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A NEW GENERALIZED CLASS OF LINEAR FAILURE RATE POWER SERIES DISTRIBUTION: MODEL, THEORY AND APPLICATION

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ABSTRACT

A new class of distributions called the beta linear failure rate power series (BLFRPS) distributions is introduced and discussed. This class of distributions contains new and existing sub-classes of distributions including the beta exponential power series (BEPS) distribution, beta Rayleigh power series (BRPS) distribution, generalized linear failure rate power series (GLFRPS) distribution, generalized Rayleigh power series (GRPS) distribution, generalized exponential power series (GEPS) distribution, Rayleigh power series (RPS) distributions, exponential power series (EPS) distributions, and linear failure rate power series (LFRPS) distribution of Mahmoudi and Jafari (2014). The special cases of the BLFRPS distribution include the beta linear failure rate Poisson (BLFRP) distribution, beta linear failure rate geometric (BLFRG) distribution of Oluyede, Elbatal and Huang (2014), beta linear failure rate binomial (BLFRB) distribution, and beta linear failure rate logarithmic (BLFRL) distribution. The BLFRL distribution is also discussed in details as a special case of the BLFRPS class of distributions. Its structural properties including moments, conditional moments, deviations, Lorenz and Bonferroni curves and entropy are derived and presented. Maximum likelihood estimation method is used for parameters estimation. Maximum likelihood estimation technique is used for parameter estimation followed by a Monte Carlo simulation study. Application of the model to a real dataset is presented.

KEYWORDS Power series distribution, Linear failure rate distribution, beta distribution, Poisson distribution, Maximum likelihood estimation.

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1. Introduction

In recent years, many techniques for the modeling lifetime data have been introduced. Beta-G family of distributions of Eugene et al. (2002) is one of the techniques used in modeling lifetime issues. These distributions were developed to model failure time of a system when the failure times of the components are independent and identically distributed random variable with a cumulative distribution function G. Moreover, due to the two additional shape parameters that have the ability to control skewness and kurtosis concurrently, and vary tail weight, the Beta-G distributions provide a reasonable parametric fit to real data. It plays a vital role in reliability analysis with its application in many real life situation including industrial, and biological studies. Its applications can be found in the studies of Eugene et al. (2002), Kong et al. (2007), Barreto-Souza et al. ((2010), (2011)), Cordeiro and Lemonte (2011a), Domma and Condino (2013), Adepoju et al. (2014), and Alshawarbeh et al. (2013), Leao et al. (2013), Oluyede and Yang (2015), Bidram et al. (2013), Zea et al. (2012), Percontini et al. (2013), Cintra et al. (2014), Paranaba et al. (2011), Cordeiro and Lemonte (2011b), Mahmoudi (2011), and Jafari and Mahmoudi (2015).

The cumulative distribution function (cdf) of the Beta-G distribution was proposed by Eugene et al. (2002) as

$$F(y; a, b, \psi) = \frac{B_{G(y;\psi)}(a, b)}{B(a, b)} = I_{G(y;\psi)}(a, b) \tag{1}$$

where $a > 0$, $b > 0$ are two additional parameters to introduce skewness and vary tail weight, $G(x;\ \psi)$ is an arbitrary baseline cdf, $\frac{B_{G(y;\psi)}(a,b)}{B(a,b)}$) is the incomplete beta function ratio, $B_{G(y;\psi)}(a, b) = \int_0^{G(y;\psi)} t^{a-1}(1-t)^{b-1}\,dt$ is the incomplete beta function and $B(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$. The derivative of the cdf in equation (1) gives its corresponding probability density function (pdf) as

$$f(y; a, b, \psi) = \frac{g(y; \psi)}{B(a, b)}[G(y; \psi)]^{a-1}[1 - G(y; \psi)]^{b-1} \tag{2}$$

In this paper, the arbitrary baseline cumulative distribution function (cdf) is the linear failure rate power series (LFRPS) distribution of Mahmoudi and Jafari (2014). The LFRPS class of distributions has the ability in fitting different hazard rate function such as increasing, decreasing, upside-down bathtub (unimodal), bathtub and increasing-decreasing-increasing (cubic) shaped. However, these class of distributions have the inability to fit skewed data. Therefore, combining this class of distribution with the beta function introduces a model that has the ability to fit skewed data. The introduction of the two shape parameters from the beta function gives the model the ability to control skewness and kurtosis simultaneously, and vary tail weight. The LFRPS distribution was used to model the lifetime in days of 40 patients suffering from leukemia from one of the Ministry of Health Hospitals in Saudi Arabia.

This paper is organized as follows. In section 2, we introduce the new model with its special case, their cdfs, pdfs, hazard and reverse hazard functions, sub-models, quantile function, density expansion, moments, conditional moments, Order Statistics and Rényi entropy are presented. The special case of the Beta Linear failure rate logarithmic distribution is presented in section 3. Parameter estimation and inference are given in section 4. Monte Carlo simulation study for the BLFRL distribution is presented in section 5 and an applications to real data is presented in sections 6. Concluding remark is given in section 7.

2. Beta Linear Failure Rate Power Series

Let $Y_1, Y_2, \ldots Y_N$ be independent and identically distributed (iid) random samples from a linear failure rate (LFR) distribution and N a discrete random variable from a Power series distribution given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1,2,\cdots, \tag{3}$$

where $C(\theta)$ is chosen such that $C(\theta)$ is finite and its first three derivatives with respect to θ exist, and $\theta \in (0,\infty)$. Then suppose that Y is a random variable representing a lifetime of a series-system of $N = n$ components, that is $Y = min\ [Y_1, Y_2, \ldots Y_N]$. It follows that the marginal cdf of Y is the compounding of the conditional cdf of Y given $N = n$ and the power series class of distributions. Hence the marginal cdf of Y is given by:

$$\begin{aligned} G_{LFRPS}(y;\alpha,\beta,\theta) &= \sum_{n=1}^{\infty} F_{Y\leq|N=n}(y) P(N = n) \\ &= \sum_{n=1}^{\infty} \left[1 - exp\left(-\alpha n y - \frac{\beta n}{2} y^2\right)\right] \frac{a_n \theta^n}{C(\theta)} \\ &= 1 - \frac{C\left(\theta e^{-\alpha y - \frac{\beta}{2} y^2}\right)}{C(\theta)} = 1 - \frac{C(\theta e^{-z})}{C(\theta)} \end{aligned}$$

where $z = -\alpha y - \frac{\beta}{2} y^2$, $\alpha > 0, \beta > 0, \theta > 0, y > 0$, and the corresponding pdf of the marginal cdf of Y is given by

$$g_{LFRPS}(y;\alpha,\beta,\theta) = \frac{dG_{LFRPS}(y;\alpha,\beta,\theta)}{dy} \theta(\alpha + \beta y) e^{-z} \frac{C(\theta e^{-z})}{C(\theta)}$$

Now taking the $G(y;\psi)$ to be the linear failure rate power series (LFRPS) distribution, that is the marginal cdf of Y, we obtain the cdf of the new class of distributions called the beta linear failure rate power series (BLFRPS) as

$$\begin{aligned} \mathrm{F}_{\mathrm{BLFRPS}}(\mathrm{y};\ \mathrm{a},\mathrm{b},\alpha,\beta,\theta) &= \frac{1}{B(a,b)}\int_0^{G_{LFRPS}(y;\alpha,\beta,\theta)} v^{a-1}(1-v)^{b-1}dv \\ &= I_{\left[1-\frac{C(\theta e^{-z})}{C(\theta)}\right]}(a,b) \end{aligned} \tag{4}$$

for $y > 0,\ a > 0,\ b > 0,\ \alpha > 0,\ \beta > 0$, and $\theta \in (0, s)$, and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is a complete beta function. If a random variable Y has the beta linear failure rate power series distribution, we write $Y \sim BLFRPS(a,\ b,\ \alpha,\ \beta,\ \theta)$. The corresponding pdf of this new distribution is given by

$$f(y;\ a,b,\alpha,\beta,\theta) = \frac{\theta(\alpha+\beta y)e^{-z}C'(\theta e^{-z})}{B(a,b)C(\theta)}\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{a-1}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{b-1}, \tag{5}$$

for $y > 0$, where $a > 0$, and $b > 0$ are shape parameters, $\alpha > 0,\ \beta > 0$ and $\theta > 0$, are scale parameters, and $C'(\theta) = \frac{d}{d\theta}C(\theta)$. The reliability function, hazard and reverse hazard functions of the BLFRPS distribution are given by

