(t) = E[(X-t^n)\,|X>t]$ $n, = 1\ 2, \ldots$, uniquely determine $F(x)$. The n th moment of the residual life of X is given by $m_n(t) = \frac{1}{1-F(t)}\int_t^\infty (x-t)^n\; dF(x)$Therefore,

$$m_n(t) = \frac{1}{1-F(t)}\sum_{k=0}^{2j} w_k^\star \int_t^\infty x^r \boldsymbol{\pi}_{k+1}(x),$$

where $w_k^\star = w_k \sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r}$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t) \mid X>t]$, which represents the expected additional life length for a unit which is alive at age t The MRL of X can be obtained by setting $n=1$ in the last equation. The n th moment of the reversed residual life, say $M_n(t) = E[(t-X)^n \mid X \le t]$ for $t>0$ and $n = 1,2\ \ ,\ldots$ uniquely determines $F(x)$. We obtain $M_n(t) = \frac{1}{F(t)}\int_0^t (t-x)^n dF(x)$.Then, the nth moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{k=0}^{2\,j} w_k^{\bigstar\bigstar} \int_0^t x^r \boldsymbol{\pi}_{k+1}(x),$$

where $w_k^{\bigstar\bigstar} = w_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$. The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by $M_1(t) = E[(t-X) \mid X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$.The MIT of the TTL-G family of distributions can be obtained easily by setting$n = 1$ in the above equation.

3.3 Stress-strength model

Stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications in physics and engineering such as strength failure and system collapse. In the stress-strength modeling, $\mathbf{R} = \Pr(X_2 < X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1. The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. $\mathbf{R}$ can be considered as a measure of system performance and naturally arise in electrical and electronic systems. Other interpretation can be that, the reliability of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent random variables wiht TTL-G $(\lambda_1, \alpha_1, \boldsymbol{\psi})$ and TTL-G $(\lambda_1, \alpha_1, \boldsymbol{\psi})$ distributions . Then, the reliability is defined by

$$\mathbf{R} = \int_0^\infty f_1\,(x; \lambda_1, \alpha_1, \boldsymbol{\psi})\, F_2\,(x; \lambda_1, \alpha_1, \boldsymbol{\psi})\, dx.$$

We can write

$$\mathbf{R} = \sum_{k=0}^{2\,j} \sum_{m=0}^{2\,w} \boldsymbol{\Omega}_{k,m} \int_0^\infty \boldsymbol{\pi}_{k+m}\,(x)\, dx,$$

where

$$\begin{aligned} \boldsymbol{\Omega}_{k,m} \quad &= \quad \sum_{j,w=0}^{\infty} \frac{(k+1)\,(-1)^{j+k+w+m}}{(k+m+1)} \binom{2\,j}{k} \binom{2\,w}{m} \\ &\quad \times \left[(1+\lambda_1) \binom{\alpha_1}{j} - \lambda_1 \binom{2\alpha_1}{j} \right] \left[(1+\lambda_2) \binom{\alpha_2}{w} - \lambda_2 \binom{2\alpha_2}{w} \right]. \end{aligned}$$

Thus, the reliability, $\mathbf{R}$, can be expressed as

$$\mathbf{R} = \sum_{k=0}^{2\,j} \sum_{m=0}^{2\,w} \boldsymbol{\Omega}_{k,m}.$$

3.4 Moments, incomplete moments and generating function

The r th ordinary moment of X is given by $\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r\, f\,(x)\, dx$. Then we obtain

$$\mu'_r = \sum_{k=0}^{2\,j} w_k E(Y_{k+1}^r). \tag{11}$$

Henceforth, Y_{k+1}denotes the exp-G random variable with power parameter $k+1$. Setting $r = 1$ in (11), we have the mean of$X$.The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The n th central moment of X , say M_n, is $M_n = E(X-\mu)^n = \sum_{h=0}^{n}(-1)^h \binom{n}{h} (\mu_1')^n \mu_{n-h}'$ The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r \, \mu_{n-r}',$$

where $\kappa 1 = \mu_1'$, $\kappa 2 = \mu_2' - \mu_1'^2$, $\kappa 3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$, etc. The skewness and kurtosis measures can also be calculated from the ordinary moments using well-known relationships. The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The r th incomplete moment, say $\varphi_r(t)$, of X can be expressed, from (9), as

$$\varphi_r(t) = \int_{-\infty}^{t} x^r f(x)\, dx = \sum_{k=0}^{2j} w_k \int_{-\infty}^{t} x^r \boldsymbol{\pi}_{k+1}(x)\, dx. \tag{12}$$

