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Adaptive Type-II Progressive Censoring Schemes based on Maximum Product Spacing with Application of Generalized Rayleigh Distribution

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Abstract

In this paper, parameters estimation for the generalized Rayleigh (GR) distribution are discussed under the adaptive type-II progressive censoring schemes based on maximum product spacing. A comparison studies with another methods as maximum likelihood, and Bayesian estimation by use Markov chain Monte Carlo (MCMC) are discussed. Also, reliability estimation and hazard function are obtained. A numerical study using real data and Monte Carlo Simulation are performed to compare between different methods.

Keywords: Generalized Rayleigh distribution, Adaptive Type-II progressive censoring, Maximum Product Spacing, Bayesian estimation and Reliability estimation.

1.Introduction

Kundu and Raqab (2005) introduced the GR distribution. A random variable X has the GR distribution with parameters α and θ, say if its cumulative distribution function (cdf) and probability density function (pdf) are given by

$$F(x;\alpha,\theta) = \left(1 - e^{-(\theta x)^2}\right)^{\alpha}; \quad x > 0, \alpha, \theta > 0 \tag{1.1}$$

and

$$f(x;\alpha,\theta) = 2\alpha\theta^2 x e^{-(\theta x)^2}\left(1 - e^{-(\theta x)^2}\right)^{\alpha-1} \tag{1.2}$$

respectively, where $\alpha, \theta > 0$ and $x > 0$.

The maximum likelihood estimators, the modified moment estimators, estimators based on percentiles, least squares estimators, weighted least squares estimators and the modified L-moment estimators by using extensive simulation techniques based on complete data are obtained to estimate parameters for GR distribution by Kundu and Raqab (2005). The Maximum likelihood and Bayesian approaches based on progressive Type-II censoring are obtained for GR distribution by Raqab and Madi (2011). The maximum likelihood and Bayesian estimation of the parameters for GR distribution and reliability function under type-I progressive hybrid censoring scheme had discussed by Tomer et al. (2014). The maximum likelihood estimation of the shape and scale parameters concerning GR distribution based on ranked set sampling and its some modifications had discussed by Esemen and Gürler (2018).

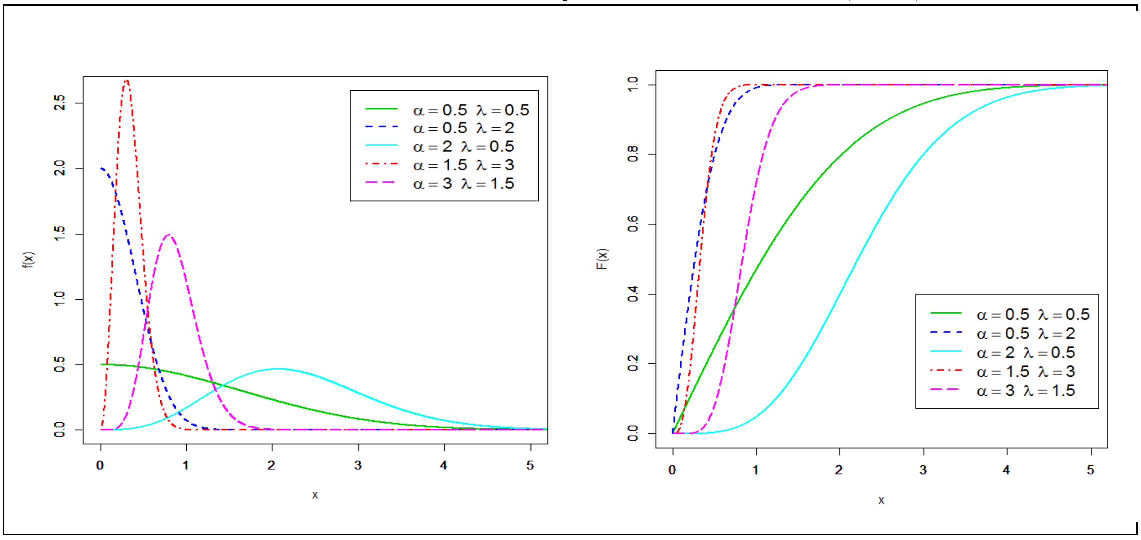
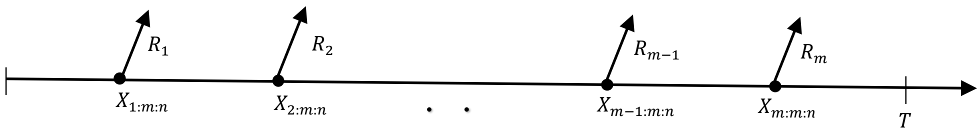
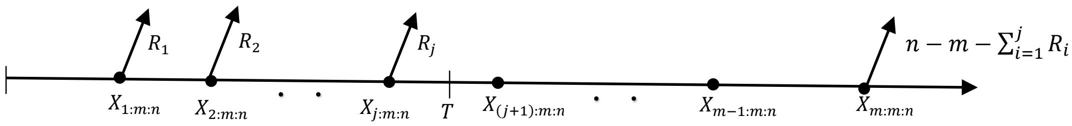


Figure (1). GR distribution with Various Value of parameters.

In censoring scheme, the most important used censored schemes are Type-I and Type-II censoring. In general, censoring scheme called Progressive censoring scheme are placed on a life testing experiment and m is a predetermined number of units to be failed, for more information see Balakrishnan and Sandhu (1995) and Balakrishnan (2007), Many authors have discussed inference under progressive Type-II censoring using different lifetime distributions. For examples, see Ng et al. (2012), Soliman et al. (2015), EL-Sagheer (2018), Singh et al. (2015), Dey at el. (2016) and Almetwaly and Almongy (2018). Ng et al. (2009) suggested an adaptive type-II progressive censoring scheme in which the effective number of failures m is fixed in advance and the progressive censoring scheme $(R_1, R_2, \dots, R_m)$ is provided, but the values of the R_i may be change accordingly during the experiment, but the experimenter consider provides a time T.