$$\bar{F}_{BLRFPS}(y;a,b,\alpha,\beta,\theta) = I_{[1-G_{LFRPS}(y;\alpha,\beta,\theta)]}(b,a) = I_{\left[\frac{C(\theta e^{-z})}{C(\theta)}\right]}(b,a),$$

$$\begin{aligned} h_{BLRFPS}(y;a,b,\alpha,\beta,\theta) &= \frac{f_{BLRFPS}(y;a,b,\alpha,\beta,\theta)}{\bar{F}_{BLRFPS}(y;a,b,\alpha,\beta,\theta)} \\ &= \frac{\theta(\alpha+\beta y)e^{-z}C'(\theta e^{-z})}{B(a,b)C(\theta)I_{\left[\frac{C(\theta e^{-z})}{C(\theta)}\right]}(b,a)}\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{a-1}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{b-1}, \end{aligned}$$

and

$$\begin{aligned} \tau_{BLRFPS}(y;a,b,\alpha,\beta,\theta) &= \frac{f_{BLRFPS}(y;a,b,\alpha,\beta,\theta)}{F_{BLRFPS}(y;a,b,\alpha,\beta,\theta)} \\ &= \frac{\theta(\alpha+\beta y)e^{-z}C'(\theta e^{-z})}{B(a,b)C(\theta)I_{\left[1-\frac{C(\theta e^{-z})}{C(\theta)}\right]}(a,b)}\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{a-1}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{b-1}, \end{aligned}$$

respectively.

2.1. Sub-Classes and Sub-Models

In this section, we look at the sub-class of distributions of the BLFRPS distribution and the sub-models of the special case BLFRL distribution. The new class of distri- butions and its special case BLFRL distribution have new and well known sub-classes of distributions and sub-models. The results are summarized in Table 2.

Table 1. Sub-Classes and Sub-Models of the BLFRPS and BLFRL Distributions

Condition	BLFRPS Sub-Classes	BLFRL Sub-Models
$\beta = 0$	Beta exponential power series (BEPS)	Beta exponential logarithmic (BEL)
$\alpha = 0$	Beta Rayleigh power series (BRPS)	Beta Rayleigh logarithmic (BRL)
$a = b = 1$	Linear failure rate power series (LFRPS)	Linear failure rate logarithmic (LFRL)
$a = b = 1, \beta = 0$	Exponential power series (EPS)	Exponential logarithmic (EL)
$a = b = 1, \alpha = 0$	Rayleigh power series (RPS)	Rayleigh logarithmic (RL)

2.2. Expansion of the Distribution

Applying the following binomial series expansion, that is, b is a positive real non-integer and $|v| < 1$, then

$$(1-v)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} v^j, \tag{6}$$

to obtain a series expansion of the BLFRPS cdf as follows:

$$\begin{aligned} F_{BLRFPS}(y; a, b, \alpha, \beta, \theta) &= \frac{1}{B(a,b)} \int_0^{G_{LFRPS}(y)} v^{a-1}(1-v)^{b-1} dv \\ &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} \int_0^{G_{LFRPS}(y)} v^{a+j-1} dv \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!\,(a+j)} \left(G_{LFRPS}(y)\right)^{a+j} \\ &= \sum_{j=0}^{\infty} \omega_j\, F_{GLRFPS}(y;\ \alpha, \beta, \theta, a+j) \end{aligned} \tag{7}$$

where $\omega_j = \frac{(-1)^j \Gamma(b)}{\Gamma(a)\Gamma(b-j)j!(a+j)}$, and $F_{GLRFPS}(y;\ \alpha, \beta, \theta, a+j)$ is the generalized linear failure rate power series (GLFRPS) distribution with parameters α, β, θ, and $a + j$.

Alternatively, for easy computation of model properties these pdfs can be expanded as follows

$$f_{BLRFPS}(y; a, b, \alpha, \beta, \theta) = \frac{\theta(\alpha + \beta y)e^{-z} C(\theta e^{-z})}{B(a,b) e^{-z} C(\theta)} \sum_{j=0}^{\infty} \binom{a-1}{j} \left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{b+j-1}, \tag{8}$$

where the expansion

$$\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{a-1} = \sum_{j=0}^{\infty}(-1)^j\binom{a-1}{j}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^j,$$

was used. Note that $C(\theta e^{-z}) = \sum_{n=1}^{\infty} na_n(\theta e^{-z})^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(\theta e^{-z})^n = \sum_{n=0}^{\infty} b_n(\theta e^{-z})^n$, where $b_n = (n+1)a_{n+1}$. Also,$C(\theta e^{-z})^k = \sum_{m=0}^{\infty} e_{m,k}(\theta e^{-z})^m$, where $k = b+j-1$,by using the result on power series raised to a positive integer, that is $(\sum_{m=0}^{\infty} a_s y^s)^k = \sum_{m=0}^{\infty} e_{s,k} y^s$, where $e_{s,k} = (sa_0)^{-1}\sum_{l=1}^{s}[k(l+1)-s]a_l e_{s-1,k}$, and $e_{0,k} = {a_0}^k$.Using the results above, the BLERPS pdf in eqution(8) can be rewritten as follows:

$$f_{BLRFPS}(y;a,b,\alpha,\beta,\theta) = \frac{(\alpha+\beta y)}{B(a,b)}\sum_{j,m,n=0}^{\infty}\frac{(-1)^j b_n e_{m,k}}{(C(\theta))^{b+j}}\binom{a-1}{j}\theta^{n+m+1}(e^{-z})^{n+m+1}$$

$$= \frac{(\alpha+\beta y)}{B(a,b)}\sum_{j,m,n=0}^{\infty}\frac{(-1)^j b_n e_{m,k}}{(C(\theta))^{b+j}}\binom{a-1}{j}\times\frac{(n+m+1)}{(n+m+1)}\theta^{n+m+1}(e^{-z})^{n+m+1}$$

$$= \frac{1}{B(a,b)}\sum_{j,m,n=0}^{\infty}\frac{(-1)^j b_n e_{m,k}\theta^{n+m+1}}{(C(\theta))^{b+j}(n+m+1)}\binom{a-1}{j}$$
$$\times (n+m+1)(\alpha+\beta y)(e^{-z})^{n+m+1}$$
$$= \sum_{j,m,n=0}^{\infty}\omega(j,m,n,a,b,\theta)(n+m+1)(\alpha+\beta y)(e^{-z})^{n+m+1} \tag{9}$$

where

$$\omega(j,m,n,a,b,\theta) = \frac{(-1)^j b_n e_{m,k}\theta^{n+m+1}}{(C(\theta))^{b+j}(n+m+1)}\binom{a-1}{j}.$$

Simplifying further, we have

$$f_{BLRFPS}(y;a,b,\alpha,\beta,\theta) = \frac{1}{B(a,b)}\sum_{i,j,m,n=0}^{\infty}\frac{(-1)^{j+i} b_n e_{m,k}\theta^{n+m+1}\beta^i}{(C(\theta))^{b+j}(n+m+1)2^i i!}\binom{a-1}{j}$$
$$\times (n+m+1)(\alpha+\beta y)y^{2i}e^{-\alpha(n+m+1)y}$$
$$= \sum_{i,j,m,n=0}^{\infty}\omega(i,j,m,n,a,b,\beta,\theta)(n+m+1)$$
$$\times (\alpha+\beta y)y^{2i}e^{-\alpha(n+m+1)y} \tag{10}$$

where

$$\omega(i,j,m,n,a,b,\beta,\theta) = \frac{(-1)^{j+i} b_n e_{m,k} \theta^{n+m+1} \beta^i}{\left(C(\theta)\right)^{b+j} (n+m+1) 2^i i!} \binom{a-1}{j} \tag{11}$$

