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The Exponentiated Weibull-Power Function Distribution

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Abstract: In this article, we introduce an extension referred to as the exponentiated Weibull power function distribution based on the exponentiated Weibull-G family of distributions. The proposed model serves as an extension of the two-parameter power function distribution as well as a generalization to the Weibull power function presented by Tahir et al. (2016 a). Various mathematical properties of the subject distribution are studied. General explicit expressions for the quantile function, expansion of density and distribution functions, moments, generating function, incomplete moments, conditional moments, residual life function, mean deviation, inequality measures, Rényi and q – entropies, probability weighted moments and order statistics are obtained. The estimation of the model parameters is discussed using maximum likelihood method. Finally, the practical importance of the proposed distribution is examined through three real data sets. It has been concluded that the new distribution works better than other competing models.

Key words: Exponentiated Weibull-G family, Power function distribution, Moments, Order statistics, Maximum likelihood estimation.

1. Introduction

Power function (PF) distribution arises in several scientific fields. It is a flexible life time distribution that may offer a suitable fit to some sets of failure data. The power function distribution is a special model from the uniform distribution. Dallas (1976) showed that the power function is the inverse of Pareto distribution, that is, if has the power function then has Pareto distribution. Meniconi and Barry (1996) mentioned in their studies that the power function distribution is preferred over exponential, lognormal and Weibull because it provides a better fit for failure data. More details on this distribution and its applications can be found in Ahsanullah and Lutful-Kabir (1974), Chang (2007) and Tavangar (2011).

The probability density function (pdf) of PF distribution is given by

$$g(x;\lambda,\theta)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}; \qquad 0<x<\lambda \quad ,\theta>0, \tag{1}$$

where, $\lambda > 0$ is the scale parameter and $\theta > 0$ is the shape parameter. The corresponding cumulative distribution function (cdf) is given by

$$G(x;\lambda,\theta)=\left(\frac{x}{\lambda}\right)^{\theta}. \qquad (2)$$

In the last few years, new generated families of continuous distributions have attracted several statisticians to develop new models as well as provide great flexibility in modelling real data. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. We list some of the generated families as follows: the beta-G by Eugene et al.(2002) and Jones (2004), gamma-G (type 1) by Zografos and Balakrishanan (2009), Kumaraswamy-G by Cordeiro and de Castro (2011), McDonald-G by Alexander et al. (2012), gamma-G (type 2) by Risti´c and Balakrishanan (2012), exponentiated generalized class by Cordeiro et al. (2013), transformed-transformer by Alzaatreh et al. (2013), Weibull- G by Bourguignon et al. (2014), the Kumaraswamy transmuted-G family by Afify et al. (2016), Kumaraswamy Weibull-G by Hassan and Elgarhy (2016a), exponentiated Weibull-G by Hassan and Elgarhy (2016 b) and additive Weibull-G by Hassan and Hemeda (2016), transmuted Weibull- G by Alizadeh et al. (2016), new Weibull-G by Tahir et al. (2016 b), new generalized Weibull-G by Cordeiro et al. (2015), beta Weibull-G by Yousof et al. (2017), generalized additive Weibull-G by Hassan et al. (2017) and generalized Marshall-Olkin Kumaraswamy-G family of distributions by Chakraborty and Handique (2017) among others.

Some recent extended distributions from power function have been studied by some authors. Beta power function distribution was presented by Cordeiro and Brito (2012) based on beta-G family. The Weibull power function was suggested by Tahir et al. (2016a) by using the Weibull-G family. Oguntunde et al. (2015) suggested the Kumaraswamy power distribution (KPF). Transmuted power function (TPF) was proposed by Haq et al. (2016). The exponentiated Kumaraswamy power function distribution was presented by Bursa and Kadilar (2017).

Based on exponentiated Weibull-generated (EW−G) presented by Hassan and Elgarhy (2016b), the cumulative distribution function of (EW−G) family is given by

$$F(x)=\left[1-\exp\left(-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right)\right]^{a}, \qquad (3)$$

where $\alpha, \beta > 0$ are the two shape parameters and $\alpha > 0$ is the scale parameter. The cdf (3) provides a wider family of continuous distributions. The pdf corresponding to (3) is given by

$$f(x)=\frac{a\alpha\beta(G(x))^{\beta-1}g(x)}{(1-G(x))^{\beta+1}}e^{-\alpha[\frac{G(x)}{1-G(x)}]^{\beta}}\left[1-\exp\left(-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right)\right]^{a-1}. \qquad (4)$$

In this study, we introduce a new five-parameter model as a competitive extension for the power function distribution using the EW−G family. The rest of the paper is outlined as follows. In Section 2, we introduce the exponentiated Weibull power function (EWPF) distribution. In Section 3, we derive a very useful representation for the EWPF density and distribution functions. In the same section, some general mathematical properties of the proposed distribution are derived. The maximum likelihood method is used to drive the estimates of the model parameters in Section 4. Simulation study is performed to obtain the maximum likelihood estimates of the model paramerets in Section 5. Applicability of the proposed model is shown and compared with other competing probability models in Sestion 6. At the end, concluding remarks are outlined.

2. The Exponentiated Weibull-Power Distribution

In this section, the five-parameter EWPF distribution is obtained based on the EW−G family.

The cdf of the exponentiated Weibull power function distribution, denoted by EWPF$(a,\alpha,\beta,\lambda,\theta)$ is obtained by inserting the cdf (2) in cdf (3) as follows

$$F(x;\Psi)=\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]^{a}; \quad a,\alpha,\beta,\lambda,\theta>0 \;,\; 0<x<\lambda, \tag{5}$$

where, $\Psi\equiv(a,\alpha,\beta,\lambda,\theta)$, is the set of parameters. The pdf of EWPF distribution is obtained by inserting the pdf (1) and cdf (2) into (4) as the following

$$f(x;\Psi)=\frac{a\alpha\beta\theta\lambda^{\theta}x^{\theta\beta-1}}{(\lambda^{\theta}-x^{\theta})^{\beta+1}}\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]^{a-1}. \tag{6}$$

Note that; for a=1, the pdf (6) reduces to Weibull power function which is carried out by Tahir et al. (2016a). The survival, hazard rate, reversed-hazard rate and cumulative hazard rate functions of EWPF distribution are respectively given by

$$R(x;\Psi)=1-\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]^{a},$$

$$h(x;\Psi)=\frac{a\alpha\beta\theta\lambda^{\theta}x^{\theta\beta-1}\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]^{a-1}}{(\lambda^{\theta}-x^{\theta})^{\beta+1}\left\{1-\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]^{a}\right\}},$$

$$\tau(x;\Psi)=\frac{a\alpha\beta\theta\lambda^{\theta}x^{\theta\beta-1}\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)}{(\lambda^{\theta}-x^{\theta})^{\beta+1}\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]},$$

and

$$H(x;\Psi)=-\ln\left|R(x;\Psi)\right|=-\ln\left|1-\left[1-\exp\left(-\alpha\left(\frac{x^{\theta}}{\lambda^{\theta}-x^{\theta}}\right)^{\beta}\right)\right]^{a}\right|.$$

Plots of the pdf and hazard rate function (hrf) of the EWPF are displayed in Figures 1 and 2 for selected parameter values. It is clear from Figure 1 that the EWPF densities take various shapes such as symmetrical, left-skewed, reversed-J, right skewed and unimodal. Also, Figure 2 indicates that EWPF hrfs can have increasing, decreasing, constant and U-shaped. This fact implies that the EWPF can be very useful for fitting data sets with various shapes.