The mean deviations about the mean $[\delta_1 = E(|X-\mu_1'|)]$ and about the median $[\delta_2 = E(|X-M|)]$ of X are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median\,(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (5) and $\varphi_1(t)$ is the first incomplete moment given by (12) with $r = 1$. A general equation for $\varphi_1(t)$ can be derived from (12) as

$$\varphi_1(t) = \sum_{k=0}^{2j} w_k I_{k+1}(x),$$

where $I_{k+1}(x) = \int_{-\infty}^{t} x\, \boldsymbol{\pi}_{k+1}(x)\, dx$ is the first incomplete moment of the exp-G distribution. The moment generating function (mgf) $M_X(t) = E\left(e^{tX}\right)$ of X can be derived from equation (9) as

$$M_X(t) = \sum_{k=0}^{2j} w_k\, M_{k+1}(t),$$

where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M_X(t)$ can be determined from the exp-G generating function.

3.5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the TTL-G family of distributions and let $X_{1:n}, \ldots, X_{n:n}$ be their corresponding order statistics. The pdf of i th order statistic, $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \tag{13}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in equation (13) and using a power series expansion,we have

$$f(x)\; F(x)^r = \sum_{k=0}^{2m+1} t_k\, \boldsymbol{\pi}_{k+1}(x),$$

where

$$\begin{aligned} t_k \;\; &= \sum_{h=0}^{j+i-1} \sum_{m=0}^{\infty} \frac{2\alpha\lambda^h (-1)^{h+m+k}}{(k+1)(1+\lambda)^{-(j+i-h-1)}} \binom{j+i-1}{h} \binom{2m+1}{k} \\ &\times \left\{ (1+\lambda) \binom{\alpha(j+i+h)-1}{m} - 2\lambda \binom{\alpha(j+i+h+1)-1}{m} \right\}. \end{aligned}$$

The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum_{j=0}^{n-i}\sum_{k=0}^{2m+1}\frac{(-1)^j\binom{n-i}{j}}{\mathrm{B}(i,n-i+1)}t_k\ \boldsymbol{\pi}_{k+1}.$$

Then, the density function of a TTL-G order statistic is a mixture of exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those of Y_{k+1}. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^q\right)=\sum_{j=0}^{n-i}\sum_{k=0}^{2m+1}\frac{(-1)^j\binom{n-i}{j}t_k}{\mathrm{B}(i,n-i+1)}E\left(Y_{k+1}^q\right). \tag{14}$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. In view of the moments in equation (14), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TTL-G order statistics. They are linear functions of the expected order statistics defined by

$$\lambda_r=\frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E\left(X_{r-d:r}\right),\ r\geq 1.$$

4 Characterizations

Characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This section deals with various characterizations of TTL-G distribution. These characterizations are based on: (i) a simple relationship between two truncated moments and (ii) the hazard function. It should be mentioned that for characterization (i) the cdf need not have a closed form.

4.1 Characterizations based on two truncated moments

In this subsection we present characterizations of TTL-G distribution in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to Glänzel (1987) see Theorem 4.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Theorem 4.1. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $(d=-\infty,\ e=\infty$ might as well be allowed$)$ Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X)\ \mid X\geq x\right]=\mathbf{E}\left[q_1(X)\ \mid X\geq x\right]\xi(x),\quad x\in H,$$

is defined with some real function ξ Assume that $q_1,q_2\in C^1(H)$, $\xi\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1=q_2$ has no real solution in the interior of H . Then F is uniquely determined by the functions q_1,q_2 and ξ , particularly

$$F(x)=\int_d^x C\left|\frac{\xi'(u)}{\xi(u)q_1(u)-q_2(u)}\right|\exp\left(-s(u)\right)\ du\ ,$$

where the functions s is a solution of the differential equation $s' = \frac{\xi' \, q_1}{\xi \, q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Here is our first characterization.

Proposition 4.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \left\{1 + \lambda - 2\lambda \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha\right\}^{-1}$ and $q_2(x) = q_1(x) \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha$ for $x > 0$. The random variable X belongs to TTL-G family(6) if and only if the function ξ defined in Theorem 4.1 has the form

$$\xi(x) = \frac{1}{2}\left\{1 + \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha\right\}, \quad x > 0. \tag{15}$$