2.3. Quantile Function

The quantile function of the BLFRPS distribution is given by

$$y = \frac{-\alpha + \sqrt{\alpha^2 - 2\beta \log\left(\frac{1}{\theta} c^{-1}\left(\left(1 - I_q^{-1}(a,b)\right) C(\theta)\right)\right)}}{\beta}$$

for $\beta > 0$. Note that we have to solve the equation

$$I_{\left[1-\frac{C(\theta e^{-z})}{C(\theta)}\right]}(a,b) = q,$$

$0 < q < 1$, to obtain the BLFRPS quantile function. That is,

$$1 - \frac{C(\theta e^{-z})}{C(\theta)} = I_q^{-1}(a,b),$$

so that,

$$\left(1 - I_q^{-1}(a,b)\right) C(\theta) = C(\theta e^{-z}),$$

Now,

$$c^{-1}\left[\left(1 - I_q^{-1}(a,b)\right) C(\theta)\right] = \theta e^{-z},$$

and

$$\frac{1}{\theta} c^{-1}\left[\left(1 - I_q^{-1}(a,b)\right) C(\theta)\right] = \exp\left(-\alpha y - \frac{\beta}{2} y^2\right).$$

Therefore,

$$\frac{\beta}{2} y^2 + \alpha y + \log\left(\frac{1}{\theta} c^{-1}\left[\left(1 - I_q^{-1}(a,b)\right) C(\theta)\right]\right) = 0.$$

Consequently,

$$y = \frac{-\alpha + \sqrt{\alpha^2 - 2\beta log(\frac{1}{\theta} c^{-1}\left(\left(1 - I_q^{-1}(a,b)\right) C(\theta)\right))}}{\beta}$$

for $\beta > 0$, is the quantile function for a general case of any distribution in the class of the BLFRPS distribution.

2.4. Moments

In the subsection, we present the moments of the BLFRPS class of distributions.The r^{th} moment of the BLFRPS class of distributions is given by

$$\mathrm{E}(Y^r) = \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta)[(n+m+1)\,\Gamma(r+2i+1) \times (\frac{\alpha^2(n+m+1)+\beta(r+2i+1)}{(\alpha(n+m+1))^{r+2i+2}})] \tag{12}$$

Note that using equation (10), we have

$$\mathrm{E}(Y^r) = \int_0^{\infty} y^r f_{BLRFPS}(y;a,b,\alpha,\beta,\theta)\,dy$$

$$= \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta)(n+m+1)\int_0^{\infty} y^{r+2i}(\alpha+\beta y)e^{-\alpha(n+m+1)y}\,dy$$

$$= \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta)(n+m+1) \times \left\{\left[\frac{\alpha(n+m+1)\Gamma(r+2i+1)}{(\alpha(n+m+1))^{r+2i+1}}\right]+\left[\frac{\beta(n+m+1)\Gamma(r+2i+2)}{(\alpha(n+m+1))^{r+2i+2}}\right]\right\}$$

$$= \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta) \times \left[(\mathrm{n}+\mathrm{m}+1)\Gamma(r+2i+1)\left(\frac{\alpha^2(n+m+1)+\beta(r+2i+1)}{\left(\alpha(n+m+1)\right)^{r+2i+2}}\right)\right]$$

$$
\begin{aligned}
\mathrm{E}(Y^r) &= \int_0^\infty y^r f_{BLRFPS}(y; a, b, \alpha, \beta, \theta)\, dy \\
&= \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta)(n+m+1) \int_0^\infty y^{r+2i}(\alpha \\
&\qquad + \beta y) e^{-\alpha(n+m+1)y}\, dy \\
&= \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta)(n+m+1) \\
&\times \left\{ \left[\frac{\alpha(n+m+1)\Gamma(r+2i+1)}{\left(\alpha(n+m+1)\right)^{r+2i+1}} \right] + \left[\frac{\beta(n+m+1)\Gamma(r+2i+2)}{\left(\alpha(n+m+1)\right)^{r+2i+2}} \right] \right\} \\
&= \sum_{i,j,m,n=0}^{\infty} \omega(i,j,m,n,a,b,\beta,\theta) \\
&\times \left[(\mathrm{n}+\mathrm{m}+1)\Gamma(r+2i+1) \left(\frac{\alpha^2(n+m+1) + \beta(r+2i+1)}{\left(\alpha(n+m+1)\right)^{r+2i+2}} \right) \right]
\end{aligned} \tag{13}
$$

2.5. Conditional Moments

Another useful quantity in survival analysis is the rth conditional moments. The conditional moment is used to calculate the mean residual function and other useful quantites in reliability and survival analysis. The rth conditional moment of the BLFRPS distribution is given by

$$
\begin{aligned}
E(Y^r | Y > t) &= \frac{1}{\bar{F}_{BLFRPS}(t)} \int_t^\infty y^r f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy \\
&= \sum_{i,j,m,n=0}^{\infty} \frac{\omega(i,j,m,n,a,b,\beta,\theta)(n+m+1)}{\left(\alpha(n+m+1)\right)^{r+2i+2} \bar{F}_{BLFRPS}(t)} \\
&\times \{ \left[\alpha^2(n+m+1)\Gamma\left((r+2i+1), \alpha(n+m+1)t\right) \right] \\
&+ \left[\beta\Gamma\left((r+2i+2), \alpha(n+m+1)t\right) \right] \}
\end{aligned} \tag{14}
$$

2.6. Mean Deviations, Bonferroni and Lorenz Curves

2.6.1. Mean Deviations

The mean deviation about the mean, and about the median are quantities that measures dispersion of observations around the mean and median. This quantities are given by

$$
\delta_1(Y) = \int_0^\infty |y - \mu|\, f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy \tag{15}
$$

and

$$\delta_2(Y) = \int_0^\infty |y - M|\, f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy \tag{16}$$

respectively, where $\mu = E(Y)$ is given by equation (12) when $r = 1$ and M denotes the median of Y . Note that

$$\delta_1(Y) = 2\mu F_{BLFRPS}(\mu; a, b, \alpha, \beta, \theta) - 2\mu + 2\int_\mu^\infty y f_{BLFRPS}(\mu; a, b, \alpha, \beta, \theta) dy, \tag{17}$$

and

$$\delta_2(Y) = -\mu + 2\int_M^\infty y f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy, \tag{18}$$

where

$$\int_\mu^\infty y\, f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy = \sum_{i.j.m.n=0}^{\infty} \frac{\omega(i, j, m, n, a, b, \beta, \theta)(n+m+1)}{\left(\alpha(n+m+1)\right)^{2i+3}} \times \{[\alpha^2(n+m+1)\Gamma(2i+2), \alpha(n+m+1)\mu] + \left[\beta\Gamma\left((2i+3), \alpha(n+m+1)\mu\right)\right]\} \tag{19}$$

2.6.2. Bonferroni and Lorenz Curves

Lorenz curve, Lorenz (1905), and Bonferroni curve, Bonferroni (1930) are graphical methods used to measure inequality. In this section, Bonferroni and Lorenz curves are presented. Bonferroni and Lorenz curves of a random variable Y with a BLFRPS distribution are given by

$$B(p) = \frac{1}{pE(Y)}\int_0^q y f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy = \frac{1}{p\mu}\sum_{i.j.m.n=0}^{\infty} \frac{\omega(i, j, m, n, a, b, \beta, \theta)(n+m+1)}{\left(\alpha(n+m+1)\right)^{2i+3}} \times \left\{[\alpha^2(n+m+1)\gamma(2i+2), \alpha(n+m+1)q] + \left[\beta\gamma\left((2i+3), \alpha(n+m+1)q\right)\right]\right\} \tag{20}$$

and

$$L(p) = \frac{1}{E(Y)}\int_0^q y f_{BLFRPS}(y; a, b, \alpha, \beta, \theta) dy = \frac{1}{\mu}\sum_{i.j.m.n=0}^{\infty} \frac{\omega(i, j, m, n, a, b, \beta, \theta)(n+m+1)}{\left(\alpha(n+m+1)\right)^{2i+3}} \times \left\{[\alpha^2(n+m+1)\gamma(2i+2), \alpha(n+m+1)g] + \left[\beta\gamma\left((2i+3), \alpha(n+m+1)q\right)\right]\right\} \tag{21}$$

respectively, where $\gamma(a, y) = \int_0^y t^{a-1} e^{-t} dt$ is the lower incomplete gamma function.

