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Transmuted Weibull Power Function Distribution: its Properties and Applications

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Abstract: In this paper, we introduce a new four-parameter distribution called the transmuted Weibull power function (TWPF) distribution which extends the transmuted family proposed by Shaw and Buckley [1]. The hazard rate function of the TWPF distribution can be constant, increasing, decreasing, unimodal, upside down bathtub shaped or bathtub shape. Some mathematical properties are derived including quantile functions, expansion of density function, moments, moment generating function, residual life function, reversed residual life function, mean deviation, inequality measures. The estimation of the model parameters is carried out using the maximum likelihood method. The importance and flexibility of the proposed model are proved empirically using real data sets.

Key words: Weibull distribution, transmuted family, maximum likelihood, moments, order statistics, entropy.

1. Introduction

There are hundreds of continuous distributions in the statistical literature. These distributions have several applications in many applied fields such as reliability, life testing, biomedical sciences, economics, finance, environmental and engineering, among others. However, these applications have proven that the real data following the well-known models are more often the exception rather than the reality. In order to increase the flexibility of the well-known distributions, many authors have proposed different transformations of these models and used these extended forms in several areas.

The power function (PF) distribution is a flexible model which can be obtained from the Pareto distribution by using a simple transformation $Y = X^{-1}$. The probability density function (pdf) and the cumulative distribution function (cdf) of the PF distribution are, respectively, given by

$$F(x;\alpha,\beta)=\left(\frac{x}{\alpha}\right)^{\beta}, \tag{1.1}$$

$$f(x)=\frac{\beta x^{\beta-1}}{\alpha^{\beta}}, \quad 0<x<\alpha, \beta>0 \tag{1.2}$$

where β is a shape parameter and α is a scale parameter.

The Power function (PF) distribution has been used over the past decades for modelling data in engineering, reliability and biological studies. Meniconi and Barry [2] investigated the performance of the PF distribution on electrical component and obtained that the PF distribution is better than the Weibull, log-normal and exponential models to measure the reliability of electronic components. The need for extended forms of the PF distribution arises in many applied areas and hence, some generalizations of the PF distribution have been proposed. Among these distributions, we refer to the beta power function (BPF) by [3], Weibull power function (WPF) by [4], Kumaraswamy power function (KWPF) distribution by [5], the modified power function distribution by [6], exponentiated power function distribution by [7], the exponentiated Kumaraswamy power function distribution by [8].

The pdf and cdf of the WPF distribution are, respectively, given by

$$\mathrm{g}(\mathrm{x},\mathrm{a},\mathrm{b},\alpha,\beta)=\frac{ab\beta\alpha^{\beta}x^{\beta\mathrm{b}-1}}{(\alpha^{\beta}-x^{\beta})^{b+1}}e^{-a\left(\frac{x^{\beta}}{\alpha^{\beta}-x^{\beta}}\right)^{b}}, 0<x<\alpha,\alpha,\beta>0 \tag{1.3}$$

and

$$\mathrm{G}(\mathrm{x})=1-\exp(-\mathrm{a}\left\{\frac{x^{\beta}}{\alpha^{\beta}-x^{\beta}}\right\}^{b}\Bigg) \tag{1.4}$$

where ‘α’ and ‘a’ are the scale parameters, ‘b’ and ‘β’ are the shape parameters.

Many authors have been recenlty deal with the generalization of some well-known distributions using the transmuted family proposed by [1]. Aryal and Tsokos [9] defined the transmuted generalized extreme value distribution and then studied some mathematical characteristics of the transmuted Gumbel distribution. Aryal and Tsokos [10] also presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Aryal [11] proposed and studied the various structural properties of the transmuted log-logistic distribution. Then, Khan and King [12] introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal and Tsokos [10]. Elbatal [13] presented transmuted modified inverse Weibull distribution. Elbatal and Aryal [14] presented transmuted additive Weibull distribution. Recently, transmuted generalized Lindley distribution has been obtained by Elgarhy et al. [15], transmuted power function (TPF) has been introduced by Haq et al. [16] and transmuted Weibull Frehet by Haq et al. [17], Elbatal and Elgarhy [18] introduced Transmuted quasi Lindley (TQL), and Elgarhy et al. [19] studied transmuted generalized quasi Lindley (TGQL).

The aim of this paper is to define and study a new flexible lifetime model called the transmuted Weibull power function (TWPF) distribution. Using the transmuted family proposed by [1], we construct the four-parameter TWPF model and give some of its mathematical properties. In fact, the TWPF model can provide better fits than other existing models.

The rest of the paper is organized as follows: In section 2 we demonstrate transmuted probability density, hazard rate and reliability functions of TWPF distribution. In section 3 we studied the statistical properties including quantile function, expansion of density function, moments, moment generating function, incomplete moments, conditional moments, residual life function, reversed residual life function, mean deviation, inequality measures. The distribution of order statistics is expressed in section 4. In section 5, we demonstrate the maximum likelihood estimates (MLEs) of the unknown parameters. Simulation study is carried out for TWPF distribution in Section 6. Two illustrative applications based on real data sets are investigated in section 7. Finally, concluding remarks are presented in section 8.

2. The TWPF Distribution

Before defining the TWPF distribution, we explain a transmuted probability distribution. Let $\boldsymbol{F_1}$and $\boldsymbol{F_2}$ be the cdfs of two distributions with a common sample space. The general rank transmutation is defined by Shaw and Buckely [1] as

$$G_{R12}(u) = F_2({F_1}^{-1}(u)) \text{ and } G_{R21}(u) = F_1({F_2}^{-1}(u)).$$

Note that the inverse cdf, also known as quantile function, is defined as

$$F^{-1}(y) = inf_{x\in R}\{F(x) \geq y\} \text{ for } y \in [0,1]$$

The functions $\boldsymbol{G_{R12}(u)}$ and $\boldsymbol{G_{R21}(u)}$ both map the unit interval $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$ into itself. Under suitable assumptions these functions are mutual inverses and they satisfy $\boldsymbol{G_{Rij}(0)} = \mathbf{0}$ and $\boldsymbol{G_{Rji}(0)} = \mathbf{1}$

A quadratic rank transmutation map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda\mu(1-\mu), |\lambda| \leq 1 \tag{2.1}$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \tag{2.2}$$

After differentiating (2.2), we have $\boldsymbol{f_2(x) = f_1(x)[(1+\lambda) - 2\lambda F_1(x)]}$ where $\boldsymbol{f_1(x)}$ and $\boldsymbol{f_2(x)}$ are the corresponding pdfs' associated with cdfs' $\boldsymbol{F_1(x)}$and $\boldsymbol{F_2(x)}$ respectively. More information about the quadratic rank transmutation map can be found in Shaw and Buckley [1]. Note that we have the baseline distribution for $\boldsymbol{\lambda = 0}$.

The following Lemma proves that the function $\boldsymbol{f_2(x)}$ in given (2.3) satisfies the property of pdfs.

Lemma: $\boldsymbol{f_2(x)}$ given in (2.3) is a well-defined pdf.