Proof. Let X be a random variable with pdf (6) , then

$$(1 - F(x)) E\left[q_1(x) \mid X \geq x\right] = 1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha, \quad x > 0,$$

and

$$(1 - F(x)) E\left[q_2(x) \mid X \geq x\right] = \frac{1}{2}\left\{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^{2\alpha}\right\}, \quad x > 0,$$

and finally

$$\xi(x) q_1(x) - q_2(x) = \frac{1}{2} q_1(x) \left\{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha\right\} > 0 \quad for \quad x > 0.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x) q_1(x)}{\xi(x) q_1(x) - q_2(x)} = \frac{\alpha g(x) \overline{G}(x) \left[1 - \left(\overline{G}(x)\right)^2\right]^{\alpha - 1}}{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha}, \quad x > 0,$$

and hence $s(x) = -\log\left\{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha\right\}, \quad x > 0.$

Now, in view of Theorem 4.1, X has density.(6)

Corollary 4.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. The pdf of X is (6) if and only if there exist functions q_2 and ξ defined in Theorem 4.1 satisfying the differential equation

$$\frac{\xi'(x) q_1(x)}{\xi(x) q_1(x) - q_2(x)} = \frac{\alpha g(x) \overline{G}(x) \left[1 - \left(\overline{G}(x)\right)^2\right]^{\alpha - 1}}{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha}, \quad x > 0. \tag{16}$$

The general solution of the differential equation in Corollary 4.1 is

$$\begin{aligned} \xi(x) &= \left\{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^\alpha\right\}^{-1} \times \\ & \left[-\int \alpha g(x) \overline{G}(x) \left[1 - \left(\overline{G}(x)\right)^2\right]^{\alpha - 1} (q_1(x))^{-1} q_2(x)\, dx + D\right] \end{aligned}$$

where D is a constant. Note that a set of functions satisfying the differential equation (16) is given in Proposition 4.1 with $D = \frac{1}{2}$. However, it should be also noted that there are other triplets (q_1, q_2, ξ) satisfying the conditions of Theorem 4.1.

4.2 Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h'_F(x)}{h_F(x)}-h_F(x). \tag{17}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization for TTL-G distribution in terms of the hazard function when $\alpha=1$, which is not of the trivial form given in (17).

Proposition 4.2. Let $X:\Omega\to(0,\infty)$ be a continuous random variable. Then for $\alpha=1$, the pdf of X is (6) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$\begin{aligned}&h'_F(x)-\frac{g'(x)}{g(x)}h_F(x)\\ =&\ \frac{2(g(x))^2\left\{(1-\lambda)^2+\lambda(1-\lambda)\left(\overline{G}(x)\right)^2+2\lambda^2\left(\overline{G}(x)\right)^4\right\}}{\left\{\overline{G}(x)\left[(1-\lambda)+\lambda\left(\overline{G}(x)\right)^2\right]\right\}^2},\end{aligned} \tag{18}$$

with the boundary condition $h_F(0)=2g(0)(1+\lambda)$.

Proof. If X has pdf (6), then clearly (18) holds. Now, if (18) holds, then

$$\frac{d}{dx}\left\{(g(x))^{-1}h_F(x)\right\}=2\frac{d}{dx}\left\{\left(\overline{G}(x)\right)^{-1}\right\}+2\lambda\frac{d}{dx}\left\{\frac{\overline{G}(x)}{\left[(1-\lambda)+\lambda\left(\overline{G}(x)\right)^2\right]}\right\},$$

or, equivalently,

$$h_F(x)=\frac{2g(x)\left\{(1-\lambda)+2\lambda\left(\overline{G}(x)\right)^2\right\}}{\overline{G}(x)\left[(1-\lambda)+\lambda\left(\overline{G}(x)\right)^2\right]},$$

which is the hazard function of the TTL-G distribution.

5 Estimation

Several approaches for parameter estimation are proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood method. Let $x_1,\ldots,x_n$ be a random sample from the TTL-G distribution with parameters λ,α and $\boldsymbol{\psi}$ Let $\Theta=(\lambda,\alpha,\boldsymbol{\psi})^\intercal$ be the $p\times 1$ parameter vector. To determine the MLE of Θ, we have the log-likelihood function

$$\begin{aligned}\ell\ =\ &\ell(\Theta)=n\log(2)+n\log\alpha+\sum_{i=1}^{n}\log g(x_i;\boldsymbol{\psi})+\sum_{i=1}^{n}\log\overline{G}(x_i;\boldsymbol{\psi})\\ &+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\bar{G}(x_i;\boldsymbol{\psi})^2\right]+\sum_{i=1}^{n}\log(s_i),\end{aligned}$$

where $s_i = \left\{1 + \lambda - 2\lambda \left[1 - \bar{G}(x_i; \boldsymbol{\psi})^2\right]^{\alpha}\right\}$.