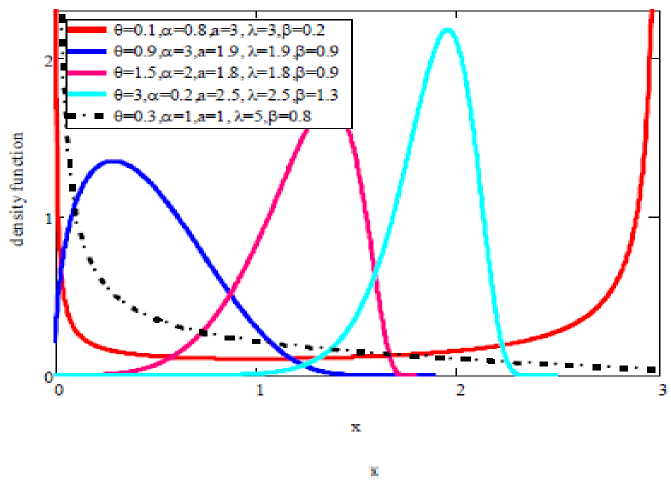


Figure 1 Plots of the pdf for some parameters

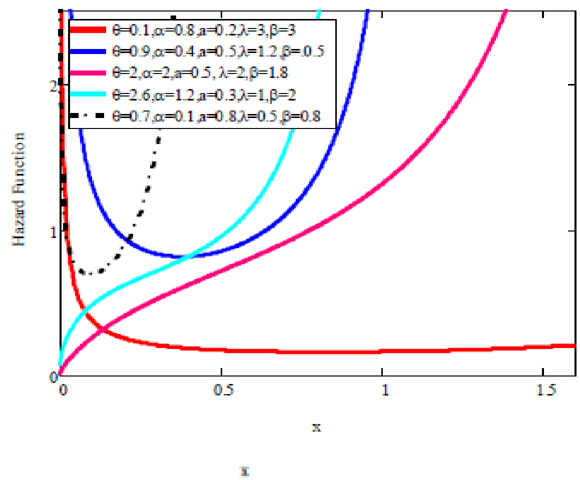


Figure 2 Plots of the hrf for some parameters

3. Statistical Properties

Here, we provide some properties of the EWPF distribution.

3.1 Important Expansions

In this subection, two usefuel representations of the pdf and cdf for exponentiated Weibull power function distribution are derived.

The pdf (6) can be rewritten as follows

$$f \qquad f(x;\Psi)=\frac{a\alpha\beta\theta}{\lambda\left(1-(\frac{x}{\lambda})^{\theta}\right)^{\beta+1}}(\frac{x}{\lambda})^{\theta\beta-1}e^{-\alpha\left[\frac{(\frac{x}{\lambda})^{\theta}}{1-(\frac{x}{\lambda})^{\theta}}\right]^{\beta}}\left[1-\exp\left(-\alpha\left[\frac{(\frac{x}{\lambda})^{\theta}}{1-(\frac{x}{\lambda})^{\theta}}\right]^{\beta}\right)\right]^{a-1}. \qquad (7)$$

The generalized binomial theorem, for β>0 is real non integer and $|Z|<1$ is given by

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}z^{i}. \qquad (8)$$

Then, by applying the binomial theorem (8) in (7), the probability density function of EWPF distribution where a is a real non integer becomes

$$f(x;\Psi)=\sum_{j=0}^{\infty}(-1)^{j}\binom{a-1}{j}\frac{a\alpha\beta\theta}{\lambda\left(1-(\frac{x}{\lambda})^{\theta}\right)^{\beta+1}}(\frac{x}{\lambda})^{\theta\beta-1}\exp\left(-\alpha(j+1)\left[\frac{(\frac{x}{\lambda})^{\theta}}{1-(\frac{x}{\lambda})^{\theta}}\right]^{\beta}\right). \qquad (9)$$

By using the power series for the exponential function and the following binomial expansion

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)z^{j}}{\Gamma(k)j!}, \qquad |z|<1, k>0. \qquad (10)$$

Hence the pdf of the EWPF distribution takes the following form

$$\left.\begin{aligned} f(x;\Psi)&=\sum_{i,j,k=0}^{\infty}w_{i,j,k}\,g_{\theta(\beta+\beta i+k)}(x),\\ w_{i,j,k}&=(-1)^{j+i}\binom{a-1}{j}\frac{a\alpha\beta(\alpha(j+1))^{i}\,\Gamma(\beta+\beta i+k)}{k!\,i!\,\Gamma(\beta+\beta i+1)},\end{aligned}\right\} \qquad (11)$$

Where $g_{\theta(\beta+\beta_i+k)}(x)$ denotes the pdf of PF distribution with parameters $\theta(\beta+\beta_i+k)$ and λ.

Furtheremore, an expansion for the cumulative distribution function is derived. Using binomial expansion for $(F(x;\Psi))^{s}$, where s is an integer and a is a real non integer, leads to:

$$(F(x;\Psi))^{s}=\sum_{p=0}^{\infty}(-1)^{p}\binom{as}{p}\exp\left(-\alpha p\left[\frac{(\frac{x}{\lambda})^{\theta}}{1-(\frac{x}{\lambda})^{\theta}}\right]^{\beta}\right).$$

Applying the power series for the exponential function and the relation (10) where β is real non integer, then the previous cdf takes the following form

$$(F(x;\Psi))^{s}=\sum_{p,q,m=0}^{\infty}(-1)^{p+q}\binom{as}{p}\frac{(\alpha p)^{q}\,\Gamma(m+\beta q)}{\Gamma(\beta q)m!}\left[(\frac{x}{\lambda})\right]^{\theta(\beta q+m)}.$$

Or, it can also be written as,

$$\left.\begin{aligned}(F(x;\Psi))^{s}&=\sum_{p,q,m=0}^{\infty}\eta_{p,q,m}\,G_{\theta(\beta q+m)}(x),\\ \eta_{p,q,m}&=(-1)^{p+q}\binom{as}{p}\frac{(\alpha p)^{q}\,\Gamma(m+\beta q)}{\Gamma(\beta q)m!}.\end{aligned}\right\} \qquad (12)$$

3.2 Quantile Function

The quantile function, say $x=Q(u)=F^{-1}(u)$ of X can be obtained by inverting (5) as follows

$$x = Q(u) = \lambda\left[1 + \{\frac{-1}{\alpha}\ln(1 - u^{\frac{1}{a}})\}^{\frac{-1}{\beta}}\right]^{\frac{-1}{\theta}}, \tag{13}$$