2.7. Distribution of Order Statistics

The order statistics of the sample from a BLFRPS class of distributions is defined by the arrangement of $Y_1, Y_2, ..., Y_n$ from the smallest to largest denoted by $Y_{1:n}, Y_{2:n}, ..., Y_{n:n}$. Then the pdf of the j[th] order statistics, that is, $Y_j = Y_{j:n}$, for j = 1, 2, ...n, is given by

$$
\begin{aligned}
f_j(y_j) &= \frac{n!}{(j-1)!\,(n-j)!}[F_{BLFRPS}(y)]^{j-1}[1-F_{BLFRPS}(y)]^{n-j} f_{BLFRPS}(y) \\
&= \frac{n!}{(j-1)!\,(n-j)!}\left[\frac{B_{G_{LFRPS}}(y_i)(a,b)}{B(a,b)}\right]^{j-1}\left[1-\frac{B_{G_{LFRPS}}(y_i)(a,b)}{B(a,b)}\right]^{n-j} \\
&\times \frac{\theta(\alpha+\beta y)e^{-z}c'(\theta e^{-z})}{B(a,b)C(\theta)}\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{a-1}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{b-1} \\
&= \sum_{p=0}^{n-j}\frac{(-1)^p n!}{(j-1)!\,(n-j)!}\binom{n-j}{p}\left[\frac{B_{G_{LFRPS}}(y_i)(a,b)}{B(a,b)}\right]^{j+p-1} \\
&\times \frac{\theta(\alpha+\beta y)e^{-z}c'(\theta e^{-z})}{B(a,b)C(\theta)}\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{a-1}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{b-1} \\
&= \sum_{p=0}^{n-j}\binom{n-j}{p}\frac{(-1)^P n!\left[B_{G_{LFRPS}}(y_i)(a,b)\right]^{j+p-1}\theta(\alpha+\beta y_i)}{[B(a,b)]^{j+p}(j-1)!\,(n-j)!\,[C(\theta)]^{a+b-1}} \\
&\times\ e^{-z}c'(\theta e^{-z})[C(\theta)-C(\theta e^{-z})]^{a-1}[C(\theta e^{-z})]^{b-1},
\end{aligned}
$$

where $z=\alpha y_i+\frac{\beta}{2}{y_j}^2$. The corresponding cdf of Y_j is given by

$$
\begin{aligned}
f_j(y_j) &= \sum_{k=j}^{n}\binom{n}{k}[F_k(y_j)]^k[1-F_j(y_k)]^{n-k} \\
&= \sum_{k=j}^{n}\binom{n}{k}\left[\frac{B_{G_{LFRPS}}(y_i)(a,b)}{B(a,b)}\right]^{k}\left[1-\frac{B_{G_{LFRPS}}(y_i)(a,b)}{B(a,b)}\right]^{n-k} \\
&= \sum_{k=j}^{n}\sum_{p=0}^{n-k}\binom{n}{k}\binom{n-k}{p}(-1)^p\left[\frac{B_{G_{LFRPS}}(y_i)(a,b)}{B(a,b)}\right]^{k+p}
\end{aligned}
$$

2.8. Rényi Entropy

Rényi entropy quantifies the amount of uncertainty or randomness and is given by

$$H_\xi(f_{BLFRPS}(y)) = \frac{1}{1-\xi}\ln\left[\int_0^\infty f^{\xi}{}_{BLFRPS}(y)dy\right],\quad \xi>0, \xi\neq 1, \tag{22}$$

where

$$f^{\xi}{}_{BLFRPS}(y) = \left(\frac{\theta(\alpha+\beta y)e^{-z}c'(\theta e^{-z})}{B(a,b)C(\theta)}\right)^{\xi}\left(1-\frac{C(\theta e^{-z})}{C(\theta)}\right)^{\xi(a-1)}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{\xi(b-1)}$$

$$= \frac{\theta^{\xi}(\alpha+\beta y)^{\xi}(e^{-z})^{\xi}\left(c'(\theta e^{-z})\right)^{\xi}}{B^{\xi}(a,b)\left(C(\theta)\right)^{\xi}}\sum_{j=0}^{\infty}(-1)^{j}\binom{\xi(a-1)}{j}\left(\frac{C(\theta e^{-z})}{C(\theta)}\right)^{\xi(b-1)+j}$$

$$= \sum_{j,m,n=0}^{\infty}\binom{\xi(a-1)}{j}\frac{(-1)^{j}{b_n}^{\xi}e_{m,k}\theta^{m+\xi(n+1)}}{B^{\xi}(a,b)\left(C(\theta)\right)^{\xi}}(\alpha+\beta y)^{\xi}(e^{-z})^{m+\xi(n+1)}$$

$$= \sum_{i,j,m,n=0}^{\infty}\sum_{p=0}^{\xi}\binom{\xi(a-1)}{j}\binom{\xi}{p}\frac{(-1)^{j}{b_n}^{\xi}e_{m,k}(m+\xi(n+1))\theta^{m+\xi(n+1)}}{B^{\xi}(a,b)\left(C(\theta)\right)^{\xi b+j}2^{i}i!}$$

$$\times\, \alpha^{\xi-p}\beta^{p+i}y^{2i+p}e^{-\alpha y(m+\xi(n+1))}$$

by using the expansion for the BLFRPS pdf, and the fact that

$$\int_0^{\infty} f^{\xi}{}_{BLFRPS}(y)dy = \sum_{i,j,m,n=0}^{\infty}\sum_{p=0}^{\xi}\binom{\xi(a-1)}{j}\binom{\xi}{p}$$

$$\times\, \frac{(-1)^{j}{b_n}^{\xi}e_{m,k}\big(m+\xi(n+1)\big)\theta^{m+\xi(n+1)}\alpha^{\xi-p}\beta^{p+i}}{B^{\xi}(a,b)\left(C(\theta)\right)^{\xi b+j}2^{i}i!}$$

$$\times\left(\frac{\Gamma(2i+p+1)}{\left[\alpha\big(m+\xi(n+1)\big)\right]^{2i+p+1}}\right)$$

we have

$$H_{\xi}\big(f_{BLFRPS}(y)\big) = \frac{1}{1-\xi}\ln(\sum_{i,j,m,n=0}^{\infty}\sum_{p=0}^{\xi}\binom{\xi(a-1)}{j}\binom{\xi}{p}$$

$$\times\, \frac{(-1)^{j}{b_n}^{\xi}e_{m,k}(m+\xi(n+1))\theta^{m+\xi(n+1)}\alpha^{\xi-p}\beta^{p+i}}{B^{\xi}(a,b)\left(C(\theta)\right)^{\xi b+j}2^{i}i!}$$

$$\times\left(\frac{\Gamma(2i+p+1)}{\left[\alpha\big(m+\xi(n+1)\big)\right]^{2i+p+1}}\right))$$

3. BLFRL Distribution

In this section, we present results on the special case of BLFRPS distribution in detail. In this setting, $C(\theta) = -\log(1-\theta)$ and $a_n = n^{-1}$, $\theta \in (0, 1)$. The BLFRL cdf given by

$$F_{BLFRL}(y; a, b, \alpha, \beta, \theta) = \frac{1}{B(a,b)} \int_0^{G_{LFRL}(y;,\alpha,\beta,\theta)} v^{a-1}(1-v)^{b-1} dv$$
$$= I_{\left[1-\frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right]}(a,b) \tag{23}$$

for y > 0. The corresponding pdf, survival, and hazard functions are given by

$$f_{BLFRL}(y; a, b, \alpha, \beta, \theta) = \frac{\theta(\alpha+\beta y)e^{-z}}{B(a,b)\left[\left(-\log(1-\theta)\,(1-\theta e^{-z})\right)\right]}$$
$$\times \left(1 - \frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right)^{a-1} \left(\frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right)^{b-1}, \tag{24}$$

$$\bar{F}_{BLFRL}(y; a, b, \alpha, \beta, \theta) = I_{[1-G_{LFRL}(y;,\alpha,\beta,\theta)]}(b,a)$$
$$= I_{\left[\frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right]}(b,a) \tag{25}$$

and

$$h_{BLFRL}(y; a, b, \alpha, \beta, \theta) = \frac{f_{BLFRL}(y; a, b, \alpha, \beta, \theta)}{\bar{F}_{BLFRL}(y; a, b, \alpha, \beta, \theta)}$$
$$= \frac{\theta(\alpha+\beta y)e^{-z}}{B(a,b)\left[\left(-\log(1-\theta)\,(1-\theta e^{-z})\right)\right] I_{\left[1-\frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right]}(b,a)}$$
$$\times \left(1 - \frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right)^{a-1} \left(\frac{\log(1-\theta e^{-z})}{\log(1-\theta)}\right)^{b-1}. \tag{26}$$

respectively. For the selected values of the model parameter, we present graphs of the BLFRL density function in Figure 1. The shapes of the graphs exhibit skewness, L-shape, uni-modal to point out a few, for the selected values of the model parameters. The shapes of the hazard graphs produced by the BLFRL distribution are also presented. The graphs of the hazard rate function are given in Figure 2. The plot shows various shapes including monotonically increasing, monotonically decreasing and bathtub shapes for the five combinations of the values of the parameters. This flexibility makes the BLFRL hazard rate function suitable for both monotonic and non-monotonic empirical hazard behaviors that are likely to be encountered in real life situations.