The components of the score vector, $\mathbf{U}(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \boldsymbol{\psi}}\right)^{\intercal}$, are

$$U_{\lambda} = \sum_{i=1}^{n} \frac{z_i}{s_i}, U_{\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[1 - \bar{G}(x_i; \boldsymbol{\psi})^2\right] + \sum_{i=1}^{n} \frac{p_i}{s_i}$$

and

$$U_{\boldsymbol{\psi}} = \sum_{i=1}^{n} \frac{g'(x_i; \boldsymbol{\psi})}{g(x_i; \boldsymbol{\psi})} - \sum_{i=1}^{n} \frac{G'(x_i; \boldsymbol{\psi})}{\overline{G}(x_i; \boldsymbol{\psi})} + (\alpha - 1) \sum_{i=1}^{n} \frac{2G'(x_i; \boldsymbol{\psi}) \bar{G}(x_i; \boldsymbol{\psi})}{\left[1 - \bar{G}(x_i; \boldsymbol{\psi})^2\right]} - 4\lambda\alpha \sum_{i=1}^{n} \frac{G'(x_i; \boldsymbol{\psi}) \bar{G}(x_i; \boldsymbol{\psi})}{s_i \left[1 - \bar{G}(x_i; \boldsymbol{\psi})^2\right]^{1-\alpha}},$$

where

$$p_i = \frac{-4\alpha\lambda g(x_i; \boldsymbol{\psi}) \bar{G}(x_i; \boldsymbol{\psi})}{\left[1 - \bar{G}(x_i; \boldsymbol{\psi})^2\right]^{1-\alpha}}, g'(x_i; \boldsymbol{\psi}) = \frac{\partial g(x_i; \boldsymbol{\psi})}{\partial \boldsymbol{\psi}},$$

$$z_i = 1 - 2\left[1 - \bar{G}(x_i; \boldsymbol{\psi})^2\right]^{\alpha} \text{ and } G'(x_i; \boldsymbol{\psi}) = \frac{\partial G(x_i; \boldsymbol{\psi})}{\partial \boldsymbol{\psi}}.$$

Setting the nonlinear system of equations $U_{\lambda} = U_{\alpha} = 0$ and $U_{\boldsymbol{\psi}} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\widehat{\lambda}, \widehat{\alpha}, \widehat{\boldsymbol{\psi}}^{\intercal})^{\intercal}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\Theta) = \{\frac{\partial^2 \ell}{\partial r\, \partial s}\}$ (for $r, s = \lambda, \alpha, \boldsymbol{\psi}$), whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\Theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\Theta)\hat{}$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

6 Simulation study

For the simulation study, we consider a specific member of the TTL-G family, by setting G as a beta (type I) distribution with parameters $a > 0$ and $b > 0$.
The associated likelihood function can be obtained for a random sample of size n drawn from pdf (6) by setting $G(x; \phi) = I_x(a, b)$, where $I_x(a, b)$ is the incomplete beta function.

We consider a random sample of size $n = 50, 100$ and 200 from our density corresponding to particular choices of the parameters as follows: $\alpha = 2$, $\lambda = 0.4, a = 1.5, b = 1.5$. Below we provide the bias and standard deviation for the estimates of all the parameters under both the methods of estimation. Table 1 provides the bias and standard error under the method of maximum likelihood. We consider $20,000$ simulations for drawing random samples each of size $n = 50$, $n = 100$, and $n = 200$ drawn from our density respectively.

Table 1. Bias and standard deviation of the parameter estimates using maximum likelihood.

Sample size	Bias$(\widehat{\alpha})$	Bias$(\widehat{\lambda})$	Bias$(\widehat{a})$	Bias$(\widehat{b})$	S.E$(\widehat{\alpha})$	S.E$(\widehat{\lambda})$	S.E$(\widehat{a})$	S.E$(\widehat{b})$
50	-0.127	0.0194	0.0032	0.0371	0.0396	0.1098	0.05606	0.0148
100	-0.087	0.0114	0.0017	0.0149	0.0272	0.0532	0.0383	0.0109
200	0.015	0.0089	0.0004	0.0065	0.0154	0.0412	0.0203	0.0095

The log-likelihood function can be maximized directly by using the R-package or by solving the nonlinear likelihood equations obtained by differentiating the pdf (6) (using optim function and Max-BFGS subroutines. One can find the estimates of the unknown parameters by setting the score vector to zero, and then using any statistical software to solve them numerically. The results show that the maximum likelihood estimation performs well. In general, the biases and standard deviations of the parameters are reasonably small. The biases and standard deviations always decrease as the sample size increases. The results suggest that the maximum likelihood method can be used to estimate the parameters of the TTLB.