Proof. Rewriting $\boldsymbol{f_2(x)}$ as $\boldsymbol{f_2(x) = f_1(x)[1 - \lambda\{F_1(x) - 1\}]}$ we observe that $\boldsymbol{f_2(x)}$ is non-negative. We need to show that the integration of the support of the random variable is equal to one. Consider the case when the support of $\boldsymbol{f_1(x)}$ is 1. In this case, we have

$$\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-\infty}^{\infty} f_1(x)[(1+\lambda) - 2\lambda F_1(x)]\,dx$$

$$= (1+\lambda)\int_{-\infty}^{\infty} f_1(x)\,dx - 2\lambda\int_{-\infty}^{\infty} F_1(x)f_1(x)\,dx$$

$$= (1+\lambda) - \lambda = 1$$

Similarly, other cases where the support of the random variable is a part of the real line follows. Hence, $\boldsymbol{f_2(x)}$ is a well-defined pdf and it is called as the transmuted probability density of a random variable with base density $\boldsymbol{f_1(x)}$. Note that when $\boldsymbol{\lambda = 0}$, we have $\boldsymbol{f_2(x) = f_1(x)}$. This proves the required result.

Now, using (1.2) and (1.4), we obtain the cdf of TWPF distribution as

$$F_{TWPF}(x) = (1 - \exp(-a\{\frac{x^\beta}{\alpha^\beta - x^\beta}\}^b))\{1 + \lambda - \lambda\left(1 - \exp\left(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right)\right)\} \quad (2.4)$$

where α and a are the scale parameters; b and β are the shape parameters and λ is the transmuted parameter. Then, the pdf of the TWPF distribution is given by

$$\left\{\begin{matrix} f_{TWPF}(x) = \dfrac{ab\beta\alpha^\beta x^{\beta b-1}}{(\alpha^\beta - x^\beta)^{b+1}} e^{-a(\frac{x^\beta}{\alpha^\beta - x^\beta})^b}\left[(1+\lambda) - 2\lambda\left\{1 - \exp\left(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right)\right\}\right], \\ 0 < x < \alpha \end{matrix}\right\} \quad (2.5)$$

The reliability (survival) function of the TWPF distribution is given by

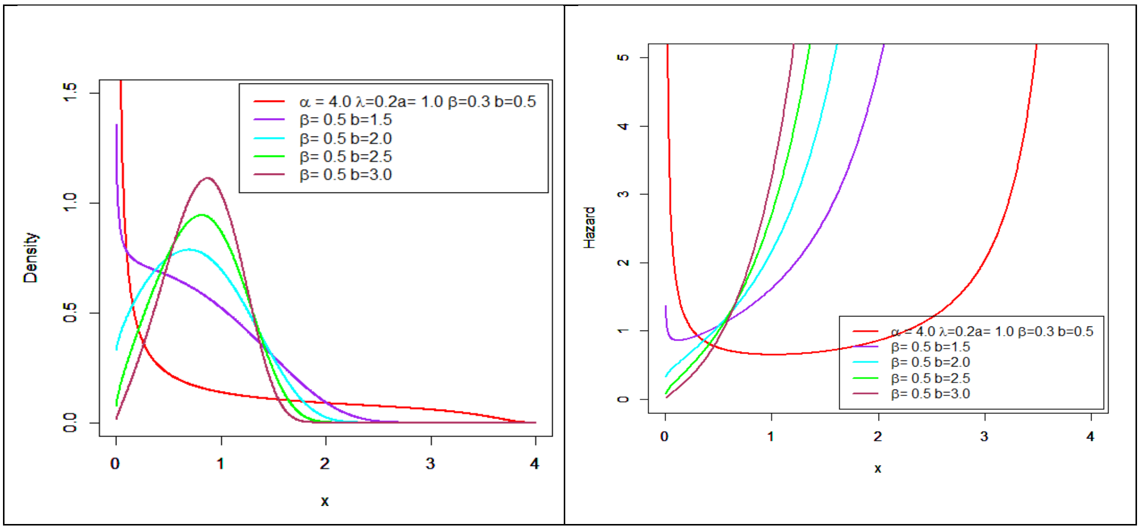
$$R_{TWPF}(x) = 1 - F_{TWPF}(x)$$
$$= 1 - (1 - \exp(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\Big))\{1 + \lambda - \lambda(1 - \exp\left(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right)\Big)\}$$

One of the characteristics in reliability analysis is the hazard rate function (hrf) defined by

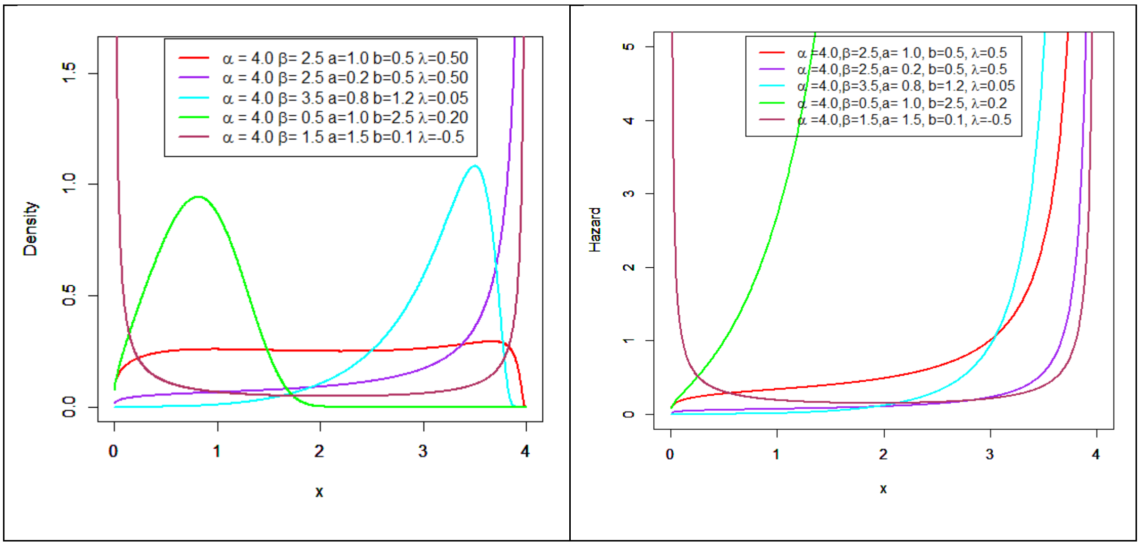
$$h_{TWPF}(x) = \frac{f_{TWPF}(x)}{1 - F_{TWPF}(x)}$$
$$= \frac{ab\beta\alpha^\beta x^{\beta b-1} e^{-a(\frac{x^\beta}{\alpha^\beta - x^\beta})^b}[(1+\lambda) - 2\lambda\{1 - \exp\left(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right)\}]}{(\alpha^\beta - x^\beta)^{b+1}(1 - (1 - \exp\left(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right)\Big)\left\{1 + \lambda - \lambda\left(1 - \exp\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right)\right)\Big\})}$$

It is important to note that the units for $h_{TWPF}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters.