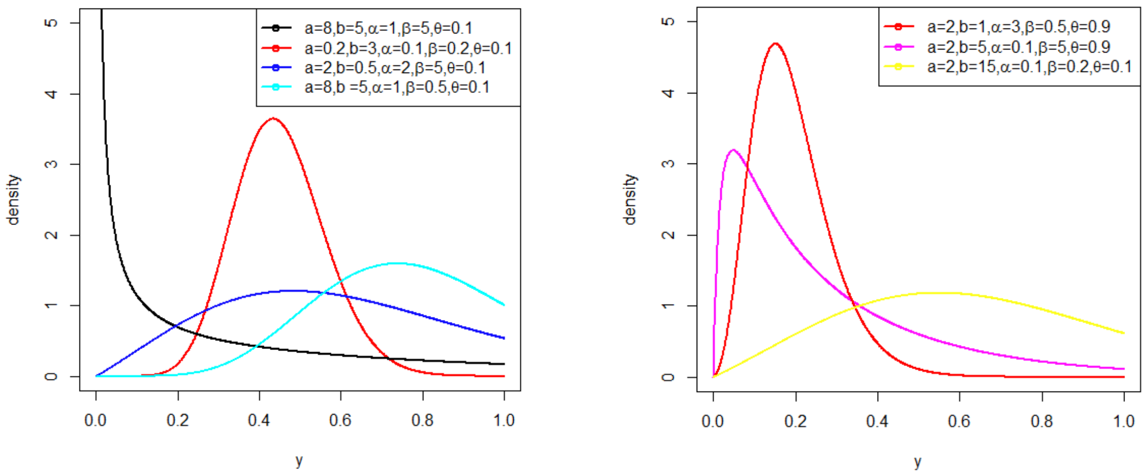

Figure 1. Plots of BLFRL Density Function

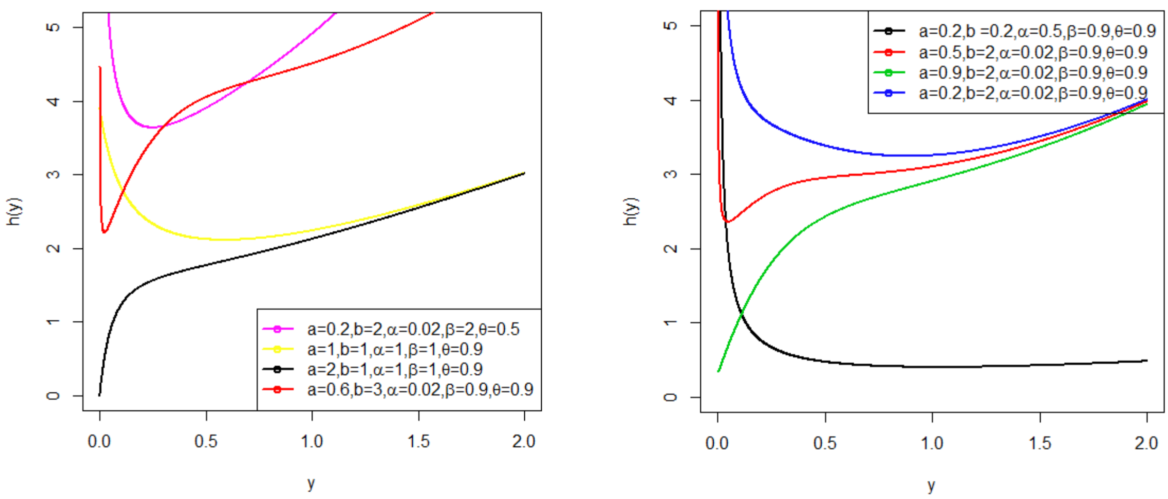

Figure 2. Plots of BLFRL Hazard Function

3.1. BLFRL Series Expansion

Note that with $C(\theta) = -log(1-\theta)$ in equation (4), the cdf of the BLFRL distribution can be written as

$$F_{BLFRL}(y; a, b, \alpha, \beta, \theta) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j! (a+j)} \left(1 - \frac{log(1-\theta e^{-z})}{log(1-\theta)}\right)^{a+j}$$

$$= \sum_{j=0}^{\infty} \omega_j F_{GLFRL}(y; \alpha, \beta, \theta, a+j), \qquad (27)$$

where $F_{GLFRL}(y; \alpha, \beta, \theta, a+j)$ is the generalized linear failure rate logarithmic (GLFRL) distribution with parameters *α, β, θ* and $a + j$. The corresponding pdf is given by

$$f_{BLFRL}(y; a, b, \alpha, \beta, \theta) = \sum_{j=0}^{\infty} \omega_j \, F_{GLFRL}(y; \alpha, \beta, \theta, a + j). \tag{28}$$

respectively. It follows that BLFRL distribution can be written as a linear combinations of the generalized linear failure rate logarithmic distribution. Hence the mathematical and statistical properties of the BLFRL distribution follows directly from those of the GLFRL distribution.

3.2. Quantile Function

Now, for *C(θ) = − log(1 − θ)*, the quantile function from equation (12) is given by

$$y = \frac{-\alpha + \sqrt{\alpha^2 - 2\beta \log\left[\frac{1}{\theta}\left(1 - exp\left(log(1-\theta) - \left[I_q{}^{-1}(a,b)[log(1-\theta)]\right]\right)\right)\right]}}{\beta} \tag{29}$$

Random numbers can now be readily generated from the BLFRL distribution. Table 2 presents the deciles generated by equation (29). We consider set of parameter values *I = (b = 0.1, α = β = 0.3, θ = 0.2)*, *II = (b = 1.0, α = β = 0.3, θ = 0.2)* and *III = (b = 1.5, α = β = 0.3, θ = 0.2)*, for different values of the parameter a. The shape parameter a varies from values *a = 0.1, 1.0* and *1.5* and b is held constant at *b = 0.1, 1.0* and *1.5*.

Table 2. Quantiles for Selected Values of Model Parameter

	I			II			III		
u	a=0.1	a=1.0	a=1.5	a=0.1	a=1.0	a=1.5	a=0.1	a=1.0	a=1.5
0.1	0.000000	1.748039	2.262965	0.000000	0.276968	0.569281	0.000000	0.191481	0.421985
0.2	0.000270	2.902045	3.3444412	0.000000	0.530959	0.882484	0.000000	0.375678	0.664722
0.3	0.15442	3.906052	4.280405	0.000020	0.778128	1.153917	0.000010	0.559821	0.877886
0.4	0.241139	4.859143	5.179657	0.000312	1.028996	1.413693	0.000175	0.749995	1.083383
0.5	1.293348	5.817666	6.095736	0.002903	1.293348	1.677872	0.001625	0.952883	1.293348
0.6	3.081593	6.833015	7.076310	0.017892	1.583395	1.961049	0.010051	1.177647	1.519166
0.7	4.942685	7.974122	8.187258	0.082029	1.918630	2.283241	0.046703	1.439582	1.776772
0.8	6.893964	9.371397	9.556360	0.296223	2.338833	2.683068	0.175172	1.770530	2.097166
0.9	9.417482	11.400632	11.555738	0.899198	2.961669	3.273022	0.570626	2.265474	2.570961

3.3. Moments

The r[th] moment of the BLFRL distribution is given by

$$E(Y^r) = \sum_{j,k,n=0}^{\infty} \sum_{i=0}^{a+j-1} \omega(i,j,k,n,a,b,\beta,\theta)$$

$$\times \left[(i(k+1)+n+1)\Gamma(r+2p+1) \left(\frac{\alpha^2(i(k+1)+n+1)+\beta(r+2p+1)}{\left(\alpha(i(k+1)+n+1)\right)^{r+2p+2}} \right) \right] \quad (30)$$

Moments are useful in calculating the standard deviation $SD = \sqrt{\mu_2 - \mu_1^1}$, coeffcient of variation $CV = \sqrt{\frac{\mu_2}{\mu_1^2} - 1}$, coeffcient of skewnwss $CS = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1{}^3}{[\mu_2 - \mu_1^2]^{\frac{3}{2}}}$ and coeffcient of kurtosis $CK = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1{}^4}{[\mu_2 - \mu_1^2]^2}$. For specified values of the parameter model, we obtain the first six moments and the SD, CV, CS, and CK. The results are given in Tables 3.