7 Special models

In this section, we provide examples of the TTL-G family. The pdf (6) will be most tractable when $g(x)$ and $G(x)$ have simple analytic expressions. These special models generalize some well-known distributions reported in the literature. Here, we provide two special models of this family corresponding to the baseline Weibull (W) and beta (B) distributions to show the flexibility of the new family.

7.1 The TTLW distribution

Consider the pdf and cdf (for $x > 0$) $g(x) = ba^b x^{b-1} e^{-(ax)^b}$ and $G(x) = 1 - e^{-(ax)^b}$, respectively, of the Weibull distribution with positive parameters a and $b$which are scale and shape parameters respectively. Then, the pdf of the TTLW model is given by

$$f(x) = 2\alpha b a^b\, x^{b-1} e^{-2(ax)^b} \left\{1 - e^{-2(ax)^b}\right\}^{\alpha-1} \left\{1 + \lambda - 2\lambda \left\{1 - e^{-2(ax)^b}\right\}^{\alpha}\right\}.$$

The TTLW density and hrf plots for selected parameter values are displayed in Figure 1.

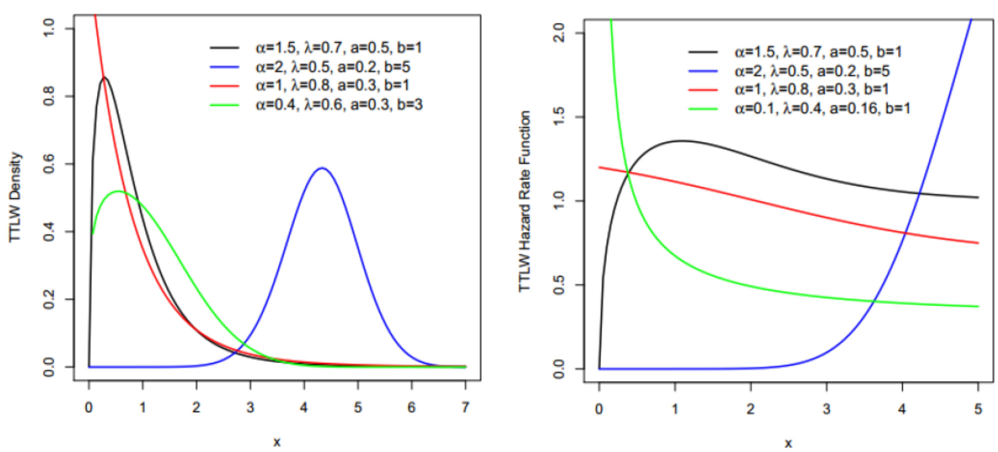

Figure 1: pdf and hrf of TTLW distribution

7.2 The TTLB distribution

Consider the pdf and cdf (for $x > 0$) $g(x) = \Gamma(a+b)\, x^{a-1}(1-x)^{b-1} / \{\Gamma(a)\Gamma(b)\}$ and $G(x) = I_x(a,b)$, respectively, of the beta distribution with positive parameters a and b . Then, the pdf of the TTLB model is given by

$$f(x) = \frac{2\alpha\,\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \left[1 - I_x(a,b)\right] \left[1 - (1 - I_x(a,b))^2\right]^{\alpha-1}$$
$$\times \left\{1 + \lambda - 2\lambda \left[1 - (1 - I_x(a,b))^2\right]^{\alpha}\right\}.$$

The TTLB density and hrf plots for selected parameter values are displayed in Figure 2.