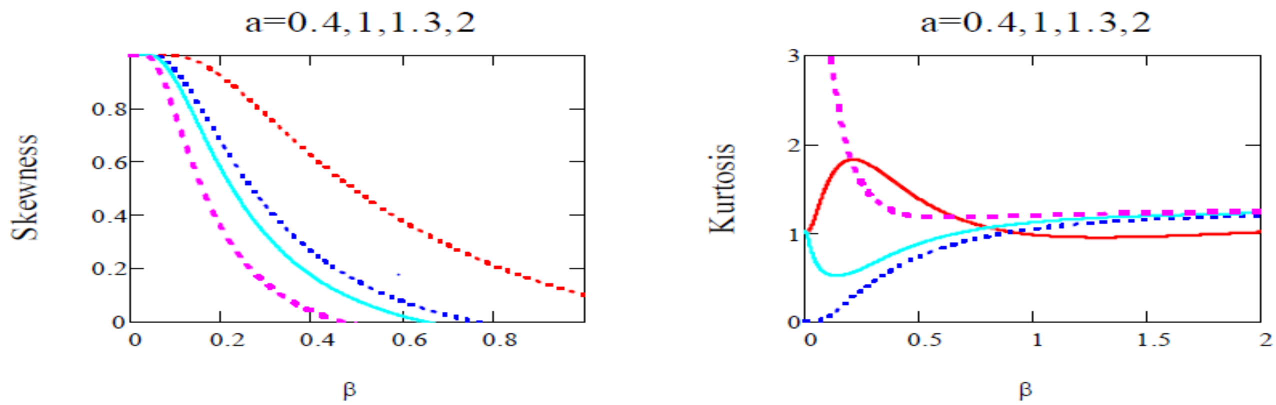
where, u is a uniform random variable on the unit interval (0,1). In particular, the first quartile, median and third quartile are obtained by subsituting u=0.25,0.5 and u=0.75 in (13).
The Bowley skewness introduced by Kenney and Keeping (1962) based on quartiles is given by

$$B = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

The Moors kurtosis (see Moors (1988)) based on octiles is given by

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

where Q(.) denotes the quantile function. Plots of the skewness and kurtosis for some choices of a as function of β are shown in Figure 3. The plots indicate the variability of these measures on the shape parameters.



(i) **(ii)**

Figure 3: Skewness (i) and kurtosis (ii) plots for EWPF distribution based on quantile function

3.3 Moments

This subsection concerns with the rth moment and the moment generating function for EWPF distribution.

If X has the pdf (11), then its rth moment can be obtained as follows

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\Psi)dx . \qquad (14)$$

Inserting (11) into (14) yields:

$$\mu_r' = E(X^r) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \int_0^{\lambda} x^r g_{\theta(\beta+\beta i+k)}(x)dx .$$

$$\mu_r' = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{\lambda^r \theta(\beta+\beta i+k)}{r+\theta(\beta+\beta i+k)}, \quad r=1,2,... \qquad (15)$$

Setting r=1,2,3,4 in (15), we can obtain the first four moments about zero. Generally, the moment generating function of EWPF distribution is obtained through the following relation

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{t^r}{r!} \frac{\lambda^r \theta(\beta+\beta i+k)}{r+\theta(\beta+\beta i+k)} .$$

3.4 Incomplete and Conditional Moments

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The incomplete moments, say $\varphi_s(t)$, is given by

$$\varphi_s(t) = \int_0^t x^s f(x;\Psi)dx .$$

Using (11), then $\varphi_s(t)$ can be written as follows

$$\varphi_s(t) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{\theta(\beta+\beta i+k)t^{s+\theta(\beta+\beta i+k)}}{\lambda^{\theta(\beta+\beta i+k)}(s+\theta(\beta+\beta i+k))}$$

Further, the conditional moments, say $r_s(t)$, is given by

$$\gamma_s(t) = \int_t^{\infty} x^s f(x;\Psi)dx .$$

Hence, by using pdf (11), we can write

$$\gamma_s(t) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \frac{\theta(\beta+\beta i+k)}{\lambda^{\theta(\beta+\beta i+k)}} \left[\frac{\lambda^{s+\theta(\beta+\beta i+k)} - t^{s+\theta(\beta+\beta i+k)}}{(s+\theta(\beta+\beta i+k))} \right].$$

Additionally, the mean deviation can be calculated by the following relation

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \qquad \text{and} \qquad \delta_2(X) = \mu - 2T(M),$$

where, $\mathrm{T}(\mathrm{q}) = \int_0^q x f(x) dx$ which is the first incomplete moment.

By using (11) then,

$$T(\mu)=\int_0^{\mu} xf(x)dx=\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\theta(\beta+\beta i+k)\,\mu^{\theta(\beta+\beta i+k)+1}}{\lambda^{\theta(\beta+\beta i+k)}(\theta(\beta+\beta i+k)+1)},$$

$$T(M)=\int_0^{M} xf(x)dx=\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\theta(\beta+\beta i+k)\,M^{\theta(\beta+\beta i+k)+1}}{\lambda^{\theta(\beta+\beta i+k)}(\theta(\beta+\beta i+k)+1)}.$$

3.5 Residual Life Function

The residual life plays an important role in life testing situations and reliability theory. The nth moment of the residual life is given by:

$$m_n(t)=E[(X-t)^n|X>t]=\frac{1}{R(t)}\int_t^{\infty}(x-t)^n f(x)dx. \quad (16)$$

The nth moment of the residual life of EWPF random variable is obtained by inserting pdf (11) in (16) as follows

$$m_n(t)=\frac{1}{R(t;\Psi)}\sum_{r=0}^{n}\sum_{i,j,k}^{\infty}(-1)^{n-r}w_{i,j,k}\binom{n}{r}t^{n-r}\int_t^{\lambda}x^r g_{\theta(\beta+\beta i+k)}(x)dx$$

So, the nth moment of the residual life of EWPF can be represented as follows

$$m_n(t)=\frac{\sum_{r=0}^{n}\sum_{i,j,k}^{\infty}(-1)^{n-r}w_{i,j,k}\binom{n}{r}t^{n-r}\frac{\theta(\beta+\beta i+k)}{\lambda^{\theta(\beta+\beta i+k)}}\left[\frac{\lambda^{r+\theta(\beta+\beta i+k)}-t^{r+\theta(\beta+\beta i+k)}}{r+\theta(\beta+\beta i+k)}\right]}{\left[1-\left[1-\exp(-\alpha(\frac{t^{\theta}}{\lambda^{\theta}-t^{\theta}})^{\beta})\right]^{a}\right]}. (17)$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age x defined by $m_1(t)=[(X-t)|X>t]$, which represents the expected additional life length for a unit which is alive at age x The MRL of the EWPF distribution can be obtained by setting n=1 in (17).