Table 3. Moments and Related Statistics

	$b = 0.1, \alpha = \beta = 0.3, \theta = 0.2$				$a = 0.1, \alpha = \beta = 0.3, \theta = 0.2$			
$\mu_r{}'$	a=0.1	a=0.2	a=0.3	a=0.4	b=0.1	b=0.2	b=0.3	b=0.4
E(X)	0.0430578	0.0777184	0.1083593	0.1366191	0.0430578	0.0586423	0.0753213	0.0931584
$E(X^2)$	0.0275633	0.0529725	0.0773937	0.1013534	0.0275633	0.0443274	0.0642599	0.0876957
$E(X^3)$	0.0251819	0.0504964	0.0763499	0.1029436	0.0251819	0.0480014	0.0787419	0.1188079
$E(X^4)$	0.0300542	0.0621738	0.0965290	0.1332228	0.0300542	0.0680208	0.1264063	0.2110455
$E(X^5)$	0.0443705	0.0940436	0.1491681	0.2098667	0.0443705	0.1193370	0.2513721	0.4645881
$E(X^6)$	0.0781247	0.1688837	0.2726914	0.3899655	0.0781247	0.2498147	0.5966282	1.2209344
SD	0.1603414	0.2166388	0.2562265	0.2875563	0.1603414	0.2022091	0.240467	0.2811001
CV	3.7238608	2.7874843	2.3645996	2.1048033	3.7238608	3.4481799	3.2135242	3.0174423
CS	5.2837488	3.8441080	3.1944037	2.7968739	5.2837488	4.9112493	4.5890684	4.3182559
CK	39.3562766	21.9218694	15.8867651	12.7638725	39.3562766	34.4765136	30.5249234	27.40577484

3.4. Conditional Moments

The r[th] conditional moment for the BLFRL distribution is given by

$$E(Y^r|Y>t)=\frac{1}{\bar{F}_{BLFRL}(t)}\int_t^\infty y^r f_{BLFRL}(y;a,b,\alpha,\beta,\theta)dy$$

$$=\sum_{j,k,n=0}^{\infty}\sum_{i=0}^{a+j-1}\frac{\omega(i,j,k,n,p,a,b,\beta,\theta)(i(k+1)+n+1)}{\left(\alpha(i(k+1)+n+1)\right)^{r+2p+2}\bar{F}_{BLFRL}(t)}$$
$$\times\{\left[\alpha^2(i(k+1)+n+1)\Gamma\left((r+2p+1),\alpha(i(k+1)+n+1)t\right)\right]$$
$$+\left[\beta\Gamma\left((r+2p+2),\alpha(i(k+1)+n+1)t\right)\right]\}. \tag{31}$$

The corresponding mean residual life function is given by $E(Y^r|Y>t)-t$.

3.5. Order Statistics and Rényi Entropy

3.5.1. Order Statistics

The pdf of the j[th] order statistics from the BLFRL distribution given by

$$f_j(y_j)=\sum_{p=0}^{n-j}\binom{n-j}{p}\frac{(-1)^p n!\left[B_{G_{LFRL}(y_i)}(a,b)\right]^{j+p-1}\theta(\alpha+\beta y)e^{-z}}{[B(a,b)]^{j+p}(j-1)!\,(n-j)!\,(-log(1-\theta))(1-\theta e^{-z})}$$
$$\times\left[1-\frac{log(1-\theta e^{-z})}{log(1-\theta)}\right]^{a-1}\left[\frac{log(1-\theta e^{-z})}{log(1-\theta)}\right]^{b-1} \tag{32}$$

3.5.2. Rényi Entropy

Rényi entropy for the BLFRL distribution is given as

$$H_\xi\left(f_{BLFRL}(y)\right)=\frac{1}{1-\xi}\ln(\sum_{j,k,n=0}^{\infty}\sum_{i=0}^{\xi(a-1)+j}\sum_{p=0}^{\xi}\binom{\xi(a-1)+j}{i}\binom{\xi}{p}$$
$$\times\left[\frac{(-1)^{j+i}\Gamma(\xi(b-1)+1)\Gamma(\xi+n)}{\Gamma(\xi(b-1)+1-j)\Gamma(\xi)2^m m!\,n!\,j!}\right]$$
$$\times\left[\frac{\theta^{i(k+1)+\xi+n}\alpha^{\xi-p}(\beta)^{p+m}[i(k+1+\xi+n)]^m}{B^\xi(a,b)(-log(1-\theta))^{-(\xi+i)}(k+1)^i}\right]$$
$$\times\left(\frac{\Gamma(2m+p+1)}{[i(k+1)+\xi+n]^{2m+p+1}}\right)).$$

4. Estimation and Inference in the BLFRL Distribution

Let $Y_1, Y_2, \ldots, Y_n$ be random samples from the BLFRL distribution with parameter vector $\psi = (a, b, \alpha, \beta, \theta)$. The log-likelihood function is given by

$$\ln L(\psi) = n\ln\theta \sum_{i=1}^{n} \ln(\alpha + \beta y_i) - \sum_{i=1}^{n} z_i + n\ln\Gamma(a+b) - n\ln\Gamma(a)$$

$$-n\ln\Gamma(b) + n\ln(\log(1-\theta)) - \sum_{i=1}^{n} \ln(1-\theta e^{-z_i})$$

$$+(a-1)\sum_{i=1}^{n} \ln\left(\frac{\log(1-\theta) - \log(1-\theta e^{-z_i})}{\log(1-\theta)}\right) + (b-1)\sum_{i=1}^{n} \ln\left(\frac{\log(1-\theta e^{-z_i})}{\log(1-\theta)}\right)$$

where $z_i = \alpha y_i + \frac{\beta}{2} {y_i}^2$. The MLEs of a, b, α, β, and θ, denoted by $\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, and\ \hat{\theta}$ respectively, can be obtained by solving the non linear equations given by setting the elements of the score vector

$$U_n(\psi) = \left(\frac{\partial lnL(\psi)}{\partial a}, \frac{\partial lnL(\psi)}{\partial b}, \frac{\partial \ln \mathrm{L}(\psi)}{\partial \alpha} \frac{\partial \ln \mathrm{L}(\psi)}{\partial \beta}, \frac{\partial \ln \mathrm{L}(\psi)}{\partial \theta}\right)$$

to zero, where

$$\frac{\partial lnL(\psi)}{\partial a} = n\psi(a+b) - n\psi(a) + \sum_{i=1}^{n} ln\left(\frac{log(1-\theta) - log(1-\theta e^{-z_i})}{log(1-\theta)}\right), \quad (33)$$

$$\frac{\partial lnL(\psi)}{\partial b} = n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n} ln\left(\frac{log(1-\theta e^{-z_i})}{log(1-\theta)}\right), \quad (34)$$

$$\frac{\partial\, ln\, L(\psi)}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\alpha + \beta y_i} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \frac{y_i e^{-z_i}}{(1-e^{-z_i})}$$

$$-(a-1)\sum_{i=1}^{n} \frac{y_i \theta e^{-z_i}}{[log(1-\theta) - log(1-\theta e^{-z_i})][(1-e^{-z_i})]}$$

$$+(b-1)\sum_{i=1}^{n} \frac{y_i \theta e^{-z_i}}{[log(1-\theta e^{-z_i})][(1-e^{-z_i})]} \quad (35)$$

$$\frac{\partial \ln \mathrm{L}(\psi)}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{\alpha + \beta y_i} - \frac{1}{2}\sum_{i=1}^{n} {y_i}^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{{y_i}^2 e^{-z_i}}{(1-e^{-z_i})}$$

$$- \frac{(a-1)}{2}\sum_{i=1}^{n} \frac{{y_i}^2 \theta e^{-z_i}}{[log(1-\theta) - log(1-\theta e^{-z_i})][(1-e^{-z_i})]}$$