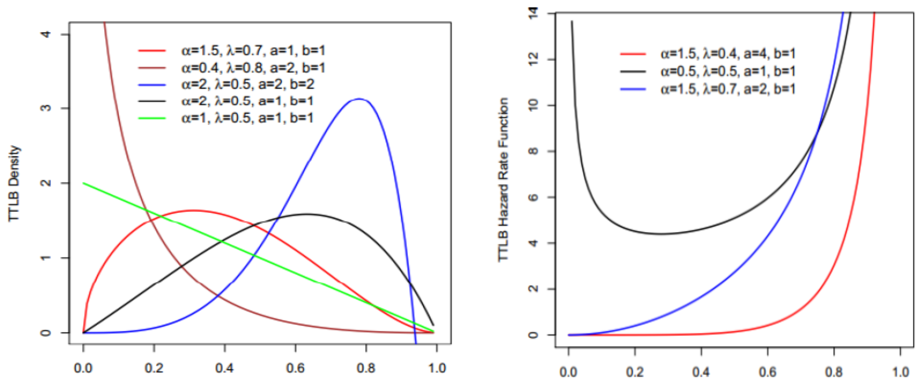

Figure 2: pdf and hrf of TTLB distribution

8 Applications

In this section, we provide two applications to real data to illustrate the flexibility and potentiality of the TTLW and TTLB models presented in Section 7. For illustrative purposes, we consider two data sets and compare with the Weibull and beta distributions. For each data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method (as discussed in Section 5) and all the computations were done using the Adequacy model package of the R software. The goodness-of-fit statistics for these models are compared with other competitive models and the MLEs of the model parameters are determined. In order to compare the fitted models, we consider some goodness-of-fit measures including the values of the Cramér-von Mises statistic (W), Anderson-Darling statistic (A), Kolmogorov-Smirnov Statistic (D), Kolmogorov-Smirnov probability value (D_{pvalue}), $-2\hat{\ell}$ where $\hat{\ell}$ is the maximized log-likelihood, the Akaike information criterion (AIC), Bayesian information criterion (BIC).

We compared the fits of the TTLW distribution with some of its special cases and other models such as Topp-Leone Weibull (TLW) (Rezaei et al., 2016), Weibull (W) (Weibull, 1951) , exponetiated Weibull (EW) (Mudholkar and Srivastava 1993), beta Weibull (BW) (Famoye et al., 2005) and McDonald Weibull (McW et al., 2014) (Cordeiro, 2014) distributions given by:

- TLW : $f\left(x\right)=2\alpha ba^bx^{b-1}e^{-2(ax)^b}\left[1-e^{-2(ax)^b}\right]^{\alpha-1}$;
- W : $f(x)=ba^bx^{b-1}e^{-(ax)^b}$;
- EW : $f(x)=\theta ba^bx^{b-1}e^{-(ax)^b}\left[1-e^{-(ax)^b}\right]^{\theta-1}$;
- BW : $f(x)=ba^bx^{b-1}e^{-\beta(ax)^b}\left[1-e^{-(ax)^b}\right]^{\alpha-1}/B(\alpha,\beta)$;
- McW : $f(x)=\gamma ba^bx^{b-1}e^{-\beta(ax)^b}\left[1-e^{-(ax)^b}\right]^{\alpha\gamma-1}\left\{1-\left[1-e^{-(ax)^b}\right]^{\gamma}\right\}^{\beta-1}/B(\alpha,\beta)$.

In addition, we compared the fits of the TTLB distributions with some of its special cases and other models such as Topp-Leone beta (TLB), beta (B), beta power (BP) (Cordeiro and Bager, 2012), Kumaraswamy (Kum) (Kumaraswamy, 1980) and exponetiated Kumaraswamy (EKum) (Lemonte et al., 2013) distributions given by:

- TLB : $2\alpha\,\Gamma(a+b)\,x^{a-1}(1-x)^{b-1}\left[1-I_x(a,b)\right]\left\{1-\left[1-I_x(a,b)\right]^2\right\}^{\alpha-1}/\left[\Gamma(a)\Gamma(b)\right]$;
- B : $f\left(x\right)=\Gamma(a+b)x^{a-1}(1-x)^{b-1}/\Gamma(a)\Gamma(b)$;

- BP : $f(x) = \alpha\beta(\beta x)^{\alpha a-1}[1-(\beta x)^{\alpha}]^{b-1}/B(a+b)$;
- Kum : $f(x) = abx^{a-1}(1-x^a)^{b-1}$,
- EKum : $f(x) = \theta abx^{a-1}(1-x^a)^{b-1}\left[1-(1-x^a)^b\right]^{\theta-1}$.

First, we describe the two data sets:

Data set I: (Breaking Stress data)

The data for breaking stress of carbon fibers of 50 mm length (GPa) was reported by Nicholas and Padgett (2006). This data was used by Cordeiro and Lemonte (2011) to illustrate the application of the four-parameter beta-Birnbaum-Saunders distribution when compared to the two-parameter Birnbaum-Saunders distribution (Birnbaum and Saunders, 1969). The data are: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Data set II: (Milk Production)

The data is about the total milk production in the first birth of 107 cows from SINDI race. These cows are property of the Carnaúba farm which belongs to the Agropecuária Manoel Dantas Ltda (AMDA), located in Taperoá City, Paraíba (Brazil). The original data is not in the interval (0,1), and it was necessary to make a transformation given by

$$x_i = [y_i - min(y_i)]/[max(y_i) - min(y_i)], \text{ for } i = 1, ..., 107.$$

The values of y_i are given in Table 3.1 of Brito (2009, p. 46) and x_i values are: 0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747.