Experiment terminates before time T



Experiment terminates after time T

Figure (2): schematic representation of adaptive type-II progressive censoring

Note that:

$if\ (x_m \le T)\ \rightarrow\ the\ experiment\ stops\ at\ this\ time(\ Progressive).$

$if\ (x_m > T)$

$\rightarrow\ adapt\ the\ number\ of\ items\ progressively\ with\ removed\ items(Adaptive)$

See for more example Ye at al. (2014), Sobhi and Soliman (2016) and Hemmati and Khorram (2017) and Almetwally and Almongy (2018).

The maximum product spacing estimation (MPSE) method was introduced by Cheng and Amin (1983) and independently discussed by Ranneby (1984) as an alternative method to maximum likelihood estimation (MLE) method for continuous distributions, for example see Singh et al. (2014) and to more information see Ekström (2006). Ng et al. (2012) introduced estimation of the parameters for a three-parameter Weibull distribution based on progressively Type-II right censored samples using the MPSE method. See for example Singh et al. (2016) and Almetwally and Almongy (2019). Basu et al. (2018) introduced product of spacing estimator for a Progressive hybrid Type-I censoring scheme with binomial removals.

The goal of this paper is parameters estimation for the GR distribution under adaptive type-II progressive censoring scheme based on MPSE. The comparison is done by using the methods (MLE, MPSE and Byesian) which are proposed in this study. The optimal censoring scheme depend on using three different methods for optimality criteria (mean squared error (MSE), Bias and relative efficiency (RE)). To evaluate the performance of the estimators, extensive simulation techniques is carried out. Application of real data set is introduced to confirm the validity of the model and the scheme.

The paper is organized as follows: In section 2, we introduce adaptive Type-II progressive censoring scheme based on MPSE. Parameters estimation of the GR distribution under adaptive Type-II progressive censoring scheme are derived in section 3, while in section 4 the Reliability Function estimation is discussed. In section 5, Monte Carlo simulation study to compare the performance of the parameters estimation for different methods is presented. In section 6, two applications of real data sets are studied. Finally, the results and conclusion of the current study are discussed.

2.Adaptive Type-II Progressive Censoring Schemes of MPSE

Based on the observed sample $x_{1:m:n} < \cdots < x_{m:m:n}$ from an adaptive Type-II progressive censoring scheme, the maximum product spacing under adaptive Type-II progressive censoring scheme can be written as:

$$S = A\prod_{i=1}^{m+1}(D_{i:m:n})\left(\prod_{i=1}^{j}(1-F(x_{i:m:n};\ \theta,\alpha))^{R_i}\right)\left(1 - F(x_{m:m:n};\ \theta,\alpha)\right)^{n-m-\sum_{i=1}^{j}R_i}, \tag{2.1}$$

where A is a constant that it doesn't depend on parameters and

$$D_{i:m:n} = \begin{cases} D_{1:m:n} = F(x_{1:m:n}) \\ D_{i:m:n} = F(x_{i:m:n}) - F(x_{(i-1):m:n});\ i = 2 \ldots m, \\ D_{(m+1):m:n} = 1 - F(x_{m:m:n}) \end{cases} \tag{2.2}$$

such that $\sum D_{i:m:n} = 1$, depending on Cheng and Amin (1983), Ng et al. (2012) and Ng et al. (2009).

Table 1: Special Cases

T	Scheme	Cases	Function
$(x_m > T)$	any different scheme R_i	MPSE under adaptive Type-II progressive censoring scheme.	Eq. (2.1)
$(x_m \le T)$	any different scheme R_i	MPSE function under progressive type-II censoring scheme. Ng et al. (2012).	$S = A\prod_{i=1}^{m+1}(D_{i:m:n})\left(\prod_{i=1}^{m}(1 - F(x_{i:m:n};\ \theta,\alpha))^{R_i}\right)$
$(x_m > T)$ or $(x_m \le T)$	$R_1 = \cdots = R_{m-1} = 0,$ $R_m = n - m$	MPSE function under type-II censoring scheme.	$S = A(1 - F(x_{m:m:n};\ \theta,\alpha))^{n-m}\prod_{i=1}^{m+1}(D_{i:m:n})$
$(x_m \le T)$	$R_1 = \cdots = R_m = 0$	MPSE function under complete censoring sample. Cheng and Amin (1983)	$\prod_{i=1}^{m+1}(D_{i:m:n})$

3.Parameters Estimation of the GR Distribution

In this section, MLE, MPSE and Bayesian estimation by use MCMC of the GR distribution under adaptive type-II progressive censored data have been discussed.

3.1.MLE Method

The likelihood function of GR distribution under adaptive type-II progressive censored data can be written as

$$L_{MLE}(\Theta) = A(2\alpha\theta^2)^m \left(1 - \left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)^{n-m-\sum_{i=1}^{j} R_i} e^{-\theta^2 \sum_{i=1}^{m} x_{i:m:n}^2} \prod_{i=1}^{m}\left(x_{i:m:n}\left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1}\right)\prod_{i=1}^{j}\left(1 - \left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)^{R_i}, \tag{3.1}$$

The natural logarithm of the likelihood function is

$$\begin{aligned}\ln L_{MLE}(\Theta) = {} & \ln A + m\ln 2 + m\ln\alpha + 2m\ln\theta - \theta^2\sum_{i=1}^{m} x_{i:m:n}^2 \\ & + \sum_{i=1}^{m}\ln(x_{i:m:n}) + (\alpha-1)\sum_{i=1}^{m}\ln\left(1 - e^{-(\theta x_{i:m:n})^2}\right) \\ & + \sum_{i=1}^{j} R_i \ln\left(1 - \left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right) \\ & + \left(n - m - \sum_{i=1}^{j} R_i\right)\ln\left(1 - \left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)\end{aligned} \tag{3.2}$$

to obtain the normal equations for the unknown parameters, we differentiate equation (3.2) partially with respect to the parameters α and θ and equate them to zero. The estimators for $\hat{\theta}$ and $\hat{\alpha}$ can be obtained as in the solution of the following equations:

$$\frac{\partial \ln L_{MLE}(\Theta)}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} \ln\left(1 - e^{-(\theta x_{i:m:n})^2}\right) - \sum_{i=1}^{j} R_i \frac{\left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha} \ln\left(1 - e^{-(\theta x_{i:m:n})^2}\right)}{\left(1 - \left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)} - \left(n - m - \sum_{i=1}^{j} R_i\right) \frac{\left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha} \ln\left(1 - e^{-(\theta x_{m:m:n})^2}\right)}{\left(1 - \left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)} \tag{3.3}$$