Plots of the pdf and hrf of the TWPF distribution for some parameter values are displayed in Figure 1 (a) and (b), respectively. As seen from Figure 1(a), the density function can take various forms depending on the parameter values. Increasing, decreasing, unimodal, upside down bathtub shaped or bathtub shapes appear to be possible. It is evident that the TWPF distribution is very flexible. Furthermore, Figure 1(b) shows that the hrf of the TWPF distribution can have very flexible shapes, such as increasing, decreasing, upside-down bathtub, bathtub. This attractive flexibility makes the hrf of the TWPF useful and suitable for non-monotone empirical hazard behaviours' which are more likely to be encountered or observed in real life situations.



α = 4.0 λ=0.2a= 1.0 β=0.3 b=0.5
β= 0.5 b=1.5
β= 0.5 b=2.0
β= 0.5 b=2.5
β= 0.5 b=3.0
Density
x
Hazard



α = 4.0 β= 2.5 a=1.0 b=0.5 λ=0.50
α = 4.0 β= 2.5 a=0.2 b=0.5 λ=0.50
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α = 4.0 β= 0.5 a=1.0 b=2.5 λ=0.20
α = 4.0 β= 1.5 a=1.5 b=0.1 λ=-0.5
Density
x
α =4.0,β=2.5,a= 1.0, b=0.5, λ=0.5
α =4.0,β=2.5,a= 0.2, b=0.5, λ=0.5
α =4.0,β=3.5,a= 0.8, b=1.2, λ=0.05
α =4.0,β=0.5,a= 1.0, b=2.5, λ=0.2
α =4.0,β=1.5,a= 1.5, b=0.1, λ=-0.5
Hazard

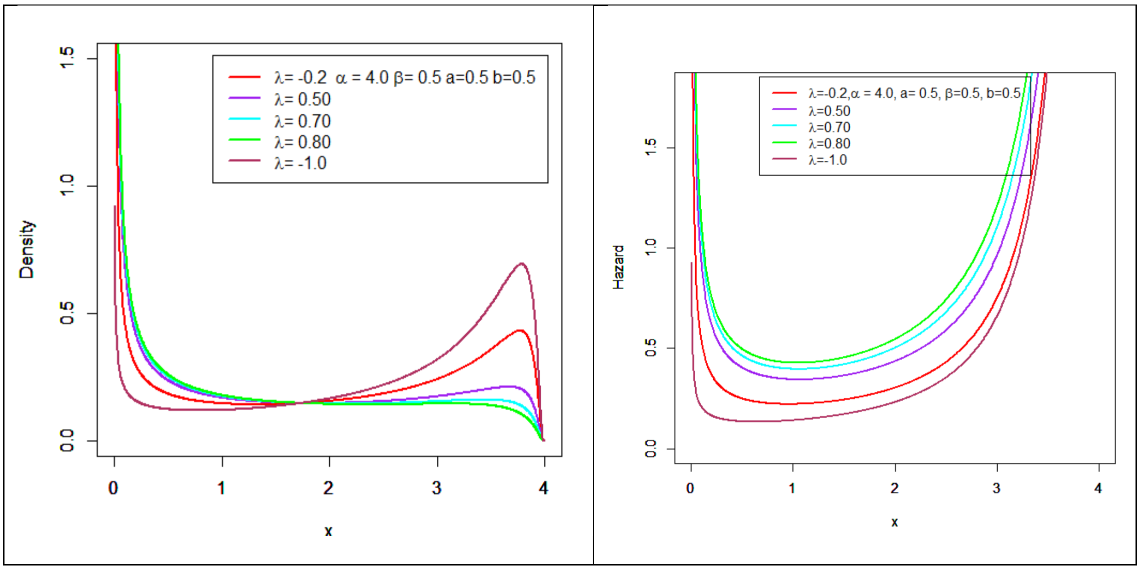


Figure 1 (a): Plots of the pdf of TWPF distribution for selected values of the parameters (left). (b): Plots of the hrf of TWPF distribution for selected values of the parameters (right).

3. Main Properties

This section is devoted to main properties of the TWPF distribution, specifically quantile function, moments and moment generating function.

3.1. Quantile Function

The quantile $\boldsymbol{x_q}$ of the TWPF distribution is obtained from (2.4) as

$$x_q = \alpha\{\ln[\frac{(\lambda-1)+\sqrt{(\lambda-1)^2-4\lambda(\mu-1)}}{2\lambda}]^{\frac{-1}{a}}\}^{\frac{1}{b\beta}}(1 + \{\ln[\frac{(\lambda-1)+\sqrt{(\lambda-1)^2-4\lambda(\mu-1)}}{2\lambda}]^{\frac{-1}{a}}\}^{\frac{1}{b}})^{\frac{-1}{\beta}}$$

We simulate the TWPF distribution by solving the equation above where u has the uniform distribution $\boldsymbol{U(0,1)}$.

3.2. A Useful Expansion

Now, a representation for the density function of the TWPF distribution will be presented. Using the power series for the exponential function, we obtain

$$\exp(-\mathrm{ax}) = \sum_{k=0}^{\infty}\frac{(-1)^k a^k x^k}{k!} \tag{3.1}$$

Inserting the expansion (3.1) in (2.5), we have

$$f(x) = (1-\lambda)ab\frac{\beta}{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k((\frac{x}{\alpha})^{\beta})^{bk+k-1}(1-(\frac{x}{\alpha})^{\beta})^{-(bk+b+1)}}{k!} + 2a\lambda b\frac{\beta}{\alpha}\sum_{k=0}^{\infty}\frac{(-2)^k((\frac{x}{\alpha})^{\beta})^{bk+b-1}(1-(\frac{x}{\alpha})^{\beta})^{-(bk+b+1)}}{k!} \quad (3.2)$$

Now, using the generalized binomial theorem, we obtain

$$(1-\mathrm{z})^{-\beta} = \sum_{j=0}^{\infty}\binom{\beta+j-1}{j}z^j \quad (3.3)$$

where $\boldsymbol{\beta} > \mathbf{0}$ is real non-integer. Then, by applying the binomial theorem (3.3) in (3.2) the pdf of the TWPF distribution becomes

$$f(x) = (1-\lambda)a^{k+1}b\frac{\beta}{\alpha^{\frac{1}{\beta}}}\sum_{j,k=0}^{\infty}\frac{(-1)^k}{j!}\frac{\Gamma(b(k+1)+j+1)}{k!\,\Gamma(b(k+1)+1)}((\frac{x}{\alpha})^{\beta})^{b(k+1)+j-\frac{1}{\beta}} + 2a^{k+1}\lambda b\frac{\beta}{\alpha^{\frac{1}{\beta}}}\sum_{j,k=0}^{\infty}\frac{(-2)^k\Gamma(b(k+1)+j+1)}{j!\,k!\,\Gamma(b(k+1)+1)}((\frac{x}{\alpha})^{\beta})^{b(k+1)+j-\frac{1}{\beta}}$$