3.6 Inequality Measures

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution. Zenga curve was presented by Zenga (2007). Here, the Lorenz, Bonferroni and Zenga curves for the EWPF distribution are derived. The Lorenz, Bonferroni and Zenga curves are obtained, respectively, as follows

$$L_F(t)=\frac{\int_0^t xf(x)dx}{E(X)}=\frac{\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\theta(\beta+\beta i+k)t^{\theta(\beta+\beta i+k)+1}}{\lambda^{\theta(\beta+\beta i+k)}\left(\theta(\beta+\beta i+k)+1\right)}}{\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\lambda\theta(\beta+\beta i+k)}{\theta(\beta+\beta i+k)+1}}.$$

$$B_F(t)=\frac{\int_0^t xf(x)dx}{E(X)F(t)}=\frac{L_F(t)}{F(t)}=\frac{\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\theta(\beta+\beta i+k)t^{\theta(\beta+\beta i+k)+1}}{\lambda^{\theta(\beta+\beta i+k)}\left(\theta(\beta+\beta i+k)+1\right)}}{\left[1-\exp(-\alpha(\frac{t^{\theta}}{\lambda^{\theta}-t^{\theta}})^{\beta})\right]^{a}\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\lambda\theta(\beta+\beta i+k)}{\theta(\beta+\beta i+k)+1}}.$$

and

$$A_F(t)=1-\frac{\mu^-(t)}{\mu^+(t)},$$

where

$$\mu^-(t)=\frac{\int_0^t xf(x)dx}{E(X)}=\frac{\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\theta(\beta+\beta i+k)t^{\theta(\beta+\beta i+k)+1}}{\lambda^{\theta(\beta+\beta i+k)}\left(\theta(\beta+\beta i+k)+1\right)}}{\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\lambda\theta(\beta+\beta i+k)}{\theta(\beta+\beta i+k)+1}}.$$

And

$$\mu^+(t)=\frac{\int_t^{\infty} xf(x)dx}{1-F(t)}=\frac{\sum_{i,j,k=0}^{\infty} w_{i,j,k}\frac{\theta(\beta+\beta i+k)}{\lambda^{\theta(\beta+\beta i+k)}}\left[\frac{\lambda^{1+\theta(\beta+\beta i+k)}-t^{1+\theta(\beta+\beta i+k)}}{(1+\theta(\beta+\beta i+k))}\right]}{\left[1-\left[1-\exp(-\alpha(\frac{t^{\theta}}{\lambda^{\theta}-t^{\theta}})^{\beta})\right]^{a}\right]}.$$

3.7 Rényi and q – Entropies

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. The Rényi entropy is defined by

$$I_{\delta}(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty} f(x)^{\delta}dx, \quad \delta>0 \text{ and } \delta\neq 1.$$

By applying the binomial theory (8), (10) and exponential expansion, then the pdf $f(x;\Psi)^{s}$ can be expressed as follows

$$(f(x;\Psi))^{\delta}=\sum_{i,j,k=0}^{\infty}\xi_{i,j,k}(\frac{x}{\lambda})^{\delta(\theta\beta-1)+\theta\beta i+\theta k}.$$

where

$$\xi_{i,j,k} = (-1)^{i+j}\binom{\delta(a-1)}{i}\frac{(\alpha(\delta+i))^{j}\,(a\alpha\beta\theta)^{\delta}\,\Gamma(\delta(\beta+1)+\beta i+k)}{j!\,\lambda^{\delta}\,\Gamma(\delta(\beta+1)+\beta i)k!}.$$

Therefore, the Rényi entropy of EWPF distribution is given by

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\left[\sum_{i,j,k=0}^{\infty}\frac{\xi_{i,j,k}\,\lambda}{(\delta(\theta\beta-1)+\theta\beta i+\theta k+1)}\right].$$

The q- entropy is defined by

$$H_q(X) = \frac{1}{1-q}\log\left(1-\int_{-\infty}^{\infty} f(x;\Psi)^q\,dx\right), q>0 \text{ and } q\neq 1.$$

Therefore, the q- entropy of distribution is given by

$$H_q(X) = \frac{1}{1-q}\log\left\{1-\left[\sum_{i,j,k=0}^{\infty}\frac{\xi_{i,j,k}\,\lambda}{(q(\theta\beta-1)+\theta\beta i+\theta k+1)}\right]\right\}.$$

3.8 The Probability Weighted Moments

A general theory of probability weighted moments (PWMs) was initially introduced by Greenwood et al. (1979) to derive estimators of the parameters and quantiles of generalized distributions.

The probability weighted moments of a random variable X, say $\tau_{r,s}$ is formally defined by

$$\tau_{r,s} = E[X^r\,\mathrm{F}(x)^s] = \int_{-\infty}^{\infty} x^r\,\mathrm{f}(x)(\mathrm{F}(x))^s\,dx. \qquad (18)$$

Therefore, PWM of the EWPF distribution is obtained by inserting (11) and (12) into (18), as follows

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty} w_{i,j,k}\sum_{p,q,m=0}^{\infty}\eta_{p,q,m}\int_{0}^{\lambda}\frac{\theta(\beta+\beta i+k)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(\beta+\beta i+k+\beta q+m)-1}dx.$$

Hence, the PWM of exponentiated Weibull power function distribution takes the following form

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty}\sum_{p,q,m=0}^{\infty}\frac{\eta_{p,q,m}\,w_{i,j,k}\,(\beta+\beta i+k)}{(\beta+\beta i+k+\beta q+m)}.$$

3.9 Order Statistics

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n following the EWPF, the pdf of the rth order statistic is given by

$$f_{r:n}(x) = \frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}F(x)^{r+v-1}, \qquad (19)$$

where, B(.,.) is the beta function. Inserting (11) and (12) into (19) by replacing s with $v + r - 1$ leads to

$$\left.\begin{aligned} f_{r:n}(x) &= \frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i,j,k,p,q,m=0}^{\infty} \mathrm{K}_{i,j,k,p,q,m}\,\frac{\theta(\beta+\beta i+k)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(\beta+\beta i+k+\beta q+m)-1}, \\ \mathrm{K}_{i,j,k,p,q,m} &= (-1)^{v+p+q}\, w_{i,j,k}\binom{n-r}{v}\binom{a(v+r-1)}{p}\frac{(\alpha p)^q\,\Gamma(m+\beta q)}{\Gamma(\beta q)\,m!}. \end{aligned}\right\} \quad (20)$$

In particuler, the the pdf of the smallest order statistic is obtained by substituting r=1 in (20) as follows

$$f_{1:n} = n\sum_{v=0}^{n-1}\sum_{i,j,p,q,m=0}^{\infty} \pi_{i,j,k,p,q,m}\,\frac{\theta(\beta+\beta i+1)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(\beta+\beta i+k+\beta q+m)-1}$$

$$\pi_{i,j,k,p,q,m} = (-1)^{v+p+q}\binom{n-1}{v} w_{i,j,k}\binom{av}{p}\frac{(\alpha p)^q\,\Gamma(m+\beta q)}{\Gamma(\beta q)\,m!}$$

Further, the the pdf of the largest order statistic is obtained by subsituting in (20) as follows

$$f_{1:n} = n\sum_{i,j,p,q,m=0}^{\infty} \varpi_{i,j,k,p,q,m}\,\frac{\theta(\beta+\beta i+1)}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta(\beta+\beta i+k+\beta q+m)-1},$$

$$\varpi_{i,j,k,p,q,m} = (-1)^{p+q}\, w_{i,j,k}\binom{av}{p}\frac{(\alpha p)^q\,\Gamma(m+\beta q)}{\Gamma(\beta q)\,m!}.$$