and

$$\frac{\partial \ln L_{MLE}(\Theta)}{\partial \theta} = \frac{2m}{\theta} - 2\theta \sum_{i=1}^{m} {x^2}_{i:m:n} + (\alpha - 1) \sum_{i=1}^{m} \frac{2\theta x_{i:m:n}^2 e^{-(\theta x_{i:m:n})^2}}{\left(1 - e^{-(\theta x_{i:m:n})^2}\right)} - \sum_{i=1}^{j} R_i \frac{2\alpha\theta x_{i:m:n}^2 e^{-(\theta x_{i:m:n})^2} \left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1}}{\left(1 - \left(1 - e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)} - \left(n - m - \sum_{i=1}^{j} R_i\right) \frac{2\alpha\theta x_{m:m:n}^2 e^{-(\theta x_{m:m:n})^2} \left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha-1}}{\left(1 - \left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)} \tag{3.4}$$

The MLE of $\hat{\alpha}$, $\hat{\theta}$ are obtained by solving simultaneously the likelihood equations $\left.\frac{\partial L_{ML}}{\partial \alpha}\right|_{\alpha = \hat{\alpha}} = 0$, $\left.\frac{\partial L_{ML}}{\partial \theta}\right|_{\theta = \hat{\theta}} = 0$.

But the equations (3.3) and (3.4) have to be performed numerically using a nonlinear optimization algorithm.

3.2.MPSE Method

The maximum product spacing under adaptive Type-II progressive censoring scheme for GR distribution can be written as:

$$L_{MPS}(\Theta) = A\left(1 - e^{-(\theta x_{1:m:n})^2}\right)^{\alpha} \left(1 - \left(1 - e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)^{n+1-m-\sum_{i=1}^{j} R_i} \tag{3.5}$$

$$\prod_{i=1}^{j}\left(1-\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)^{R_i}$$

$$\prod_{i=2}^{m}\left(\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}-\left(1-e^{-\left(\theta x_{(i-1):m:n}\right)^2}\right)^{\alpha}\right),$$

the natural logarithm of the likelihood function is

$$\begin{aligned}\ln L_{MPS}(\Theta) = {} & \ln A+\alpha\ln\left(1-e^{-(\theta x_{1:m:n})^2}\right) \\ & +\sum_{i=1}^{m}\ln\left(\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha} -\left(1-e^{-\left(\theta x_{(i-1):m:n}\right)^2}\right)^{\alpha}\right) \\ & +\ln\left(1-\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right) \\ & +\sum_{i=1}^{j}R_i\ln\left(1-\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right) \\ & +\left(n-m-\sum_{i=1}^{j}R_i\right)\ln\left(1-\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)\end{aligned} \tag{3.6}$$

To obtain the normal equations for the unknown parameters, differentiate (3.6) partially with respect to the parameters α and θ and equate them to zero. The estimators for $\hat{\theta}$ and $\hat{\alpha}$ can be obtained as illustrative in the solution of the following equations:

$$\begin{aligned}\frac{\partial \ln L_{MPS}(\Theta)}{\partial\alpha} & \\ = {} & \ln\left(1-e^{-(\theta x_{1:m:n})^2}\right) \\ & -\frac{\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\ln\left(1-e^{-(\theta x_{m:m:n})^2}\right)}{\left(1-\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)}\left(1+\left(n-m-\sum_{i=1}^{j}R_i\right)\right) \\ & -\sum_{i=1}^{j}R_i\frac{\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\ln\left(1-e^{-(\theta x_{i:m:n})^2}\right)}{\left(1-\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)} \\ & +\sum_{i=1}^{m}\frac{\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\ln\left(1-e^{-(\theta x_{i:m:n})^2}\right)-\left(1-e^{-\left(\theta x_{(i-1):m:n}\right)^2}\right)^{\alpha}\ln\left(\right.}{\left(\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}-\left(1-e^{-\left(\theta x_{(i-1):m:n}\right)^2}\right)^{\alpha}\right)}\end{aligned} \tag{3.7}$$

and

$$\frac{\partial \ln L_{MPS}(\Theta)}{\partial\theta} = \alpha\sum_{i=1}^{m}\frac{2\theta x_{1:m:n}^2 e^{-(\theta x_{1:m:n})^2}}{\left(1-e^{-(\theta x_{1:m:n})^2}\right)} - \frac{2\alpha\theta x_{m:m:n}^2 e^{-(\theta x_{m:m:n})^2}\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha-1}}{\left(1-\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)}\left(1 + \left(n-m-\sum_{i=1}^{j}R_i\right)\right)$$
$$+\sum_{i=1}^{m}\frac{2\alpha\theta x_{i:m:n}^2 e^{-(\theta x_{i:m:n})^2}\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1} - 2\alpha\theta x_{(i-1):m:n}^2 e^{-(\theta x_{(i-1):m:n})^2}\cdots}{\left(\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha} - \left(1-e^{-(\theta x_{(i-1):m:n})^2}\cdots\right.\right.} - \sum_{i=1}^{j}R_i\frac{2\alpha\theta x_{i:m:n}^2 e^{-(\theta x_{i:m:n})^2}\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1}}{\left(1-\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)} \quad (3.8)$$

The MPSE for $\hat{\alpha}, \hat{\theta}$ are obtained by solving simultaneously equations

$$\left.\frac{\partial L_{ML}}{\partial\alpha}\right|_{\alpha=\hat{\alpha}} = 0, \quad \left.\frac{\partial L_{ML}}{\partial\theta}\right|_{\theta=\hat{\theta}} = 0,$$

but the equations (3.7) and (3.8) have to be performed numerically using a nonlinear optimization algorithm.