$$f(x) = (1-\lambda)ab\frac{\beta}{\alpha}\sum_{j,k=0}^{\infty}\frac{(-1)^k\Gamma(b(k+1)+j+1)}{j!\,k!\,\Gamma(b(k+1)+1)}((\frac{x}{\alpha})^{\beta})^{b(k+1)+j-1} + 2a\lambda b\frac{\beta}{\alpha}\sum_{j,k=0}^{\infty}\frac{(-2)^k\Gamma(b(k+1)+j+1)}{j!\,k!)^k\Gamma(b(k+1)+1)}((\frac{x}{\alpha})^{\beta})^{b(k+1)+j-1}$$

and after simplification, the TWPF density can be expressed as

$$f(\mathrm{x}) = \sum_{j,k=0}^{\infty}\frac{(-1)^k a^{k+1}b\beta\Gamma(b(k+1)+j+1)\{1-\lambda+2^{k+1}\lambda\}}{j!\,k!\,\alpha^{\frac{1}{\beta}}\Gamma(b(k+1)+1)}((\frac{x}{\alpha})^{\beta})^{b(k+1)+j-\frac{1}{\beta}}$$

or equivalently, we can write

$$\mathrm{f(x)} = \sum_{j,k=0}^{\infty}\cup_{j,k}\,(\frac{x}{\alpha})^{\beta(b(k+1)+j)-1} \quad (3.4)$$

where

$$\cup_{j,k} = \frac{(-1)^k a^{k+1}b\beta\Gamma(b(k+1)+j+1)\{1-\lambda+2^{k+1}\lambda\}}{j!\,k!\,\alpha^{\frac{1}{\beta}}\Gamma(b(k+1)+1)}$$

If $\boldsymbol{\beta}$ is an integer, the index i in the previous sum stops at $\boldsymbol{\beta}-\mathbf{1}$.

3.3. Moments

This subsection concerns with the $\boldsymbol{\mu'_r}$ moment and moment generating function for the TWPF distribution. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness, and kurtosis).

If X has the pdf in (2.5), then its rth moment can be obtained through the following relation

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

Substituting (3.4) in above relation, we get

$$\mu'_r = E(X^r) = \sum_{j,k=0}^{\infty} U_{j,k} \int_0^{\alpha} x^r \left(\frac{x}{\alpha}\right)^{\beta(b(k+1)+j)-1} \mathrm{dx} \tag{3.5}$$

Then, we obtain

$$\mu'_r = \sum_{j,k=0}^{\infty} \cup_{j,k} \frac{\alpha^{r+1}}{r+\beta(b(k+1)+j)} \tag{3.6}$$

Based on the first four moments of the TWPF distribution, the measures of skewness $\mathbf{A}(\boldsymbol{\varphi})$ and kurtosis $\mathbf{k}(\boldsymbol{\varphi})$ of the TWPF distribution can be obtained as

$$\mathrm{A}(\varphi) = \frac{\mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1}{[\mu'_2 - \mu'^2_1]^{\frac{3}{2}}},$$

and

$$\mathrm{k}(\varphi) = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1}{[\mu'_2 - \mu'^2_1]^2}.$$

Using the relation between the non-central and central moments, we can obtain the nth central moment, denoted by M_n, of TWPF random variable as follows:

$$M_n = E(X-\mu)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} E(X^r)$$

Then, we can also write

$$M_n = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} (\mu'_1)^{n-r} \mu'_r$$

and the cumulants of the random variable X can be obtained as

$$k_n = \mu'_n - \sum_{r=0}^{n-1} \binom{n-1}{r-1} k_1 \mu'_{n-r}$$

where $\boldsymbol{k_1 = \mu'_1, k_2 = \mu'_2 - (\mu'_1)^2, k_3 = \mu'_3 - 3\mu'_2\mu'_1 + (\mu'_1)^3}$ etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

3.4. Moment Generating Function

In this subsection, we derived the moment generating function of TWPF distribution. The moment generating function is given by the relation

$$M_x(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x)\, dx,$$

then, we have

$$M_x(t) = \sum_{j,k=0}^{\infty} U_{j,k} \int_0^{\alpha} (\frac{x}{\alpha})^{\beta(b(k+1)+j)-1} e^{tx}\, dx$$

The moment generating function of the TWPF distribution is obtained by

$$M_x(t) = \sum_{j,k=0}^{\infty} U_{j,k} \frac{\gamma(\beta(k+1)+j, \alpha t)}{(\alpha t)^{\beta(b(k+1)+j)-1}} \tag{3.7}$$

where $\boldsymbol{\gamma}(\mathbf{s},\mathbf{t}) = \int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{x}^{\boldsymbol{s}-\mathbf{1}} \boldsymbol{e}^{-\boldsymbol{x}}\, \boldsymbol{dx}$is the lower incomplete gamma function.
Another formula for moment generating function can be given as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{j,k,r=0}^{\infty} U_{j,k} \frac{t^r}{r!} \frac{\alpha^{r+1}}{r+\beta(b(k+1)+j)}$$

3.5. Incomplete and Conditional Moments

The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance, and medicine. The incomplete moments, say $\boldsymbol{\varphi_s(t)}$, is given by

$$\varphi_s(t) = \int_0^t x^s f(x)\, dx.$$

Using (2.5), $\boldsymbol{\varphi_s(t)}$can be written as

$$\varphi_s(t) = \sum_{j,k=0}^{\infty} U_{j,k} \int_0^t x^s (\frac{x}{\alpha})^{\beta(b(k+1)+j)-1}\, dx.$$

Then,

$$\varphi_s(t) = U_{j,k} \frac{t^{s+\beta(b(k+1)+j)}}{[s+\beta(b(k+1)+j)]\alpha^{\beta(b(k+1)+j)-1}}$$

Further, the conditional moments, say $\boldsymbol{\tau_s(t)}$, is given by

$$\tau_s(t) = \int_t^{\alpha} x^s f(x)\, dx$$

Hence, by using pdf (2.5), we can write

$$\tau_s(t) = \sum_{j,k=0}^{\infty} U_{j,k} \int_t^{\alpha} x^s (\frac{x}{\alpha})^{\beta(b(k+1)+j)-1}\, dx$$

Then, we have

$$\tau_s(t) = \sum_{j,k=0}^{\infty} U_{j,k} \frac{\alpha^{s+\beta(b(k+1)+j)} - t^{s+\beta(b(k+1)+j)}}{[s+\beta(b(k+1)+j)]\alpha^{\beta(b(k+1)+j)-1}}$$

Additionally, the mean deviation can be calculated by the following relation $\boldsymbol{\delta_1(X) = 2\mu F(\mu) - 2T(\mu)}$ **and** $\boldsymbol{\delta_2(X) = \mu - 2T(M)}$ where, $\boldsymbol{T(q) = \int_0^q x f(x)\, dx}$.