4. Maximum Likelihood Estimation

The maximum likelihood estimators of the unknown parameters for the exponentiated Weibull power function distribution are determined based on complete samples. Let $X_1, \ldots, X_n$ be observed values from the EWPF distribution with set of parameters $\Psi \equiv (a, \alpha, \beta, \lambda, \theta)^T$. The total log-likelihood function for the vector of parameters Ψ can be expressed as

$$\begin{aligned} \ln L(\Psi) &= n\ln a + n\ln\alpha + n\ln\beta + n\ln\theta + n\theta\ln\lambda + (\theta\beta-1)\sum_{i=1}^{n}\ln x_i - (\beta+1)\sum_{i=1}^{n}\ln(\lambda^\theta - x_i^\theta) \\ &\quad -\alpha\sum_{i=1}^{n} z_i^{\beta} + (a-1)\sum_{i=1}^{n}\ln\left[1-\exp(-\alpha z_i^{\beta})\right], \end{aligned}$$

where, $Z_i = \left(\frac{x_i^\theta}{\lambda^\theta - x_i^\theta}\right)$. The λ is known and we estimate it from the sample maxima. The elements of the score function $U(\Psi) = (U_a, U_\alpha, U_\beta, U_\theta)$ are given by

$$U_a = \frac{n}{a} + \sum_{i=1}^{n}\ln\left[1-\exp(-\alpha z_i^{\beta})\right], \quad (21)$$

$$U_{\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}z_i^{\beta}+(a-1)\sum_{i=1}^{n}\frac{z_i^{\beta}\exp(-\alpha z_i^{\beta})]}{\left[1-\exp(-\alpha z_i^{\beta})\right]}, \qquad (22)$$

$$U_{\beta}=\frac{n}{\beta}+\theta\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\ln(\lambda^{\theta}-x_i^{\theta})-\alpha\sum_{i=1}^{n}z_i^{\beta}\ln z_i+\alpha(a-1)\sum_{i=1}^{n}\frac{z_i^{\beta}\exp\left[-\alpha z_i^{\beta}\right]\ln z_i}{1-\exp\left[-\alpha z_i^{\beta}\right]}, \quad (23)$$

and

$$U_{\theta}=\frac{n}{\theta}+n\ln\lambda+\beta\sum_{i=1}^{n}\ln x_i-(\beta+1)\sum_{i=1}^{n}\frac{\lambda^{\theta}\ln\lambda-x_i^{\theta}\ln x_i}{(\lambda^{\theta}-x_i^{\theta})}-\alpha\beta\sum_{i=1}^{n}z_i^{\beta-1}\left[\frac{\lambda^{\theta}x_i^{\theta}(\ln x_i-\ln\lambda)}{(\lambda^{\theta}-x_i^{\theta})^2}\right]$$
$$+(a-1)\sum_{i=1}^{n}\frac{\alpha\beta z_i^{\beta-1}\exp(-\alpha z_i^{\beta})}{\left[1-\exp(-\alpha z_i^{\beta})\right]}\left[\frac{\lambda^{\theta}x_i^{\theta}(\ln x_i-\ln\lambda)}{(\lambda^{\theta}-x_i^{\theta})^2}\right]. \qquad (24)$$

Then the maximum likelihood estimates (MLEs) of the parameters a, α, β and θ are obtained by setting equations (21 -24) to be zero and solving them numerically.

5. Simulation Study

In this section, a simulation study is carried out to evaluate the performance of the MLEs of the EWPF parameters with respect to sample size. The numerical procedures are described through the following algorithm.

Step(1): A random sample $X_1,\ldots,X_n$ of sizes n=(10, 20, 30, 50, 100, 200) are selected, these random samples are generated from the EWPF distribution.

Step(2): Assume that λ to be known and we will take it to be one in all experiments. Eight selected parameter combinations are considered as follows: set I (α=0.5, β=1, $\theta=1$, a=0.5), set II (α=1.5, β=1, $\theta=1.5$, a=0.5), set III (α=0.5, β=0.5, $\theta=1.5$, a=0.5), set IV (α=1.5, β=0.5, $\theta=0.5$ a=1), set V (α=0.5, β=0.5, $\theta=0.5$, a=0.5), set VI (α=0.5, β=0.5, $\theta=0.5$, a=1.5), set VII (α=0.5, β=0.5, $\theta=0.5$, a=1.5), set VIII (α=0.5, β=0.5, $\theta=1$, a=1.5).

Step (3): For each model parameters and for each sample size, the MLEs of α, β, θ and a are computed.

Step (4): Steps from 1 to 3 are repeated 1000 times for each sample size and for selected sets of parameters. Then, the MLEs of the parameters, their biases and standard errors (SE) are computed.

From simulation results, the following observations can be made:

1. The standard errors for each parameter generally decrease as the sample size increases for all set of parameters as shown in Figure (4).

2. The SEs of α for set I have the smallest values corresponding to the other set of parameters (see Figure 4(i)), the SEs of β for set VI have the smallest values corresponding to the other set of parameters (see Figure 4(ii)). Also, the SEs of θ for set II have the smallest values corresponding to the other set of parameters (see Figure 4(iii)), the SEs of a for set IV have the smallest values corresponding to the other set of parameters (see Figure 4(iv)).