3.3.Bayesian Estimation

We consider the Bayesian estimation of the unknown parameters α and θ. The Bayesian estimates is considered under assumption that the random variables α and θ have an independent gamma distribution is a conjugate prior to the GR distributions under adaptive type-II progressive censored scheme. Assumed that $\theta \sim Gamma(a, b)$ and $\alpha \sim Gamma(c.d)$ according to Kundu and Howlader (2010) and Madi and Raqab (2009) then, the joint prior density of α and θ can be written as

$$g(\alpha,\theta) \propto \theta^{a-1}\, e^{-\frac{\theta}{b}}\, \alpha^{c-1}\, e^{-\frac{\alpha}{d}}; \quad a, b, c, and\ d > 0, \quad (3.9)$$

where all the hyper parameters $a, b, c\ and\ d$ are known and non-negative. According to Kundu and Howlader (2010), the hyper-parameters can be chosen to suit the prior belief of the experimenter in terms of location and variability of the gamma prior distribution.

Based on the likelihood function in Equation (3.1) and the joint prior density in Equation (3.9), the joint posterior of adaptive type-II progressive censored of GR distribution with parameters α and θ is

$$g(\alpha,\theta|x) = K\,2^m \alpha^{m+c-1}\theta^{2m+a-1} e^{-\theta^2\sum_{i=1}^{m} x_{i:m:n}^2}\left(1 - \left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)^{n-m-\sum_{i=1}^{j} R_i} \prod_{i=1}^{m}\left(x_{i:m:n}\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1}\right) e^{-\frac{\theta}{b}} e^{-\frac{\alpha}{d}} \prod_{i=1}^{j}\left(1 - \left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)^{R_i}, \tag{3.10}$$

where K is normalizing constant

MCMC Approach: A wide variety of MCMC techniques are available, and it can be difficult to choose among them. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within Gibbs samplers. For more information of MCMC techniques see Nassar et al. (2018).

The conditional posterior densities of $\Theta = (\alpha, \theta)$ are as follows:

$$\pi_1^*(\alpha|\theta,x) \propto \alpha^{m+c-1}\left(1-\left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)^{n-m-\sum_{i=1}^{j} R_i} \prod_{i=1}^{m}\left(\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1}\right) e^{-\frac{\alpha}{d}} \prod_{i=1}^{j}\left(1-\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)^{R_i}, \tag{3.11}$$

$$\pi_2^*(\theta|\alpha,x) \propto \theta^{2m+a-1} e^{-\theta^2\sum_{i=1}^{m} x_{i:m:n}^2}\left(1 - \left(1-e^{-(\theta x_{m:m:n})^2}\right)^{\alpha}\right)^{n-m-\sum_{i=1}^{j} R_i} \prod_{i=1}^{m}\left(x_{i:m:n}\left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha-1}\right) e^{-\frac{\theta}{b}} \prod_{i=1}^{j}\left(1 - \left(1-e^{-(\theta x_{i:m:n})^2}\right)^{\alpha}\right)^{R_i} \tag{3.12}$$

Therefore, to generate these distributions by this method, we use the Metropolis-Hastings method (Metropolis et al. (1953) with normal proposal distribution). For more details regarding the implementation of Metropolis-Hasting algorithm, the readers may refer to Robert and Casella (2004), Mahanta et al. (2018) and Almetwally et al. (2018).

4.Reliability Function

In this section, we propose the estimation of reliability and hazard function using MLE and MPSE methods of time, say $(t = 0.25)$ and $(t = 0.5)$. So on, on this basis and by using the invariance property we estimate the reliability and hazard function. The estimates of the reliability and hazard function is given as:

$$\hat{S}(t) = 1 - \left(1 - e^{-(\hat{\theta} t)^2}\right)^{\hat{\alpha}} \tag{4.1}$$

$$\hat{h}(t)=\frac{2\hat{\alpha}\hat{\theta}^{2}te^{-(\hat{\theta}t)^{2}}\left(1-e^{-(\hat{\theta}t)^{2}}\right)^{\hat{\alpha}-1}}{1-\left(1-e^{-(\hat{\theta}t)^{2}}\right)^{\hat{\alpha}}} \tag{4.2}$$

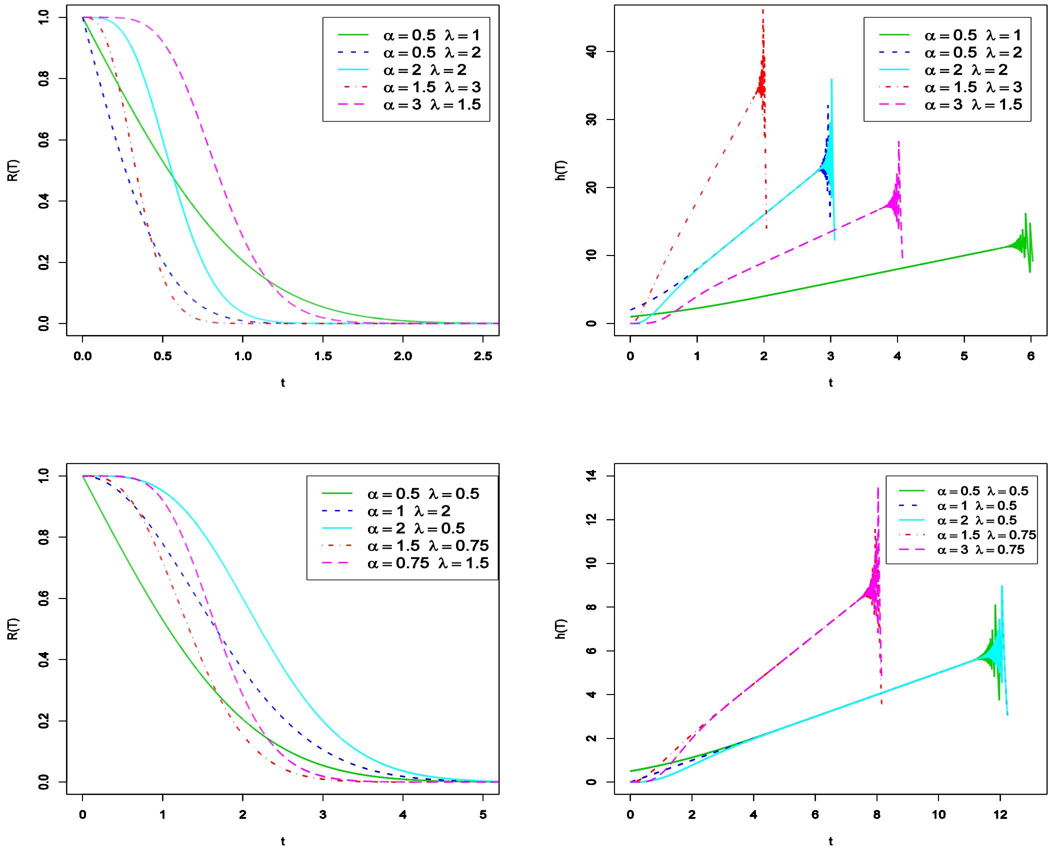


Figure (3): Reliability and Hazard Function for GR Distribution with Various Values of Parameters.