$$+ \frac{(b-1)}{2}\sum_{i=1}^{n} \frac{{y_i}^2 \theta e^{-z_i}}{[log(1-\theta e^{-z_i})][(1-e^{-z_i})]} \tag{36}$$

and

$$\frac{\partial \ln L(\psi)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{n}{log(1-\theta)} + \sum_{i=1}^{n} \frac{e^{-z_i}}{\log(1-e^{-z_i})}$$

$$+ (a-1)\sum_{i=1}^{n} \frac{1}{log(1-\theta) - log(1-\theta e^{-z_i})} [\left(\frac{1}{1-\theta} + \frac{e^{-z_i}}{1-e^{-z_i}}\right)$$

$$+ \left(\frac{log(1-\theta) - log(1-\theta e^{-z_i})}{[log(1-\theta)]\,[(1-\theta)]}\right)] + (b-1)\sum_{i=1}^{n} \frac{1}{log(1-\theta)}$$

$$\times \left[\frac{-\,log(1-\theta e^{-z_i})}{[log(1-\theta)]\,[(1-\theta)]} - \frac{log(1-\theta e^{-z_i})}{[log(1-\theta)]\,[1-e^{-z_i}]}\right], \tag{37}$$

respectively. A statistical software such as R software may be used to solve these nonlinear equations via Newton-Raphson iterative methods. The maximum likelihood estimates are asymptotically normal distributed with a joint multivariate normal distribution, that is, $\hat{\psi}_{MLE} \sim N_5[\psi, I^{-1}(\psi)]$, for $n \to \infty$, where $I(\psi)$ is Fisher information matrix. An approximate $100(1-\alpha)\%$ confidence interval for a, b, α, β and θ are given by, $\hat{a} \pm Z_{\frac{\alpha}{2}}\sqrt{{I_{aa}}^{-1}(\hat{\psi})}$, $\hat{b} \pm Z_{\frac{\alpha}{2}}\sqrt{{I_{bb}}^{-1}(\hat{\psi})}$, $\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{{I_{\alpha\alpha}}^{-1}(\hat{\psi})}$, $\hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{{I_{\beta\beta}}^{-1}(\hat{\psi})}$, and $\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{{I_{\theta\theta}}^{-1}(\hat{\psi})}$, respectively.

5. Simulation

In this section, for different sample sizes (n = 25, n = 50, n = 100, n = 200, n = 400, and n = 800), we simulate 1000 samples for the parameter values I : a = 0.6, b = 0.5, α = 1, β = 1, θ = 0.1 and II : a = 0.8, b = 0.5, α = 2, β = 2, θ = 0.1, from the BLFRL distribution to illustrate the performance of the maximum likelihood estimation method. The mean MLEs of the model parameters, root mean squared errors (RMSE) and bias are presented in the table below. The RMSE and average bias are given by

$$RMSE = \sqrt{\frac{\sum_{I=1}^{n}\left(\hat{\psi}_i - \psi\right)^2}{N}}, \qquad and \quad Bias(\psi) = \frac{\sum_{I=1}^{N}(\hat{\psi}_i - \psi)}{N}$$

respectively. It is noted from the results in Table 4 that as the sample size increases the parameter estimates approach the assumed parameter values. Moreover, it can be noted that as the sample size increase the RMSE and the bias decrease for all parameters of the model, indicating that the method of maximum likelihood performs very well in estimating the parameters of the BLFRL distribution.

Table 4. Monte Carlo Simulation Results: Mean Estimates, RMSEs and Biases

		I			II		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
a	25	0.5667303	0.2196942	-0.03326970	0.7906198	0.5276053	-0.0093802
	50	0.5454118	0.1856732	-0.05458816	0.7559085	0.2628620	-0.0440915
	100	0.5391788	0.1496564	-0.06082118	0.7664498	0.2413086	-0.0335502
	200	0.5456076	0.1230125	-0.05439239	0.7514404	0.2049054	-0.0485597
	400	0.5681366	0.0924053	-0.03186340	0.7708524	0.1528934	-0.0291476
	800	0.5828294	0.0662992	-0.01717061	0.7848079	0.1093169	-0.0151921
b	25	0.7371660	0.6601909	0.2371660	0.9183608	1.0086900	0.4183608
	50	0.6315966	0.459617	0.13159660	0.7435293	0.6387887	0.2435293
	100	0.5869022	0.3000755	0.08690218	0.6353985	0.4386437	0.1353985
	200	0.5593105	0.2400349	0.05931052	0.5861316	0.2860557	0.0861316
	400	0.5519164	0.2131936	0.05191638	0.5885012	0.2074687	0.0885012
	800	0.5412241	0.1762263	0.04122412	0.5897044	0.1808715	0.0897044
α	25	0.2353700	0.8466786	-0.7646300	0.4441945	1.7469620	-1.5558050
	50	0.2949097	0.8437929	-0.7050903	0.5542308	1.6834120	-1.4457690
	100	0.3139438	0.814134	-0.6860562	0.7419085	1.6077780	-1.2580910
	200	0.3551126	0.7655385	-0.6448874	0.7332672	1.5378390	-1.2667330
	400	0.4174307	0.6954618	-0.5825693	0.7680053	1.4357860	-1.2319950
	800	0.4682227	0.6218293	-0.5317773	0.8375324	1.3472280	-1.1624680
β	25	1.447482	1.2381868	0.4474818	2.8198210	2.6584830	0.8198208
	50	1.427661	1.0093050	0.4276608	2.7911620	2.0540480	0.7911620
	100	1.382312	0.8623746	0.3823121	2.7700340	1.6477040	0.7700337
	200	1.375109	0.7572548	0.3751095	2.7742350	1.4835560	0.7742347
	400	1.305728	0.6051623	0.3057279	2.6653290	1.2620240	0.6653289
	800	1.260722	0.5103913	0.2607224	2.5371400	0.9846641	0.5371401
θ	25	0.7279280	0.6543416	0.6279280	0.8043969	0.7213363	0.7043969
	50	0.7223881	0.6483626	0.6223881	0.7907216	0.7094214	0.6907216
	100	0.7154845	0.6427775	0.6154845	0.7802740	0.7004634	0.680274
	200	0.7148680	0.6378526	0.6148680	0.7847273	0.6997667	0.6847273
	400	0.6923729	0.6235551	0.5923729	0.7714467	0.6880321	0.6714467
	800	0.6839722	0.5839722	0.4038085	0.7395452	0.6642651	0.6395452

6. Application

In this section, the applicability and performance of the BLFRL distribution is illustrated using a real data set. The data set consist of the lifetimes of 20 electronic components which has been previously used by Teimouri and Gupta (2013) and Nasiru (2015) and is given in Table 5. Using this data sets the performance of the BLFRL distribution is compared to its some of the sub-models and the non-nested five parameter exponentiated Kumaraswamy Dagum (EKD) distribution of Huang and Oluyede (2014). The cdf and pdf of the EKD distribution are given by

$$F_{EKD}(x;\alpha,\lambda,\delta,\phi,\theta)=\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi}\right\}^{\theta} \quad (38)$$

and

$$f_{EKD}(x;\alpha,\lambda,\delta,\phi,\theta) = \alpha\lambda\delta\phi\theta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\alpha-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi-1} \times\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi}\right\}^{\theta-1}, \quad (39)$$

for $\alpha, \lambda, \delta, \varphi, \theta > 0$ and $x > 0$, respectively.

The maximum likelihood estimates (MLEs) of the distributions parameters are computed by maximizing $L(\psi)$ via library'bbmle' in R software. The estimated values of the parameters, standard error (in parenthesis), -2log-likelihood statistic ($-2\ \ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), and Consistent Akaike Information Criterion ($AICC = AIC + 2\frac{p(p+1)}{n-p-1}$), where p is the number of estimated parameters, n is the number of observations, and $L = L(\hat{\psi})$ is the likelihood function evaluated at the parameter estimates are presented in Table 6. The log-likelihood ratio test statistic ω is given by

$$\omega = -2\ln\left(\frac{L_{nestedmodel}(\hat{\psi})}{L_{BLFRL}(\hat{\psi})}\right).$$

The Cramer Von-Mises (W^*) statistic, the Anderson Darling (A^*) statistic, and Kolmogorov-Smirnov (KS) statistic and its P-value are also presented. The smaller the values of the W^* and the A^*, the better the fit. Also, presented is the sum of squares $SS = \sum_{j=1}^{n}\left[F_{BLFRL}\left(x_{(j)};\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{\theta}\right)-\left(\frac{j-0.375}{n+0.25}\right)\right]^2$. Plots of the fitted densities, the histogram and probability plots (Chambers et al. (1983)) of the data are given for the Turbocharger data and for the lifetime of components dataset, repectively. For the probability plot, we plotted $F_{BLFRL}\left(x_{(j)};\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{\theta}\right)$ against $\frac{j-0.375}{n+0.25}$, $j = 1,2,\cdots,n$, where $x_{(j)}$ are the ordered values of the observed data.