By using (3.4) we have the following equations:

$$T(\mu) = \int_0^{\mu} x f(x)\, dx = \sum_{j,k=0}^{\infty} U_{j,k} \frac{\mu^{\beta(b(k+1)+j)+1}}{[\beta(b(k+1)+j)+1]\alpha^{\beta(b(k+1)+j)-1}},$$

$$T(M) = \int_0^{M} x f(x)\, dx = \sum_{j,k=0}^{\infty} U_{j,k} \frac{M^{\beta(b(k+1)+j)+1}}{[\beta(b(k+1)+j)+1]\alpha^{\beta(b(k+1)+j)-1}},$$

3.6. Residual Life Function

Several functions are defined related to the residual life. The failure rate function, mean residual life function, and the left-censored mean function, also called vitality function. It is well known that these three functions uniquely determine $F(x)$, see [20-22]. Moreover, the nth moment of the residual life, say $\boldsymbol{m_n(t) = E[(X-t)^n | X > t], \mathbf{n} = 1, 2, \ldots}$,uniquely determine $F(x)$. The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{R(t)} \int_t^{\alpha} (x-t)^n f(x)\, dx$$

Applying the binomial expansion of $(x-t)^n$ into the above formula, we get

$$m_n(t) = \frac{1}{R(t)} \sum_{j,k=0}^{\infty} \sum_{d=0}^{n} U_{j,k} (-t)^d \binom{n}{d} \frac{\alpha^{n+\beta(b(k+1)+j)} - t^{n+\beta(b(k+1)+j)}}{[n+\beta(b(k+1)+j)]\alpha^{\beta(b(k+1)+j)-1}} \quad (3.8)$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age x defined by $\boldsymbol{m_1(t) = E[(X-t)|X > t]}$, which represents the expected additional life length for a unit which is alive at age' x '. The MRL of the TWPF distribution can be obtained by setting $n=1$ in (3.8).

Furthermore, the nth moment of the reversed residual life, say $\boldsymbol{M_n(t)E[(X-t)^n | X \le t], \textbf{for t} > 0, \mathbf{n} = 1, 2, \ldots}$,uniquely determines *F(X).* Hence, the nth moment the reversed residual life of X is given by

$$M_n(t) = \frac{1}{R(t)} \int_0^{t} (x-t)^n f(x)\, dx$$

Applying the binomial expansion of $(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{n}}$ into the above formula, we get

$$M_n(t)=\frac{1}{R(t)}\sum_{d=0}^{n}\sum_{j,k=0}^{\infty}U_{j,k}(-t)^d\binom{n}{d}\frac{t^{n+\beta(b(k+1)+j)}}{[n+\beta(b(k+1)+j]\alpha^{\beta(b(k+1)+j)-1}}$$

The mean inactivity time (MIT) or mean waiting time (MWT), also called the mean reversed residual life function, is defined by $\boldsymbol{M_1(t)=E[(X-t)|X\leq t]}$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(\boldsymbol{0},\boldsymbol{x})$.

3.7 Rényi and q-Entropies

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering, and economics. According to Rényi [23], the Rényi entropy is defined by

$$I_\delta(X)=\frac{1}{1-\delta}log\int_{-\infty}^{\infty}f(x)^\delta\,dx,\delta>0\;and\;\delta\neq1.$$

By applying the binomial theory (3.3) in the pdf (3.4), then the pdf $f(x)^\delta$ can be expressed as follows

$$\mathrm{f(x)}^\delta=\sum_{i,k,j=0}^{\infty}W_{i,k,j}x^{\delta(b\beta-1)+\beta(j+k)},$$

Where

$$W_{i,k,j}=\frac{(ab\beta)^\delta(-1)^k(1-\lambda)^{\delta-i}(2\lambda)^i[(i+\delta)a]^k}{\alpha^{\beta[k+b\delta+j]}k!}\binom{\delta}{i}\binom{\delta(b+1)+k+j-1}{j}.$$

Therefore, the Rényi entropy of the TWPF distribution is given by

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\sum_{i,k,j=0}^{\infty}W_{i,k,j}\int_0^\alpha x^{\delta(b\beta-1)+\beta(j+k)}\,dx\right],$$

then,

$$I_\delta(X)=\frac{1}{1-\delta}\log[\sum_{i,k,j=0}^{\infty}W_{i,k,j}\frac{\alpha^{\delta(b\beta-1)+\beta(j+k)+1}}{\delta(b\beta-1)+\beta(j+k)+1}].$$

The q-entropy is defined by

$$H_q(X)=\frac{1}{1-\delta}\log(1-\int_{-\infty}^{\infty}f(x)^q\,dx),q>0\;and\;q\neq1.$$

Therefore, the q-entropy of the TWPF distribution is given by

$$H_q(X)=\frac{1}{1-\delta}\log\left[1-\sum_{i,k,j=0}^{\infty}W_{i,k,j}\frac{\alpha^{\delta(b\beta-1)+\beta(j+k)+1}}{\delta(b\beta-1)+\beta(j+k)+1}\right].$$

When we replace q with, $\boldsymbol{\delta}$ the index I stop at $\boldsymbol{\delta}$ when $\boldsymbol{\delta}$ is an integer.

4. Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $\boldsymbol{X_1}, \boldsymbol{X_2}, \dots, \boldsymbol{X_n}$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function F(x).

Let $\boldsymbol{X_{(1)}}, \boldsymbol{X_{(2)}}, \dots, \boldsymbol{X_{(n)}}$the corresponding ordered random sample from a population of size n.

David [24] defined the pdf of the k^{th} order statistic as

$$f_{X_{(k)}}(x) = \frac{f(x)}{B(k, n-k+1)} \sum_{v=0}^{n-k} (-1)^v \binom{n-k}{v} F(x)^{v+k-1}, \tag{4.1}$$

where $\boldsymbol{B}(.,.)$ stands for beta function.

The pdf of the $\mathbf{k}^{th}$order statistic for TWPF distribution is derived by substituting (2.4) and (3.4) in (4.1), replacing h with v+k-1,

$$f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{j,k=0}^{\infty} \eta^* (\frac{x}{\alpha})^{\beta(b(k+1)+j)-1}$$

$$\times ((1-\exp(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b)) \{1+\lambda-\lambda(1-\exp\left(-a\left\{\frac{x^\beta}{\alpha^\beta - x^\beta}\right\}^b\right))\})^{v+k-1}$$

where

$$\eta^* = (-1)^v \binom{n-k}{v} U_{j,k}$$

This section deals with the MLEs of the unknown parameters for the TWPF distribution on the basis of complete samples. Let $\boldsymbol{X_1}, \boldsymbol{X_2}, \dots, \boldsymbol{X_n}$ be the observed values from the TWPF distribution. The log-likelihood function for parameter vector $\boldsymbol{\varphi} = (\mathbf{a}, \mathbf{b}, \boldsymbol{\beta}, \boldsymbol{\lambda})^{\boldsymbol{T}}$is obtained as

$$\begin{aligned} \ln L(\varphi) &= n \ln a + n \ln b + n\, \beta \ln \alpha \\ &+ (\beta b - 1) \sum_{i=1}^{n} \ln(x_i) - (b+1) \sum_{i=1}^{n} \ln\left(\alpha^\beta - x_i^{\beta}\right) - a \sum_{i=1}^{n} (\frac{x_i^{\beta}}{\alpha^\beta - x_i^{\beta}})^b \\ &+ \sum_{i=1}^{n} \ln((1-\lambda) + 2\lambda \exp(-a\{\frac{x_i^{\beta}}{\alpha^\beta - x_i^{\beta}}\}^b)) \end{aligned} \tag{5.1}$$