5.Simulation Study

In this section; Monte Carlo simulation is done to estimate the parameters based on adaptive Type-II progressive censoring scheme for MLE, MPSE and Bayesian methods. Using R language and using the following steps.

Simulation Algorithm Scheme: Monte Carlo experiments were carried out based on the following data generated form GR Distribution by using the quantile function $x_u=\frac{1}{\theta}\left(-\ln\left(1-u^{1/\alpha}\right)\right)^{1/2}$; $0<u<1$, where x are distributed as GR for different shape parameters $(\theta,\alpha)=$ $(0.5,0.5)$ *and* $(3,1.5)$, different samples sizes $n\ =20,\ 40,60,80,100$ *and* 120, different effective sample sizes (m) by using ratio of effective sample size $r=\frac{m}{n}=0.4$ *and* 0.5, and set of different samples schemes.

Scheme 1: $R_1 = R_2 = \cdots = R_{\mathrm{m}-1} = 0, and\ R_{\mathrm{m}} = n - m$. It is type-II scheme

Scheme 2: $R_1 = n - m\ and\ R_2 = R_3 = \cdots = R_{\mathrm{m}-1} = 0$.

Scheme 3: $R_1 = R_2 = \cdots = R_{\mathrm{m}-1} = 1, and\ R_{\mathrm{m}} = n - 2m - 1$.

We could define the best scheme as the scheme which minimizes the Bias, mean squared error (MSE) and the relative efficiency (RE) of the estimator. That is, the objective function (to be minimized in this case) would be

$$MSE = Mean(\delta - \hat{\delta})^2. \tag{5.1}$$

where $\widehat{\Theta}$ is the estimated value of $\Theta = (\theta, \alpha)$.

$$Bias = \Theta - \widehat{\Theta} \tag{5.2}$$

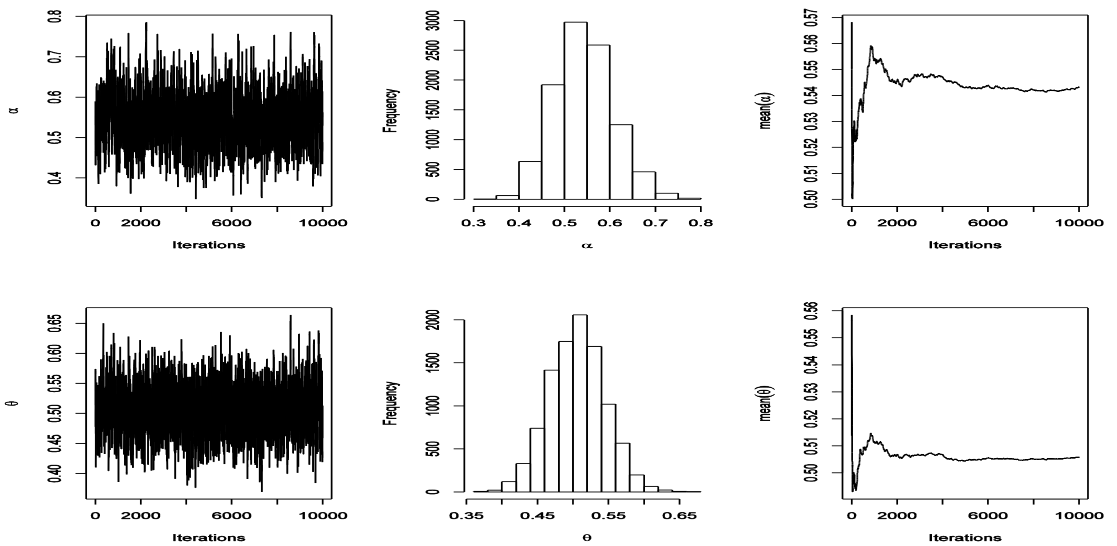
the relative efficiency

$$RE1 = \frac{MSE(MLE)}{MSE(MPS)}, RE2 = \frac{MSE(MLE)}{MSE(Bays)} \tag{5.3}$$

The simulation methods are compared using Bias, MSE and RE for the parameters and their performance was assessed through a Monte Carlo simulation study, we defined the number of repeated-samples to 1000.

The convergence of MCMC estimation of α and θ can be showed in Figure (4) in cases of complete censoring.