Tables 2 and 4 provide the values of W , A, D, D_{pvalue}, $-2\hat{\ell}$, AIC and BIC . Since the values of considered statsitics are smaller and probability values of Kolmogorov-Smirnov statistics are greater than the TTLW distribution compared with those values of the other models, this new distribution seems to be a very competitive model for these data. In addition, the MLEs and their corresponding standard errors (in parentheses) of the considered model parameters are given in Tables 3 and 5. Plots of the pdf and cdf of the TTLW and TTLB against other fitted models to these data are displayed in Figures 3 and 4. They indicate that the TTLW and TTLB distributions are superior to the other distributions in terms of model fitting. Based on these plots, we conclude that the TTLW and TTLB distributions provide a better fit to these data than other models.

Table 2: The statistics W, A, D, D_{pvalue}, $-2\hat{\ell}$, AIC and BIC for breaking stress data

Model	Goodness of fit criteria						
	W	A	D	D_{pvalue}	$-2\hat{\ell}$	AIC	BIC
W	9.233	60.628	0.9771	0.0000	86.067	176.13	180.51
TLW	0.0858	0.5084	0.0810	0.7794	85.994	177.88	184.45
TTLW	0.0689	0.4122	0.0759	0.8414	85.375	178.85	187.50
EW	0.0858	0.5084	0.0810	0.7791	85.944	177.88	184.45
BW	0.0805	0.4802	0.0829	0.7547	85.685	179.37	188.13
McW	0.0748	0.4559	0.0963	0.5729	85.407	180.81	191.76

Table 3: MLEs and their standard errors (in parentheses) for breaking stress data

Model	Estimates				
TTLW	$\widehat{\alpha}$= 0.7059 (0.4406)	$\widehat{\lambda}$= −0.6996 (0.4039)	$\widehat{a}$= 0.2742 (0.0231)	$\widehat{b}$= 3.358 (0.9479)	
TLW	$\widehat{\alpha}$= 0.8001 (0.3535)	$\widehat{a}$= 0.2592 (0.0170)	$\widehat{b}$= 3.912 (1.069)		
W	$\widehat{a}$= 0.3265 (0.0122)	$\widehat{b}$= 3.441 (0.3309)			
EW	$\widehat{\theta}$= 0.8004 (0.3535)	$\widehat{a}$= 0.3094 (0.0331)	$\widehat{b}$= 3.911 (1.069)		
BW	$\widehat{\alpha}$= 0.8602 (0.2007)	$\widehat{\beta}$= 0.1490 (0.0204)	$\widehat{a}$= 0.5356 (0.0061)	$\widehat{b}$= 3.6989 (0.0062)	
McW	$\widehat{\alpha}$= 0.9259 (0.2281)	$\widehat{\beta}$= 0.1412 (0.0187)	$\widehat{\gamma}$= 0.7972 (0.0452)	$\widehat{a}$= 0.5368 (0.0025)	$\widehat{b}$= 3.690 (0.0038)

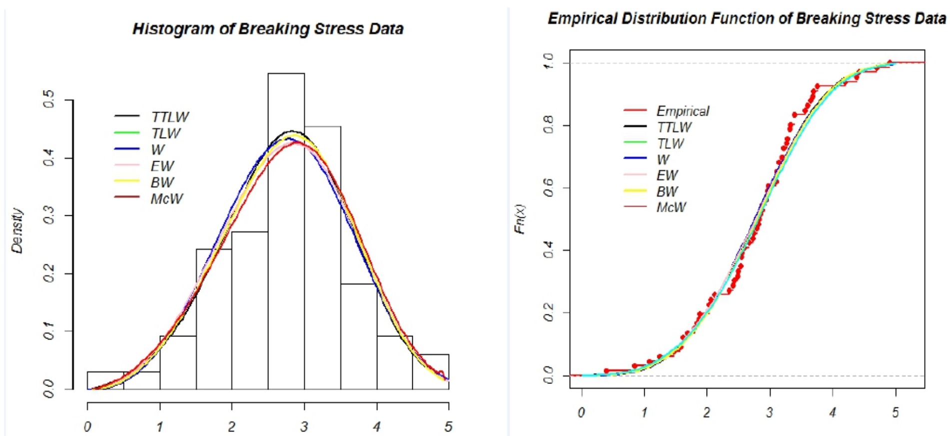

Figure 3: Histogram (left) and cdf (right) of the breaking stress data.