The elements of the score function $\mathbf{U}(\boldsymbol{\varphi}) = (\boldsymbol{U_a}, \boldsymbol{U_b}, \boldsymbol{U_\beta}, \boldsymbol{U_\lambda})$ are given by

$$U_a = \frac{n}{a} - \sum_{i=1}^{n} (\frac{x_i^{\beta}}{\alpha^\beta - x_i^{\beta}})^b - 2\lambda \sum_{i=1}^{n} \frac{\{\frac{x_i^{\beta}}{\alpha^\beta - x_i^{\beta}}\}^b \exp(-a\{\frac{x_i^{\beta}}{\alpha^\beta - x_i^{\beta}}\}^b)}{1-\lambda+2\lambda \exp(-a\{\frac{x_i^{\beta}}{\alpha^\beta - x_i^{\beta}}\}^b)},$$

$$U_b = \frac{n}{b} + \beta\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\ln\left(\alpha^\beta - x_i{}^\beta\right) - a\sum_{i=1}^{n}\left(\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right)^b \ln\left(\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right)$$

$$-2\mathrm{a}\lambda\sum_{i=1}^{n}\frac{\left(\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right)^b \ln\left(\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right)\exp\left(-a\left\{\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right\}^b\right)}{1-\lambda+2\lambda\exp\left(-a\left\{\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right\}^b\right)}$$

$$U_\beta = \frac{n}{\beta} + n\,ln\alpha + b\sum_{i=1}^{n}\ln(x_i) - (b+1)\sum_{i=1}^{n}\frac{\alpha^\beta ln\alpha - x_i{}^\beta ln x_i}{\alpha^\beta - x_i^\beta}$$

$$-ab\alpha^\beta\sum_{i=1}^{n}\frac{x_i{}^{b\beta}(ln x_i - ln\alpha)}{(\alpha^\beta - x_i^\beta)^{b+1}} - 2\mathrm{ab}\lambda\alpha^\beta\sum_{i=1}^{n}\frac{\frac{x_i{}^{b\beta}(ln x_i - ln\alpha)}{(\alpha^\beta - x_i^\beta)^{b+1}}\exp\left(-a\left\{\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right\}^b\right)}{1-\lambda+2\lambda\exp\left(-a\left\{\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right\}^b\right)}$$

and

$$U_\lambda = \sum_{i=1}^{n}\frac{2\exp\left(-a\left\{\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right\}^b\right) - 1}{1-\lambda+2\lambda\exp\left(-a\left\{\frac{x_i{}^\beta}{\alpha^\beta - x_i{}^\beta}\right\}^b\right)}$$

Since $\mathbf{x} \le \boldsymbol{\alpha}$, the MLE of $\boldsymbol{\alpha}$ is the last-order statistic $\mathbf{x}_{(\mathbf{n})}$. Setting $\boldsymbol{U_a}, \boldsymbol{U_b}, \boldsymbol{U_\beta}$ and $\boldsymbol{U_\lambda}$ equal to zero and solving these equations simultaneously yield the MLE $\hat{\boldsymbol{\varphi}} = (\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{\beta}}, \boldsymbol{\lambda})$of $\boldsymbol{\varphi} = (\mathbf{a}, \mathbf{b}, \boldsymbol{\beta}, \boldsymbol{\lambda})^{\boldsymbol{T}}$ These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

6. Simulation study

In this section, an extensive numerical investigation will be carried out to evaluate the performance of MLE for TWPF model. Performance of estimators is evaluated through their biases, and mean square errors (MSEs) for different sample sizes. A numerical study is performed using Mathematica (9) software. Different sample sizes are considered through the experiments at size n = 100, 150 and 200. In addition, the different values of parameters a, b, α, β and λ.

The experiment will be repeated 1000 times. In each experiment, the estimates of the parameters will be obtained by maximum likelihood methods of estimation. The means, MSEs and biases for the different estimators will be reported from these experiments.

n	Par	Parameters	MLE	Bias	MSE	Parameters	MLE	Bias	MSE
50	a	2.0	2.0203	0.0203	0.0333	2.0	2.0177	0.0177	0.0325
	b	2.0	2.0414	0.0414	0.0914	1.5	1.5451	0.0451	0.0564
	α	0.5	0.5827	0.0827	0.6697	0.5	0.5309	0.0309	0.0689
	β	2.0	2.2117	0.2117	1.0595	2.0	2.1276	0.1276	0.4035
	λ	0.5	0.5008	0.0008	0.0002	0.5	0.5006	0.0006	0.0002
100	a	2.0	2.0141	0.0141	0.0241	2.0	2.0104	0.0104	0.0235
	b	2.0	2.0313	0.0313	0.0630	1.5	1.5220	0.0220	0.0329
	α	0.5	0.5704	0.0704	0.1708	0.5	0.5300	0.0300	0.0217
	β	2.0	2.1375	0.1375	0.4572	2.0	2.0844	0.0844	0.2128
	λ	0.5	0.5006	0.0006	0.0002	0.5	0.5004	0.0004	0.0002
150	a	2.0	2.0124	0.0124	0.0181	2.0	2.0058	0.0058	0.0163
	b	2.0	2.0315	0.0315	0.0455	1.5	1.5147	0.0147	0.0219
	α	0.5	0.5336	0.0336	0.0341	0.5	0.5212	0.0212	0.0137
	β	2.0	2.0926	0.0926	0.2563	2.0	2.0553	0.0553	0.1172
	λ	0.5	0.5006	0.0005	0.0001	0.5	0.5001	0.0001	0.0001

Table 1: The parameter estimation from TWPF distribution using MLE

7. Application

This section provides two applications to show how the TWPF distribution can be applied in practice. For this aim, the TWPF distribution is compared with other competitive distributions. In these applications, the model parameters are estimated by the method of maximum likelihood. The Akaike information criterion (AIC), Bayesian information criterion, Anderson-Darling (A*) and Cramer‒von Mises (W*) statistics are computed to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data.