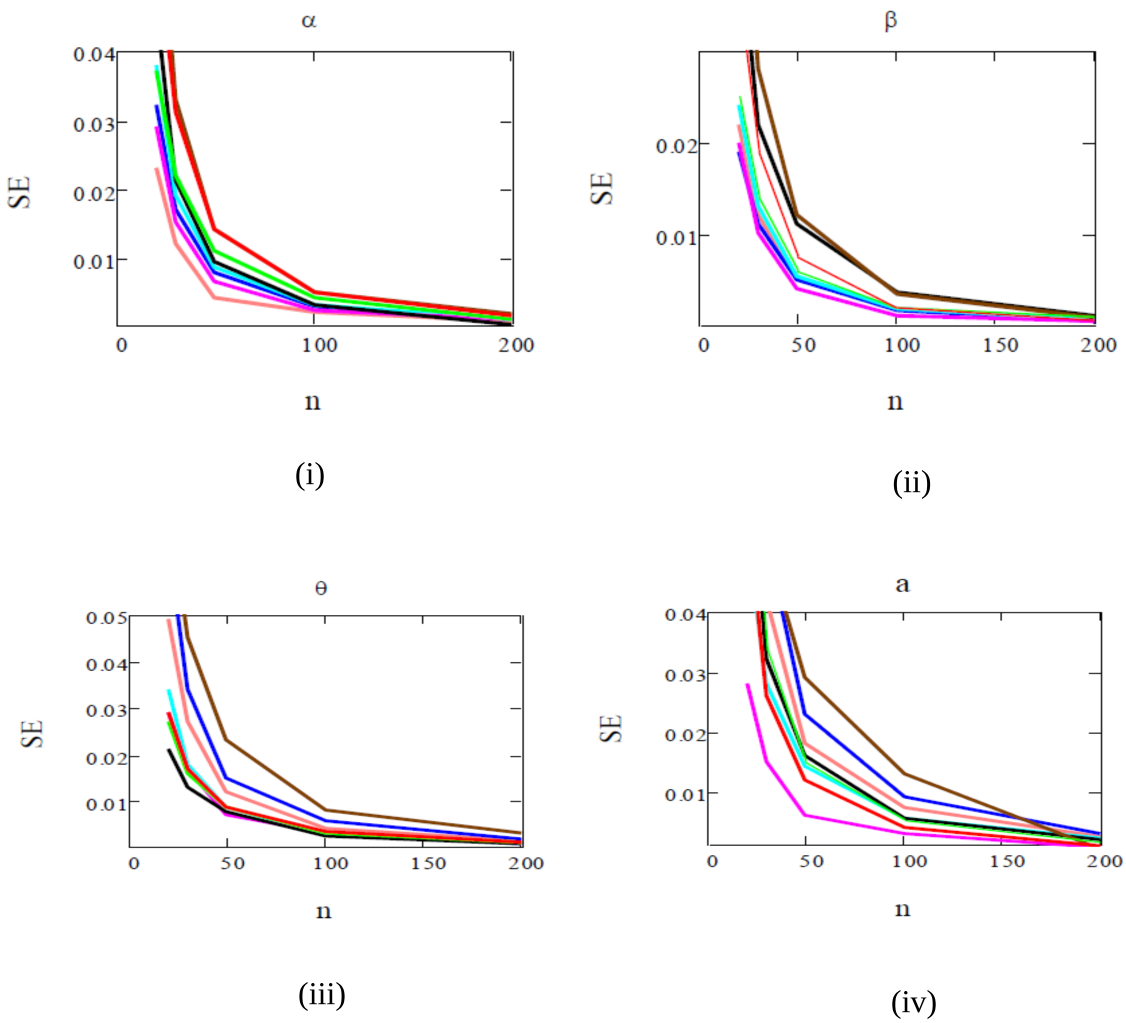


Figure 4: Standard errors of α, β, θ and a for eight set of parameters

3. Most of the estimates for different parameters are positively and negatively biased which indicates that the estimates are overestimate and underestimates respectively as shown in Figure 5. Generally, the biases for each parameter decrease as the sample size increases expect for few cases (see Figure 5).

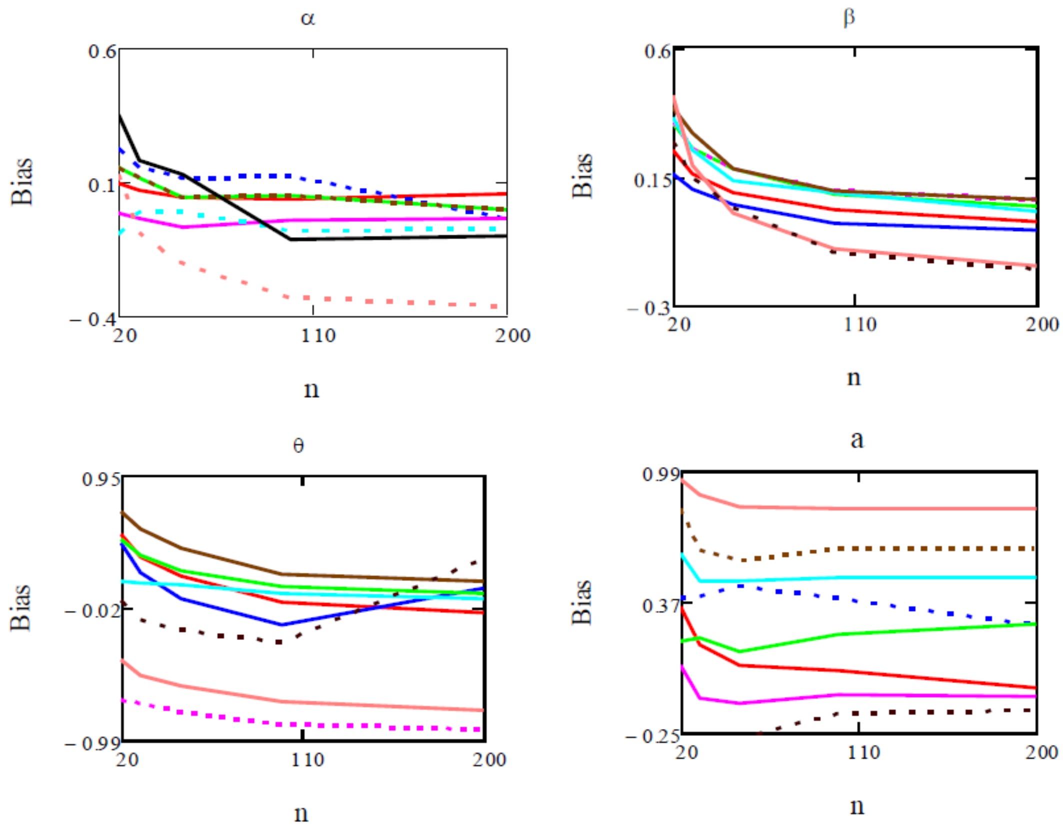


Figure 5: Biases of α, β, θ and a for eight set of parameters

4. The SEs for β in sets III, V, VI, VII and VIII take the smallest values compared to the corresponding SEs for θ, α, a ; while the SEs for θ in sets I, II, IV take the smallest values compared to the corresponding SEs for β, α, a (see Figure 6 as particular cases).

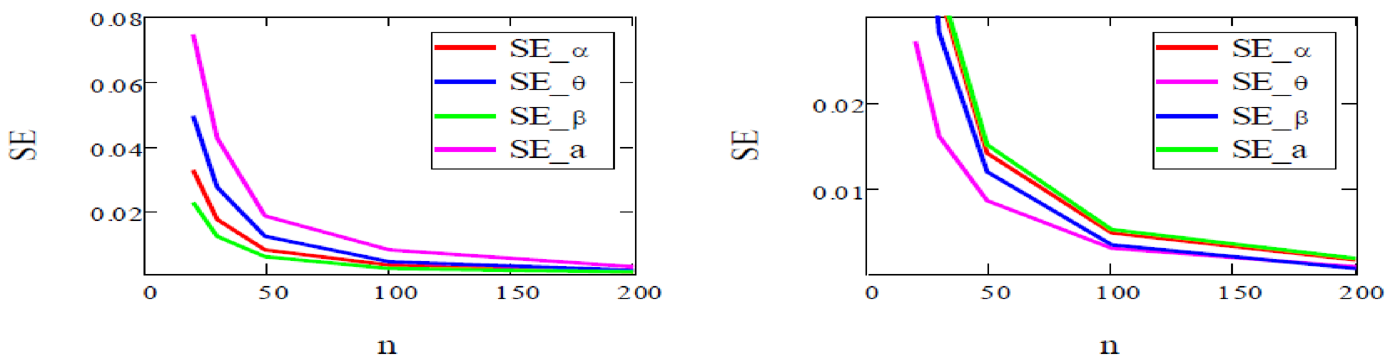


Figure 6: Standard errors of α, β, θ and a for set VI and set II

6. Data Analysis

In this section, three real data sets are analyzed to illustrate the merit of EWPF distribution compared with some other models; namely, Weibull power function (WPF), beta Weibull (BW), beta modified Weibull (BMW) (see Silva et al. (2010)), exponentiated generalized modified Weibull (EGMW) (see Aryal and Elbatal (2015)), Kumerswmay power function (KPF), PF and transmuated power function (TPF).