$\alpha = 0.5$; $\theta = 0.5$



$\alpha = 1.5$; $\theta = 3$

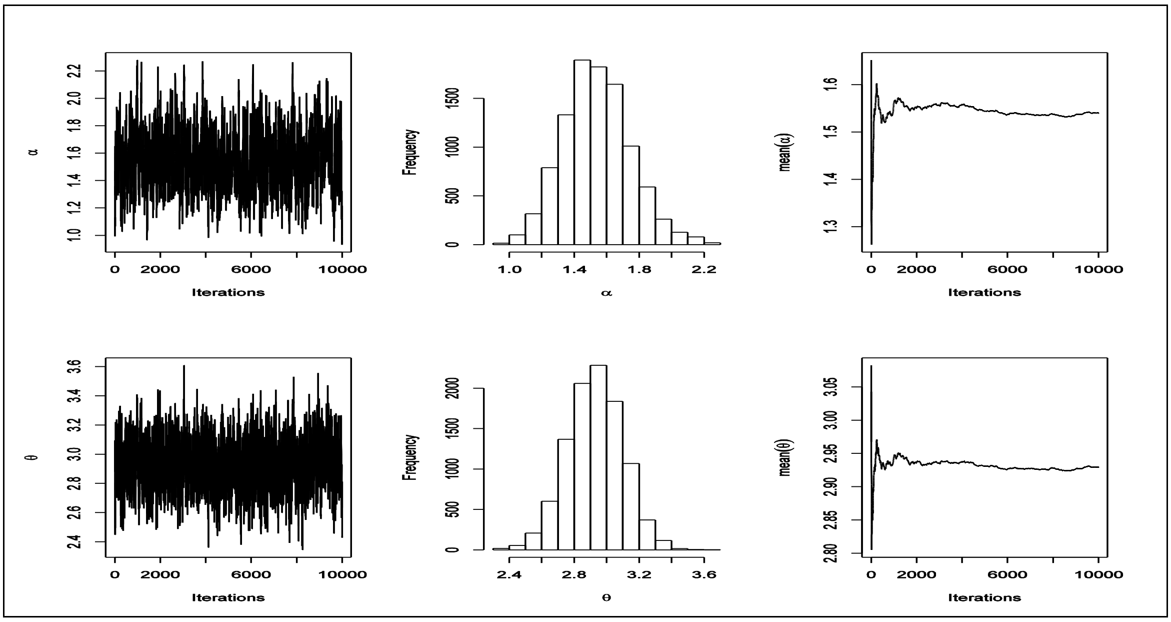


Figure (4): Convergence of MCMC estimation of α and θ

Table 2: Parameter Estimation of GR Distribution in Scheme 1 when $\alpha = 0.5$; $\theta = 0.5$

$r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
n		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	0.2274	0.4471	0.108	0.2106	-0.0610	0.2117	2.1230	2.1115
	$\hat{\theta}$	0.2163	0.2107	0.0944	0.1365	0.0684	0.1292	1.5436	1.6313
40	$\hat{\alpha}$	0.0795	0.0367	0.0365	0.0260	0.4581	0.0249	1.4115	1.4766
	$\hat{\theta}$	0.0917	0.0513	0.0367	0.0391	0.4153	0.0296	1.3120	1.7311
60	$\hat{\alpha}$	0.0519	0.0203	0.0249	0.016	0.1710	0.0142	1.2688	1.4247
	$\hat{\theta}$	0.0644	0.0307	0.0284	0.0252	0.3330	0.0278	1.2183	1.1036
80	$\hat{\alpha}$	0.04	0.014	0.0203	0.0116	0.0000	0.0047	1.2069	2.9874
	$\hat{\theta}$	0.0463	0.0214	0.0199	0.0185	-0.0930	0.0153	1.1568	1.3988
100	$\hat{\alpha}$	0.0254	0.0093	0.0102	0.0081	0.0368	0.0060	1.1481	1.5554
	$\hat{\theta}$	0.0358	0.0163	0.0147	0.0144	0.0465	0.0106	1.1319	1.5323
120	$\hat{\alpha}$	0.0234	0.0078	0.0107	0.007	0.1732	0.0063	1.1143	1.2304
	$\hat{\theta}$	0.0346	0.0133	0.017	0.0119	0.1596	0.0123	1.1176	1.0810
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.1344	0.1011	0.0524	0.0581	0.0087	0.0085	1.7401	11.8660
	$\hat{\theta}$	0.1254	0.0855	0.0364	0.0581	0.0044	0.0108	1.4716	7.9274
40	$\hat{\alpha}$	0.0666	0.0292	0.0311	0.0217	0.1917	0.0256	1.3456	1.1426
	$\hat{\theta}$	0.0637	0.0311	0.0214	0.0249	0.4010	0.1738	1.2490	0.1790
60	$\hat{\alpha}$	0.0366	0.0137	0.0145	0.0113	-0.0383	0.0057	1.2124	2.4214
	$\hat{\theta}$	0.0392	0.0181	0.0114	0.0155	-0.0793	0.0117	1.1677	1.5465
80	$\hat{\alpha}$	0.0296	0.0096	0.0133	0.0082	-0.0759	0.0083	1.1707	1.1540
	$\hat{\theta}$	0.0347	0.013	0.0138	0.0114	-0.1329	0.0121	1.1404	1.0707
100	$\hat{\alpha}$	0.0239	0.0077	0.011	0.0068	0.4158	0.0075	1.1324	1.0204
	$\hat{\theta}$	0.0276	0.0096	0.011	0.0086	0.2018	0.0077	1.1163	1.2479
120	$\hat{\alpha}$	0.016	0.0062	0.0054	0.0056	0.0008	0.0023	1.1071	2.7203
	$\hat{\theta}$	0.0159	0.007	0.0022	0.0065	-0.0362	0.0044	1.0769	1.5735

Table 3: Parameter Estimation of GR Distribution in Scheme 2 when $\alpha = 0.5$; $\theta = 0.5$

$r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
n		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	0.1059	0.0955	-0.0079	0.0407	-0.0281	0.0142	2.3464	6.7483
	$\hat{\theta}$	0.0987	0.0604	-0.0594	0.035	0.0487	0.0146	1.7257	4.1455
40	$\hat{\alpha}$	0.0439	0.0213	-0.0207	0.0141	0.2768	0.0115	1.5106	1.8496
	$\hat{\theta}$	0.0386	0.0146	-0.0582	0.0135	0.0687	0.0103	1.0815	1.4223
60	$\hat{\alpha}$	0.031	0.014	-0.018	0.0104	0.0717	0.0134	1.3462	1.0441
	$\hat{\theta}$	0.0284	0.0093	-0.0453	0.009	-0.0009	0.0058	1.0333	1.6100
80	$\hat{\alpha}$	0.0259	0.0104	-0.0155	0.0081	-0.0758	0.0093	1.2840	1.1195
	$\hat{\theta}$	0.0193	0.0064	-0.0417	0.0063	0.0632	0.0061	1.0159	1.0479