The uncensored real data of 20 observations of the lifetime of electronic components given in Table 5. Table 6 gives the summarized results of the data with parameter estimates for the EKD distribution, BLFRL distributions and its related sub-models with standard error in parentheses, 2 log(L), W^*, A^*, KS, KS-P-value, SS, AIC, AICC, and BIC statistics.

Table 5. The Lifetime of 20 Electronic Components

0.03	0.12	0.22	0.35	0.73	0.79	1.25	1.41	1.52	1.79
1.80	1.94	2.38	2.40	2.87	2.99	3.14	3.17	4.72	5.09

Table 6. Estimates of Models for Lifetime of 20 Electronic Components Data Set

	Estimates				
Model	a	b	α	β	θ
BLFRL	0.7339	0.1309	1.8944	1.19102	0.2780
	(0.4086)	(0.5701)	(10.3111)	(4.3891)	(1.7532)
BEL	1.1959	0.09134	5.5408	0	0.6776
	(1.3963)	(0.1486)	(8.9304)	-	(1.4919)
RL	1	1	0	0.3505	2.1734×10^{-6}
	-	-	-	(0.0841)	(0.4275)
Rayleigh	1	1	0	03505	0
	-	-	-	(0.07838)	-
Non Nested Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\theta}$
EKD	0.0123	39.4299	4.1875	4.1875	9.17377
	(0.0177)	(0.0169)	(1.5477)	(0.3417)	(0.0102)

	Statistics								
Model	-2logL	AIC	AICC	BIC	A^*	W^*	KS	P-Value	SS
BLFRL	63.1643	73.1644	77.4501	78.1430	0.1996	0.0303	0.1027	0.9699	0.02967
BEL	66.33644	74.3364	77.0031	78.3194	0.4756	0.08135	0.1807	0.4767	0.1180
RL	75.0385	79.0385	79.7444	81.0299	0.2510	0.040657	0.1964	0.3742	0.14746
Rayleigh	75.0385	77.0385	77.2607	78.0342	0.2510	0.04066	0.1964	0.3742	0.1475
EKD	64.2254	74.2254	78.5112	79.2040	0.2188	0.0321	0.0896	0.9925	0.0286

The inverse of the observed Fisher information matrix $\left(I^{-1}(\psi)\right)$ of the MLEs is given by

$$\begin{pmatrix} 0.16696 & -0.08926 & 1.87436 & 0.58456 & 0.33382 \\ -0.08926 & 0.32507 & -5.83286 & -2.47005 & -0.94020 \\ 1.87436 & -5.83286 & 106.31895 & 43.69200 & 16.92978 \\ 0.58456 & -2.47005 & 43.69200 & 19.26394 & 7.13945 \\ 0.33382 & -0.94020 & 16.92978 & 7.13945 & 3.07377 \end{pmatrix}$$

and the 95% confidence intervals for the parameters of the BLFRL distribution are given by: $a \in (0.7339 \pm 1.96 \times 0.4086)$, $b \in (0.1309 \pm 1.96 \times 0.5701)$, $\alpha \in (1.8944 \pm 1.96 \times 10.3111)$, $\beta \in (1.1910 \pm 1.96 \times 4.3891)$, and $\theta \in (0.2780 \pm 1.96 \times 1.7532)$, respectively. Plots of the fitted densities and histogram, as well as the observed probability versus the expected probability for the data are presented in Figures 6.1 and 6.2.

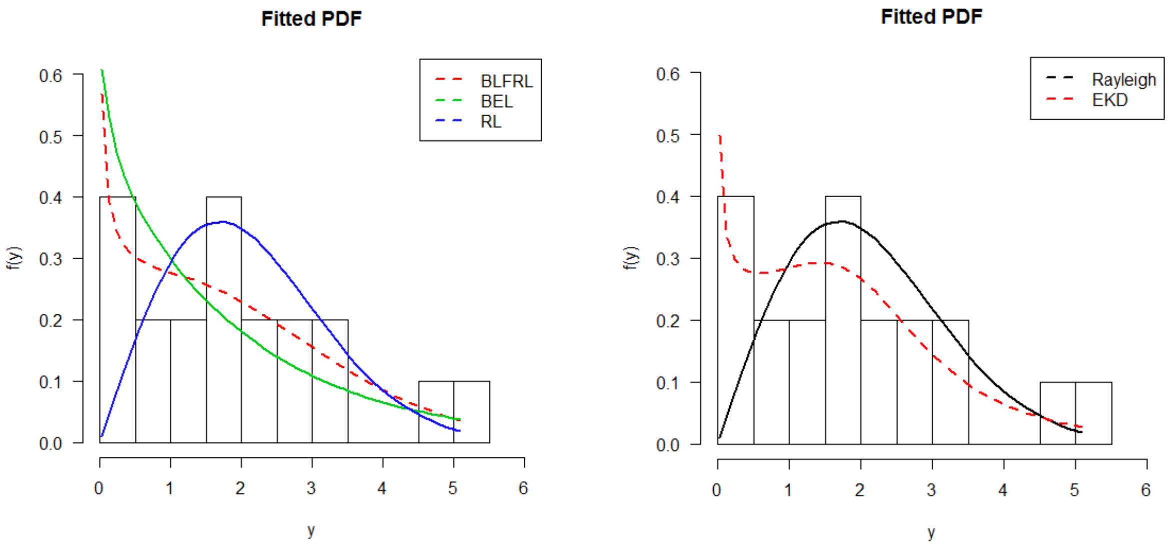

Figure 3. Fitted Densities

The values of SS from the probability plots clearly indicate that the BLFRL distribution has the smallest value and provides the better fit for this data. The log-likelihood ratio tests of the hypotheses H_0 : BEL against H_a : BLFRL, H_0 : RL against H_a : BLFRL, H_0 : EL against H_a : BLFRL and H_0 : R against H_a : BLFRL indidates that the BLFRL distribution is significantly better than the considered sub-models. The observed AIC, AICC, and BIC values of the BLFRL distribution are smaller than its sub-models and the non-nested EKD distribution. The values of the goodness-of-fit statistics clearly supports the BLFRL distribution. Therefore, there is significant evidence to conclude that the BLFRL distribution gives a better fit for this data.

7. Conclusion

A new class of distributions called the BLFRPS distribution and the special case of the beta linear failure rate logarithmic (BLFRL) distribution has been proposed. The model properties are discussed and presented. The maximum likelihood method is used for parameter estimation. Lastly, the BLFRL distribution is competitive when fitted to a real data.

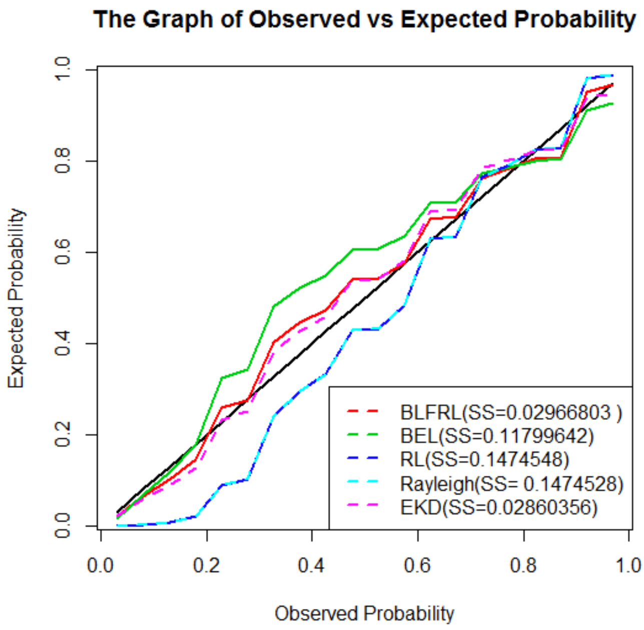

Figure 4. Probability Plot

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