We obtain the MLE and their corresponding standard errors (in parentheses) of the model parameters. To compare the distribution models, we consider criteria like; Kolmogorov-Smirnov (K-S) statistic, Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). However, the better distribution corresponds to the smaller values of AIC BIC, CAIC, HQIC criteria and K-S statistic. Further, we plot the histogram and empirical cdfs for each data set and the estimated pdf of the EWPF, WPF, BMW, BW, EGMW, KPF, PF and TPF models.

6.1 Data Set 1: Acute Myelogenous data

The first data were first analyzed by Feigl and Zelen (1965). The data represent the survival times, in weeks, of 33 patients suffering from Acute Myelogenous Leukaemia. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3,8, 4, 3, 30, 4, 43.

Table 1 lists the values of MLEs of parameters and their standard errors (SE) in parenthesis. Table 2 lists the values of AIC, CAIC, BIC, HQIC and K-S.

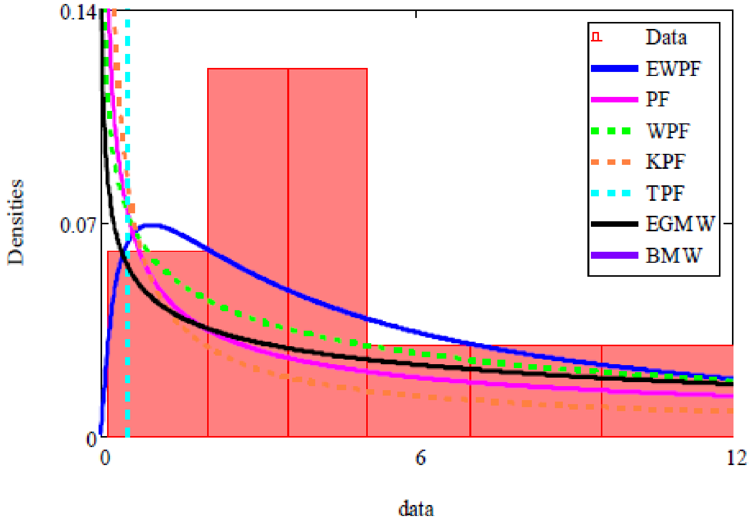
Models	Estimated Parameters							
	a	α	β			b		
EWPF	63.968 (0.0042)	6.082 (0.376)	0.083 (0.01)	156	1.681 (0.000088)	-	-	-
BMW	0.079 (0.1)	37.15 (46.903)		0.00006 (0.003)		0.883 (0.177)	0.001 (0.00014)	
EGMW	-	0.229 (0.008)	0.678 (0.0003)	1 (0.016)	0.082 (0.153)	-	-	.00031 (4.838)
WPF	2.681 (0.631)	156	4.391 (2.377)		4.391 (2.377)	0.16 (0.076)		
PF	-	0.449 (0.078)	-	156	-	-	-	-
KP	0.5 (0.1180)	0.392 (0.124)	-	156	-	0.392 (0.124)	-	-
TPF	-	0.817 (0.108)	156	-	1.194 (0.064)	-	-	-

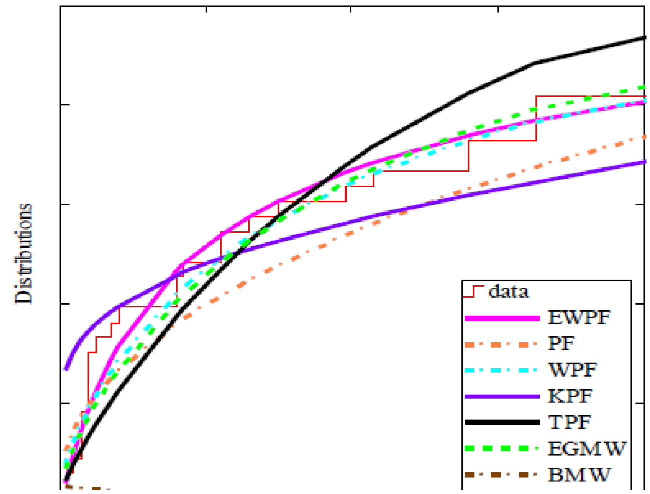
Models	***AIC***	***BIC***	***CAIC***	***HQIC***	***K-S***
EWPF	305.852	313.335	308.074	308.37	0.1199
BMW	318.967	326.449	321.189	321.484	0.939
EGMW	317.303	324.786	319.525	318.821	0.138
WPF	307.804	313.79	309.232	309.818	0.1214
PF	965.418	968.411	965.818	966.425	0.145
KP	329.734	335.72	331.162	331.748	0.2645
TPF	335.131	339.62	335.959	336.642	0.183

Table 1: MLEs and their SEs (in parentheses) for Acute Myelogenous data

Table 2: Statistics measures for Acute Myelogenous data

It is clear from Table 2 that the EWPF distribution provides a better fit than the other competitive models. It has the the smallest values for K-S, AIC, CAIC, BIC and HQIC among those considered here. Plots of the fitted densities and the histogram are given in Figure 7.





(a) Estimated pdfs (b) Estimated cdfs

Figure 7. Plots of the estimated pdfs and cdfs for the EWPF, BMW, EGMW, WPF, KPF, PF and TPF models for Acute Myelogenous data

6.2 Data Set 2: Actual Taxes data

The second real data have been used by Nassar and Nada (2011). The data represent the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The distribution is highly skewed to the right. The actual taxes revenue data (in 1000 million Egyptian pounds) are: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1,6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9,16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6,12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

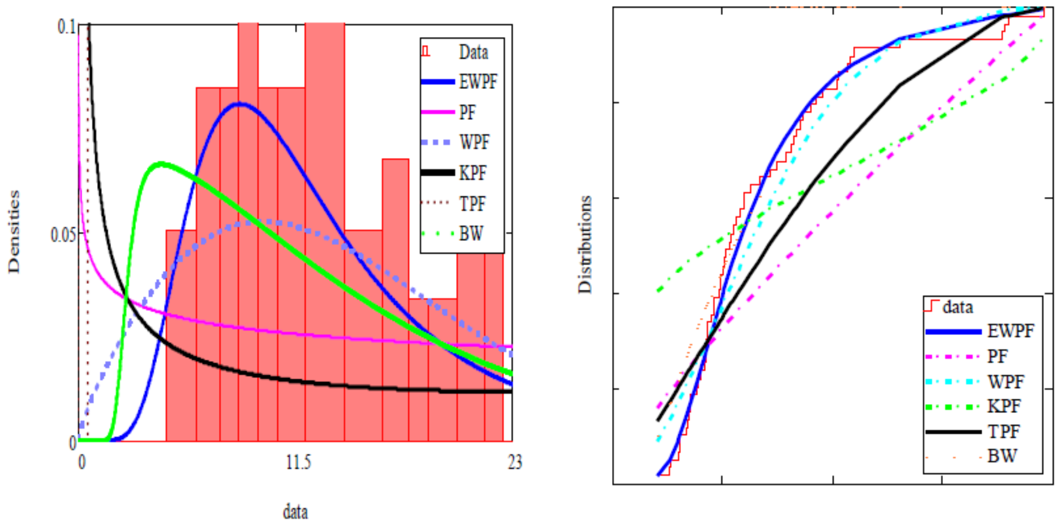
Table 3: MLEs and their SEs (in parentheses) for the Actual Taxes data