100	$\hat{\alpha}$	0.0146	0.0075	-0.0205	0.0065	0.1581	0.0064	1.1538	1.1733
	$\hat{\theta}$	0.0164	0.0051	-0.0363	0.0052	0.1190	0.0050	1.0196	1.0164
120	$\hat{\alpha}$	0.0146	0.0068	-0.017	0.0058	0.0637	0.0058	1.1724	1.1807
	$\hat{\theta}$	0.0167	0.0042	-0.0302	0.00419	0.0777	0.0041	1.0024	1.0231
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.0772	0.0509	-0.0208	0.0264	-0.0355	0.0142	1.9280	3.5835
	$\hat{\theta}$	0.0664	0.0321	-0.0672	0.0234	-0.0012	0.0087	1.3718	3.6834
40	$\hat{\alpha}$	0.0445	0.0205	-0.0155	0.0137	-0.0435	0.0103	1.4964	1.9978
	$\hat{\theta}$	0.0321	0.0117	-0.0515	0.0113	0.0292	0.0112	1.0354	1.0402
60	$\hat{\alpha}$	0.0247	0.011	-0.0201	0.0085	0.0684	0.0082	1.2941	1.3487
	$\hat{\theta}$	0.0203	0.0074	-0.0436	0.0073	-0.0672	0.0066	1.0137	1.1257
80	$\hat{\alpha}$	0.0207	0.0082	-0.0166	0.0067	0.0902	0.0062	1.2239	1.3150
	$\hat{\theta}$	0.0188	0.0053	-0.0342	0.00528	0.0086	0.0031	1.0038	1.6865
100	$\hat{\alpha}$	0.0175	0.0069	-0.0151	0.0058	0.1024	0.0052	1.1897	1.3253
	$\hat{\theta}$	0.0149	0.0038	-0.0313	0.0036	0.0127	0.0036	1.0556	1.0591
120	$\hat{\alpha}$	0.011	0.0059	-0.0172	0.0052	-0.0505	0.0052	1.1346	1.1421
	$\hat{\theta}$	0.0079	0.0028	-0.032	0.00279	0.0680	0.0028	1.0036	1.0080

Table 4: Parameter Estimation of GR Distribution in Scheme 3 when $\alpha = 0.5$; $\theta = 0.5$

$t = 0.5, r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
n		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	0.1339	0.182	0.0382	0.093	0.1736	0.0659	1.9570	2.7632
	$\hat{\theta}$	0.1199	0.1432	0.0101	0.0968	0.1304	0.0393	1.4793	3.6456
40	$\hat{\alpha}$	0.0223	0.0192	-0.0147	0.0156	-0.1262	0.0142	1.2308	1.3565
	$\hat{\theta}$	-0.0032	0.0331	-0.0518	0.0311	-0.2032	0.0248	1.0643	1.3322
60	$\hat{\alpha}$	-0.0004	0.0109	-0.0238	0.0101	-0.0478	0.0092	1.0792	1.1814
	$\hat{\theta}$	-0.0295	0.0209	-0.0612	0.0202	-0.1827	0.0192	1.0347	1.0896
80	$\hat{\alpha}$	-0.0117	0.0078	-0.0188	0.0078	-0.0980	0.0071	1.0000	1.0938
	$\hat{\theta}$	-0.0498	0.0166	-0.0495	0.0158	-0.0802	0.0142	1.0506	1.1712
100	$\hat{\alpha}$	-0.0235	0.0061	-0.0268	0.0055	0.1335	0.0054	1.1011	1.1251
	$\hat{\theta}$	-0.0587	0.0148	-0.0671	0.0142	0.1202	0.0142	0.9595	1.0435
120	$\hat{\alpha}$	-0.0254	0.0051	-0.0264	0.0051	-0.1163	0.0049	1.0000	1.0374
	$\hat{\theta}$	-0.0606	0.0124	-0.066	0.0121	-0.0948	0.0120	1.0248	1.0301
$t = 0.5, r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	-0.0174	0.0276	-0.0756	0.0249	-0.1080	0.0153	1.1084	1.7986
	$\hat{\theta}$	-0.0826	0.0499	-0.1541	0.0475	-0.0874	0.0162	1.0505	3.0797
40	$\hat{\alpha}$	-0.086	0.0158	-0.1107	0.0139	-0.1343	0.0125	1.1367	1.2617
	$\hat{\theta}$	-0.184	0.0465	-0.2043	0.045	-0.1121	0.0239	1.0333	1.9447
60	$\hat{\alpha}$	-0.1156	0.0175	-0.1008	0.016	-0.0967	0.0168	1.0938	1.0396

	$\hat{\theta}$	-0.2206	0.0568	-0.2297	0.0551	-0.1529	0.0297	1.0309	1.9149
80	$\hat{\alpha}$	-0.1294	0.0199	-0.124	0.0175	-0.1437	0.0162	1.1371	1.2256
	$\hat{\theta}$	-0.2411	0.064	-0.2547	0.0626	-0.3077	0.0620	1.0224	1.0328
100	$\hat{\alpha}$	-0.1408	0.0222	-0.1403	0.0209	0.0148	0.0099	1.0622	2.2321
	$\hat{\theta}$	-0.2587	0.071	-0.2492	0.0698	0.0117	0.0550	1.0172	1.2915
120	$\hat{\alpha}$	-0.1491	0.0241	-0.146	0.023	-0.0750	0.0095	1.0478	2.5398
	$\hat{\theta}$	-0.2712	0.0767	-0.2696	0.0761	-0.2478	0.0637	1.0079	1.2037

Table 5: Parameter Estimation of GR Distribution in Scheme 3 when $\alpha = 0.5$; $\theta = 0.5$