Table 4: The statistics W, A, D, D_{pvalue}, $-2\hat{\ell}$, AIC and BIC for milk production data

Model	Goodness of fit criteria						
	W	A	D	D_{pvalue}	$-2\hat{\ell}$	AIC	BIC
TTLB	0.0768	0.4956	0.0710	0.6520	−27.925	−47.850	−37.159
TLB	0.1191	0.7552	0.0819	0.4690	−26.790	−47.580	−39.561
B	0.2082	1.3263	0.0909	0.3384	−23.777	−43.554	−38.208
BP	0.1339	0.8413	0.0840	0.4361	−26.356	−46.713	−38.695
Kum	0.1560	1.0090	0.0762	0.5626	−25.394	−46.789	−41.443
EKum	0.0939	0.5950	0.0748	0.5868	−27.557	−49.114	−41.095

Table 5: MLEs and their standard errors (in parentheses) for milk production data

Model	Estimates		
TTLB	$\widehat{\alpha}$= 0.1290 (0.0337)	$\widehat{\lambda}$= −0.6947 (0.3239)	$\widehat{a}$= 9.970 (0.0025)
$\widehat{b}$= 3.165 (0.0025)			
TLB	$\widehat{\alpha}$= 0.2127 (0.0205)	$\widehat{a}$= 9.374 (0.0091)	$\widehat{b}$= 3.218 (0.0091)
B	$\widehat{a}$= 2.4125 (0.3144)	$\widehat{b}$= 2.8296 (0.3744)	
BP	$\widehat{\alpha}$= 6.6402 (1.5643)	$\widehat{\beta}$= 0.7756 (0.0754)	$\widehat{a}$= 0.2704 (0.0802)
$\widehat{b}$= 42.022 (3.3415)			
Kum	$\widehat{a}$= 2.194 (0.2223)	$\widehat{b}$= 3.436 (0.5820)	
EKum	$\widehat{\theta}$= 0.3361 (0.1446)	$\widehat{a}$= 5.315 (1.870)	$\widehat{b}$= 7.140 (3.092)

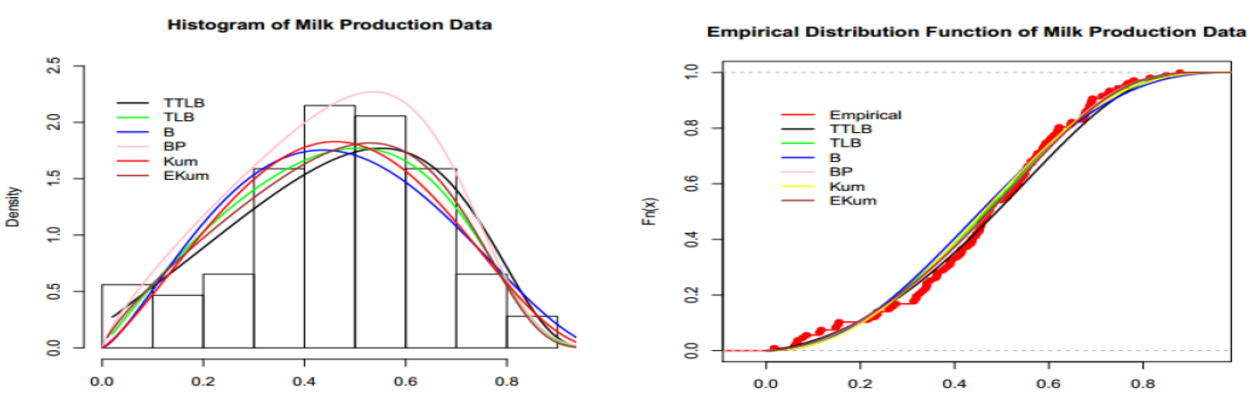

Figure 4: Histogram (left) and cdf (right) of the milk production data.

9 Conclusions

In this paper, we introduce a new family of continuous distributions called the transmuted Topp-Leone G family which extends the transmuted class pioneered by Shaw and Buckley (2007). Some of its mathematical properties including probability weighted moments, moments, generating functions, order statistics, incomplete moments, mean deviations, stress-strength model, moment of residual and reversed residual life are studied. Some useful characterization results based on two truncated moments as well as based on hazard function are presented. The maximum likelihood method is used to estimate its parameters. The Monte Carlo simulation is used for assessing the performance of the maximum likelihood method. The usefulness of the family model is illustrated by means of two real data set. The new famly is suitable for fitting different real data sets, as explained below:

1- The transmuted Topp-Leone Weibull model and the transmuted Topp-Leone Beta model are suitable for modelling unimodal and symmetric data sets.

2- It is better to use transmuted Topp-Leone G family (Weibull and Beta models case) is case of modelling big data sets.

3- For the bimodal data sets it is better to use transmuted Topp-Leone G family (Normal model case), as a future work we will consider the transmuted Topp-Leone normal (TTLN) distribution for modelling bimodal data sets and the bivariate version.

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