Models	Estimated Parameters						
	a	α	β			*b*	
EWPF	63.574 (0.00455)	8.364 (0.5902)	0.066 (0.00621)	39.2	7.377 (0.00019)	-	-
BW	55.574 (42.263)	1.391 (0.403)	-	-	-	0.031 (0.00561)	1.238 (0.222)
WPF	5.666 (1.036)	-	39.2	-	15.321 (10.509)	0.118 (0.078)	-
PF	-	0.81 (0.1055)	-	39.2	-	-	-
KPF	0.5 (0.09934)	0.44 (0.104)	-	39.2	-	0.44 (0.104)	-
TPF	-	1.172 (0.1194)	-	39.2	0.962 (0.093)	-	-

Table 4: Statistics measures for the Actual Taxes data.

Models	*AIC*	*BIC*	*CAIC*	*HQIC*	*K-S*
EWPF	383.784	394.172	384.916	387.84	0.08234
BW	401.277	409.588	402.018	404.521	5.212
WPF	401.037	409.347	401.778	404.28	0.14317
PF	12600	126600	126200	126300	0.304
KPF	499.825	508.135	500.565	503.06	0.4087
TPF	417.729	423.961	418.165	420.16	0.195

It is clear from Table 4 that the EWPF distribution provides a better fit than the other comptitive models. It has the the smallest K-S, AIC, CAIC, BIC and HQIC values among those considered here. Plots of the fitted densities and the histogram are given in Figure 8.



(a) Estimated pdfs (b) Estimated cdfs

Figure 8. Plots of the estimated pdfs and cdfs for the EWPF, BW, WPF, KPF, PF and TPF models for the Actual Taxes data

6.3 Data set 3: Failure Time Data

The third data set are provided in Murthy et al. (2004) about time between failures for 30 repairable items. The data are listed as the following: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

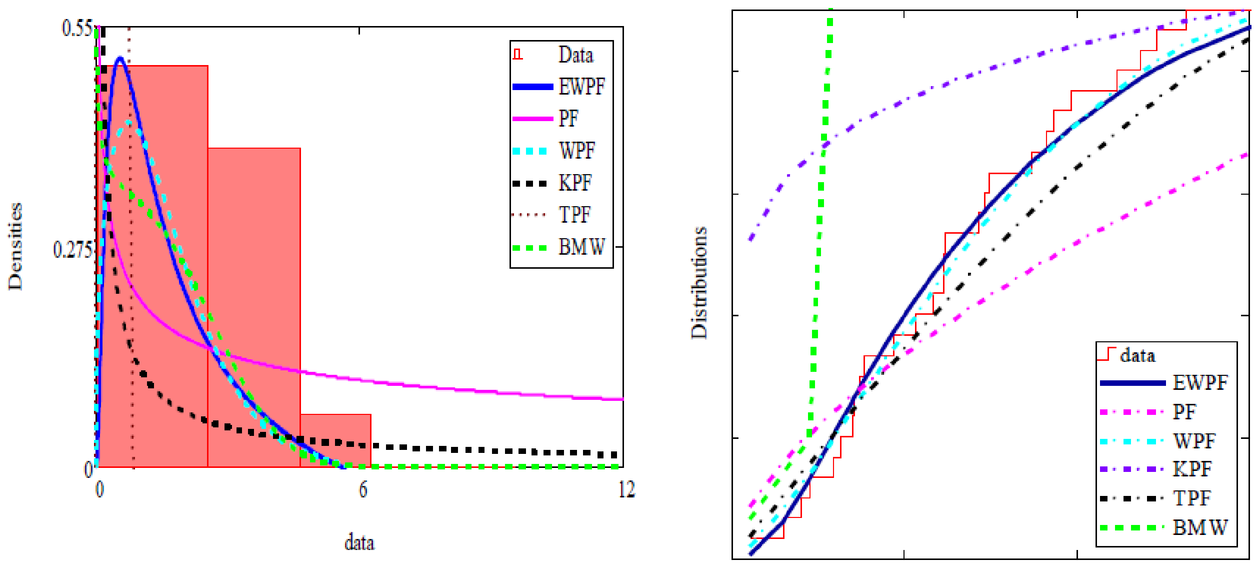
Table 5: MLEs and their SEs (in parentheses) for the failure time data

Models	***Estimated Parameters***						
	a	α	β			*b*	
EWPF	39.68 (0.02)	5.629 (0.3977)	0.248 (0.032)	4.73	1.033 (0.00012)		
BMW	0.539 (0.934)	50.552 (112.102)	-	0.274 (0.189)	-	772 (0.016)	0.857 (0.333)
WPF	5.636 (1.5679)	4.73	-	-	6.539 (6.845)	0.22 (0.221)	-
PF	-	0.631 (0.1151)	-	4.73	-		
KPF	1 (0.2939)	0.646 (0.177)	-	4.73	-	0.646 (0.177)	-
TPF	-	1.075 (0.1637)	-	4.73	1.44 (0.209)		

Table 6: Statistics measures for the failure time data

Models	***AIC***	***BIC***	***CAIC***	***HQIC***	***K-S***
EWPF	89.248	96.254	91.748	91.489	0.07328
BMW	94.406	101.412	96.906	96.647	0.86667
WPF	92.324	97.929	93.924	94.117	0.07705
PF	445.571	448.373	446.015	446.46	0.288
KPF	109.747	115.352	111.347	111.54	0.26293
TPF	101.458	105.662	102.381	102.80	0.139

As shown in Table 6; the EWPF distribution provides a better fit than the other competitive models, since it has the the smallest K-S, AIC, CAIC, BIC and HQIC values. Plots of the fitted densities and the histogram are given in Figure 9.



(a) Estimated pdfs (b) Estimated cdfs

Figure 9. Plots of the estimated pdfs and cdfs for the EWPF, BMW, WPF, KPF, PF and TPF models for failure time data

7. Concluding Remarks

In this paper, a new five-parameter, called the exponentiated Weibull power function distribution is introduced based on exponentiated Weibull-G family. The exponentiated Weibull power function distribution includes the Weibull power function distribution presented by Tahir et al. (2016 a). Some mathematical properties are derived. The maximum likelihood method is employed for estimating the model parameters. A simulation study is presented to evaluate the maximum likelihood estimates for model parameters. The practical importance of the EWPF distribution was demonstrated in three applications to show superior performance in comparison with several other former lifetime distributions. Applications showed that the

EWPF model can be performed better than the Weibull power function, power function, Kumerswmay power function, beta Weibull, beta modified Weibull, exponentiated generalized modified Weibull and the transmuted power function distributions.

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