The plots of the fitted pdfs, cdfs of some distributions given below are displayed for visual comparison. The required computations are carried out in the R-language. The density functions of the compared distributions are given by

- Beta Exponential (BE) distribution with the pdf

$$f(x) = \frac{\lambda}{Beta[a, b]} e^{-(\lambda b x)}(1 - e^{-\lambda x})^{a-1}$$

- The gamma exponentiated exponential (GEE) distribution with the pdf

$$f(x) = \frac{\lambda a^{\delta}}{\Gamma(\delta)} e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}(-log(1 - e^{-\lambda x}))^{\delta-1}$$

- Weibull Fréchet (WFr) distribution with the pdf

$$f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}exp[-b(\frac{\alpha}{x})^{\beta}](1-[-b(\frac{\alpha}{x})^{\beta}])^{-b-1}exp[-a(exp\left[(\frac{\alpha}{x})^{\beta}\right]-1)^{-b}\right]$$

- Weibull power function (WPF) distribution with the pdf

$$f(x) = \frac{ab\beta\alpha^{\beta}x^{\beta b-1}}{(\alpha^{\beta}-x^{\beta})^{b+1}}e^{-a(\frac{x^{\beta}}{\alpha^{\beta}-x^{\beta}})^{b}}, 0 < x < \alpha, a, b, \alpha, \beta > 0$$

The first data set represents the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England [25]. The data are as follows: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

The second data consisting of 100 observations on breaking stress of carbon fibres Nichols and Padgett [26] are given below: 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

The descriptive statistics are presented in Table 2 for both data sets.

n	Minimum	Median	Mean	Maximum	Variance	Skewness	Kurtosis
63	0.550	1.590	1.507	2.240	0.105	-0.900	3.924
100	0.390	2.675	2.611	5.560	1.016578	0.3925	3.1775

Table 1: The parameter estimation from TWPF distribution using MLE

The MLEs of unknown parameters of the distributions for the both data set are given in Tables 3 and 5. The numerical values of the statistics AIC, BIC, A*, W* are listed in Tables 4 and 6.

Distribution	Estimates				
BE(a, b, λ)	17.443	870.58	0.0132	-	-
GEE(λ, α, δ)	0.4339	24.666	18.803	-	-
WFr(a, b, α, β)	1.4762	16.856	0.3865	0.2436	-
WPF(α, β, a, b)	2.2400	3.2311	1.9916	1.2633	-
TWPF(α, β, a, b, λ)	2.240	2.22202	0.5771	1.58054	0.7786

Table 3: The MLEs for the first data set

Model	Goodness of fit criteria				
	AIC	BIC	-L	A*	W*
BE(a, b, λ)	53.904	60.333	-23.952	3.1256	0.5703
GEE(λ, α, δ)	55.019	61.448	-24.510	3.2248	0.5885
WFr(a, b, α, β)	39.000	47.573	-15.207	1.3410	0.2326
WPF(α, β, a, b)	25.699	33.081	-9.8490	0.8078	0.1593
TWPF(α, β, a, b, λ)	23.300	31.809	-7.850	0.6085	0.1045

Table 4: Some statistics for models fitted to the first data set

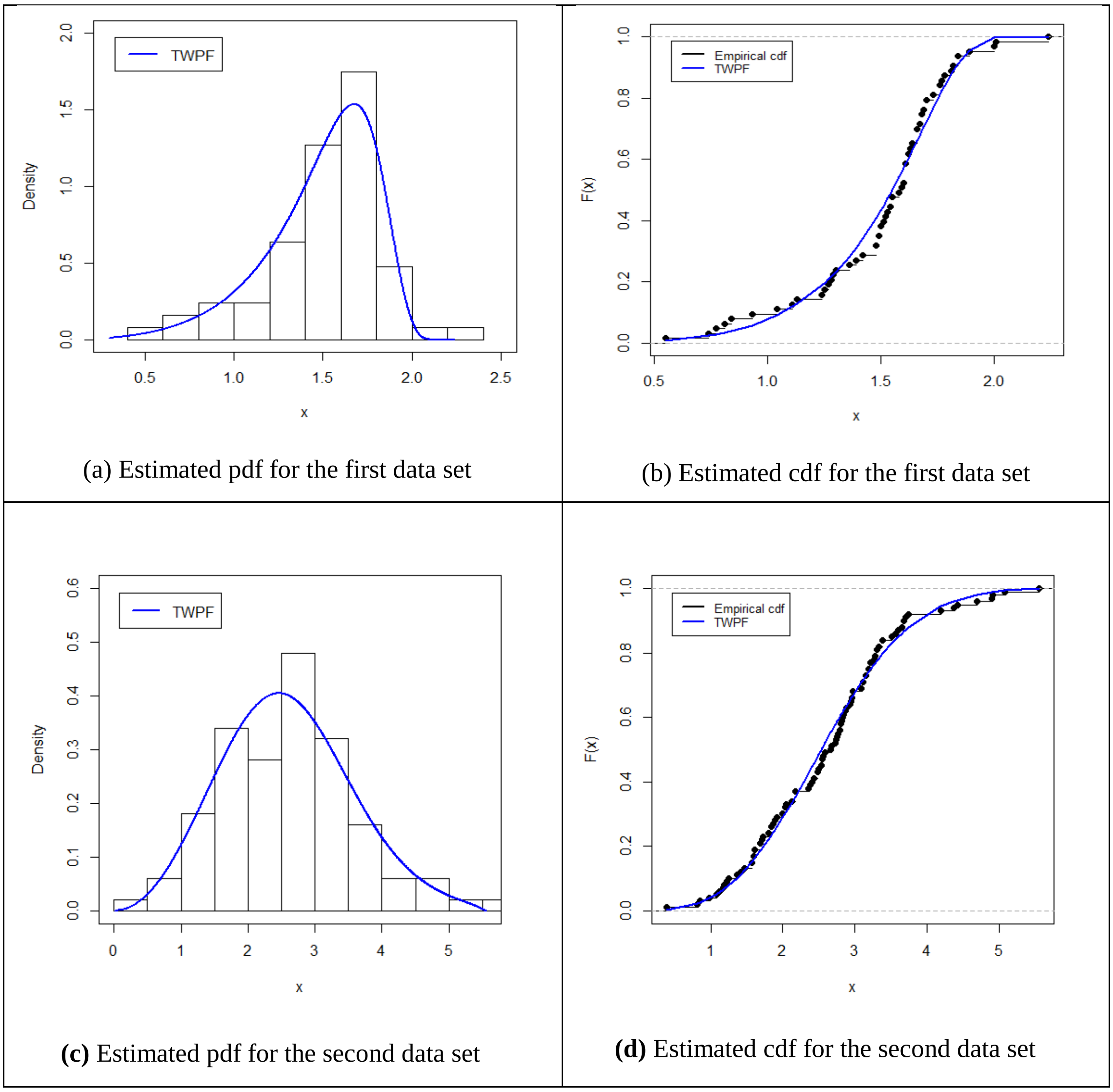
Distribution	Estimated Parameters				
BE (a, b, λ)	5.99635	191.696	0.0118252		
GEE(λ, α, δ)	0.27192	8.13102	6.17272		
WFr(α, β, a, b)	4.97914	0.241625	170.755	6.82992	
WPF(α, β, a, b)	5.56010	13.5877	6.0266060	0.2065125	
TWPF(α, β, b, a, λ)	5.56000	10.5680033	0.2879722	4.5781689	0.6952126

Table 5: The MLEs for the second data set

Distribution	AIC	BIC	-L	A*	W*
BE (a, b, λ)	291.056	298.871	-142.528	0.77823	0.15443
GEE(λ, α, δ)	292.054	299.870	-143.027	0.85420	0.16904
WFr(α, β, a, b)	289.599	300.019	-140.799	0.44782	0.07509
WPF(α, β, a, b)	284.674	292.489	-139.337	0.47192	0.06761
TWPF(α, β, a, b, λ)	278.585	288.965	-135.292	0.42123	0.07171

Table 6: Some statistics for models fitted to the second data set

Based on the Tables 4 and 6, we conclude that the new TWPF model provides adequate fits as compared to other models in both applications with small values for AIC, BIC, A*, W*. In the two applications, the proposed TWPF model is much better than the four models. The histograms of the two data sets and the estimated pdfs and cdfs of the proposed and competitive models are displayed in Figure 2. Figure 2 also supports the results in Tables 4 and 6.



(a) Estimated pdf for the first data set

(b) Estimated cdf for the first data set

(c) Estimated pdf for the second data set

(d) Estimated cdf for the second data set

Figure 2: Plots of the estimated pdfs and cdfs of the models for both data sets.

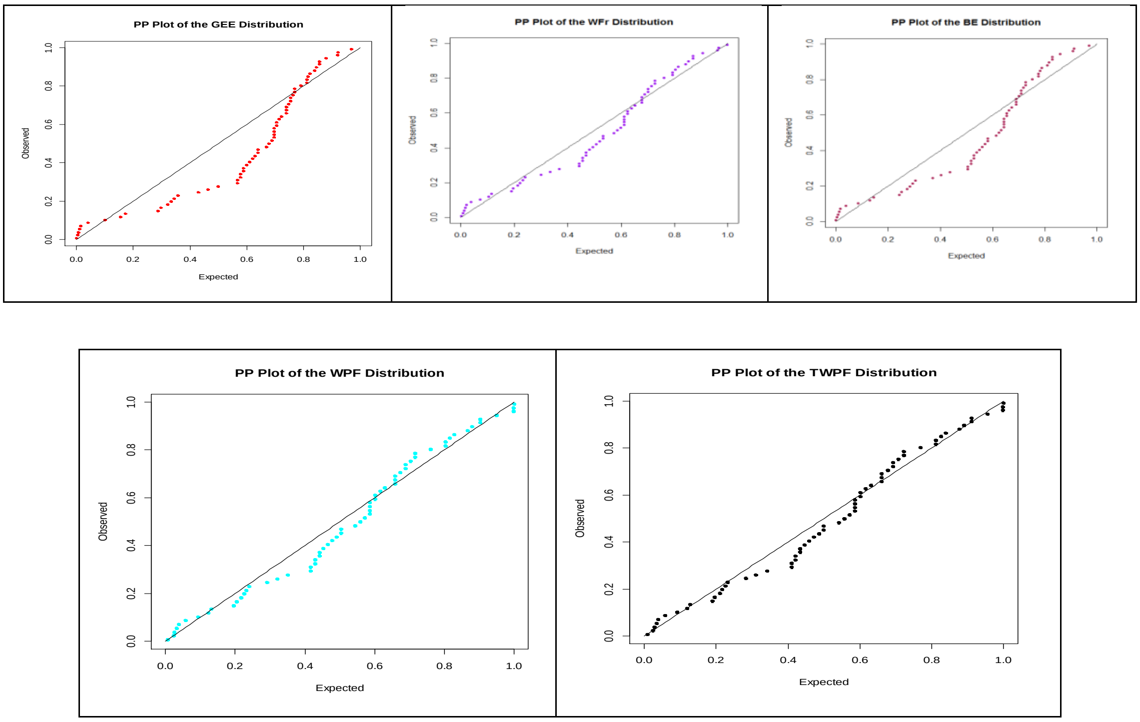


Figure 3**:** PP plots of the TWPF and other fitted distributions for the first data set

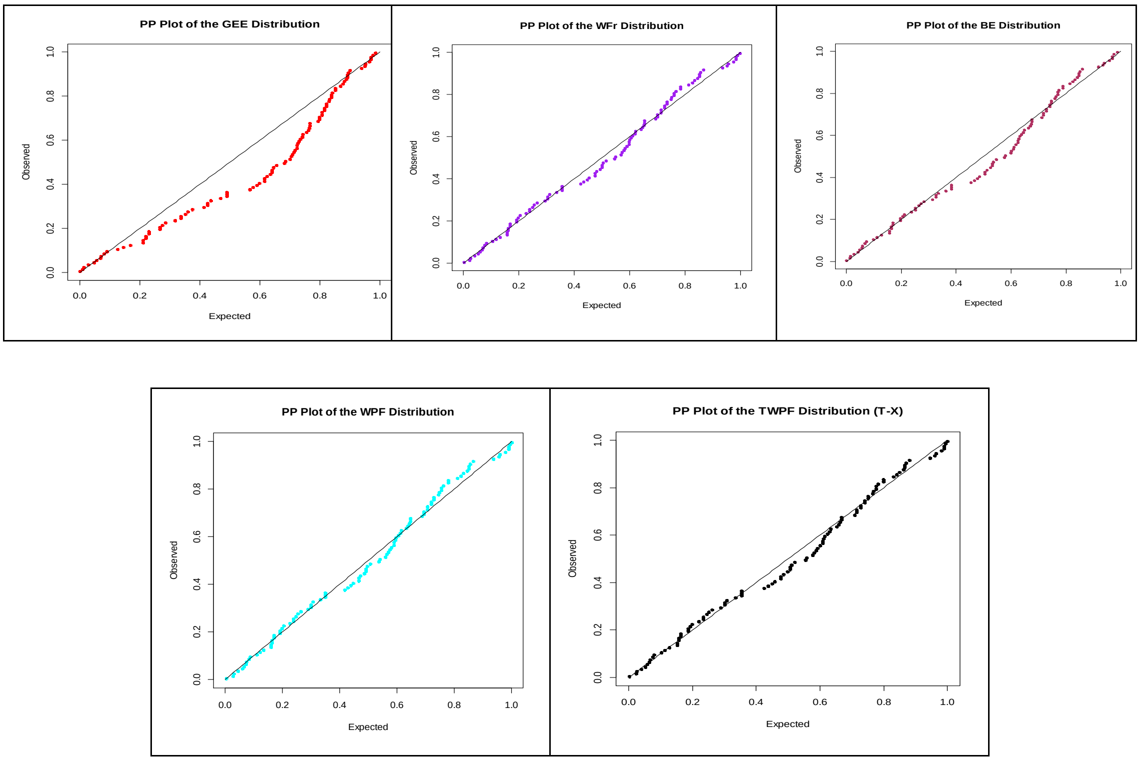


Figure 4: PP plots of the TWPF distribution and other fitted distributions for the second data set

In Figures 3 and 4, the probability-probability (P-P) plots of the TWPF distribution and other fitted distribution are also presented for both data set. As seen in Figures 3 and 4, both data sets fits well the TWPF distribution than the other fitted distributions.

8. Concluding remarks

In this paper, we propose and study the new distribution called as transformed Weibull power function (TWPF) distribution. We investigate some of its mathematical properties including an expansion for the density function and explicit expressions for the quantile function, ordinary and incomplete moments, moment generating function, entropies, reliability function and order statistics. The maximum likelihood method is employed to estimate the model parameters. We fit TWPF model to two real data sets to demonstrate the flexibility of it. We hope that the new distribution will attract wider application in areas such as engineering, survival and lifetime data, hydrology, economics, among others.

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