$t = 1.5, r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
n		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	0.192	0.3003	0.0818	0.1431	-0.0367	0.1070	2.0985	2.8070
	$\hat{\theta}$	0.1861	0.1637	0.0648	0.1027	0.1981	0.0921	1.5940	1.7768
40	$\hat{\alpha}$	0.0697	0.0306	0.0271	0.0215	0.1987	0.0207	1.4233	1.4817
	$\hat{\theta}$	0.0784	0.0399	0.0213	0.0298	0.0498	0.0288	1.3389	1.3876
60	$\hat{\alpha}$	0.0458	0.0172	0.0186	0.0135	0.0562	0.0137	1.2741	1.2595
	$\hat{\theta}$	0.0555	0.024	0.0177	0.0193	0.0593	0.0144	1.2435	1.6651
80	$\hat{\alpha}$	0.0356	0.0119	0.0157	0.0099	-0.0675	0.0090	1.2020	1.3266
	$\hat{\theta}$	0.0398	0.0166	0.0117	0.0141	0.0204	0.0103	1.1773	1.6115
100	$\hat{\alpha}$	0.0224	0.008	0.0068	0.007	0.0138	0.0053	1.1429	1.5172
	$\hat{\theta}$	0.0313	0.0118	0.0087	0.0112	0.0209	0.0106	1.0536	1.1176
120	$\hat{\alpha}$	0.0205	0.0067	0.0076	0.006	-0.0616	0.0058	1.1167	1.1538
	$\hat{\theta}$	0.0302	0.0103	0.0113	0.0091	-0.0343	0.0060	1.1319	1.7080
$t = 1.5, r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.0981	0.0693	0.0089	0.037	-0.0629	0.0276	1.8730	2.5108
	$\hat{\theta}$	0.0905	0.061	-0.0225	0.0394	0.0685	0.0251	1.5482	2.4313
40	$\hat{\alpha}$	0.0375	0.0211	-0.0076	0.0155	0.0761	0.0185	1.3613	1.1415
	$\hat{\theta}$	0.0255	0.0241	-0.0371	0.0206	0.0207	0.0176	1.1699	1.3658
60	$\hat{\alpha}$	0.0046	0.0098	-0.025	0.0088	-0.0060	0.0054	1.1136	1.8032
	$\hat{\theta}$	-0.0066	0.0162	-0.0503	0.0161	-0.0080	0.0060	1.0062	2.6947
80	$\hat{\alpha}$	-0.0051	0.0071	-0.028	0.0070	-0.0536	0.0064	1.0143	1.1044
	$\hat{\theta}$	-0.0176	0.0129	-0.0519	0.0126	-0.0707	0.0094	1.0246	1.3758
100	$\hat{\alpha}$	-0.0158	0.006	-0.0344	0.0059	0.0611	0.0041	1.0169	1.4591
	$\hat{\theta}$	-0.0334	0.0106	-0.0616	0.0104	0.0145	0.0056	1.0192	1.8794
120	$\hat{\alpha}$	-0.0265	0.0055	-0.0417	0.0054	-0.0824	0.0043	1.0185	1.2819
	$\hat{\theta}$	-0.0503	0.0105	-0.0735	0.0095	-0.0132	0.0081	1.1053	1.2974

Table 6: Parameter Estimation of GR Distribution in Scheme 1 when $\alpha = 1.5$; $\theta = 3$

$r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
		Bias	MSE	Bias	MSE	Bias	MSE		

20	$\hat{\alpha}$	0.6598	5.4762	0.5761	2.1522	0.0173	0.0933	2.5445	58.6757
	$\hat{\theta}$	0.5599	1.4729	0.1396	1.0201	0.2141	0.0801	1.4439	18.3984
40	$\hat{\alpha}$	0.3398	0.661	0.1376	0.4071	0.3784	0.1919	1.6237	3.4445
	$\hat{\theta}$	0.2459	0.4476	0.0432	0.3592	0.6463	0.3501	1.2461	1.2787
60	$\hat{\alpha}$	0.2171	0.3283	0.0927	0.2347	-0.0198	0.2154	1.3988	1.5241
	$\hat{\theta}$	0.1739	0.2845	0.0393	0.2423	-0.1525	0.2353	1.1742	1.2091
80	$\hat{\alpha}$	0.1624	0.2186	0.0724	0.1689	-0.2080	0.0564	1.2943	3.8736
	$\hat{\theta}$	0.1250	0.1992	0.0245	0.1768	0.0205	0.1381	1.1267	1.4429
100	$\hat{\alpha}$	0.1054	0.1361	0.0366	0.1116	0.4650	0.1027	1.2195	1.3256
	$\hat{\theta}$	0.0942	0.1539	0.0139	0.1404	0.4583	0.1232	1.0962	1.2493
120	$\hat{\alpha}$	0.0986	0.1146	0.0413	0.0962	0.3980	0.0822	1.1913	1.3942
	$\hat{\theta}$	0.0956	0.1318	0.0284	0.12	0.2379	0.1058	1.0983	1.2453
$r = 0.5$									
		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.6377	2.7378	0.2182	1.1789	0.0173	0.0933	2.3223	29.3345
	$\hat{\theta}$	0.3505	0.7578	0.0135	0.5585	0.2141	0.0801	1.3568	9.4658
40	$\hat{\alpha}$	0.2813	0.5016	0.1126	0.3284	0.3784	0.1919	1.5274	2.6139
	$\hat{\theta}$	0.1823	0.3151	0.0155	0.2639	0.6463	0.2501	1.1940	1.2601
60	$\hat{\alpha}$	0.1539	0.2186	0.0509	0.1641	-0.0198	0.0540	1.3321	4.0459
	$\hat{\theta}$	0.1111	0.1903	0.0001	0.1696	-0.1525	0.0530	1.1221	3.5907
80	$\hat{\alpha}$	0.1202	0.14	0.0448	0.1116	-0.2079	0.0564	1.2545	2.4808
	$\hat{\theta}$	0.098	0.1351	0.0147	0.1213	0.0205	0.0381	1.1138	3.5496
100	$\hat{\alpha}$	0.1018	0.1136	0.0418	0.0943	0.4650	0.0927	1.2047	1.2258
	$\hat{\theta}$	0.0846	0.1063	0.0175	0.0967	0.4583	0.0923	1.0993	1.1514
120	$\hat{\alpha}$	0.0662	0.0872	0.0177	0.0759	0.3980	0.0722	1.1489	1.2078
	$\hat{\theta}$	0.0451	0.0818	-0.0103	0.0779	0.2379	0.0711	1.0501	1.1512

Table 7: Parameter Estimation of GR Distribution in Scheme 2 when $\alpha = 1.5$; $\theta = 3$

$r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
n		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	0.6234	7.0693	-0.0534	1.179	0.1369	0.5806	5.9960	12.1751
	$\hat{\theta}$	0.3295	0.7462	-0.3476	0.5862	0.0070	0.2202	1.2729	3.3881
40	$\hat{\alpha}$	0.1936	0.3409	-0.1182	0.1857	0.5998	0.1454	1.8358	2.3442
	$\hat{\theta}$	-0.283	0.2616	0.1301	0.2281	0.2324	0.1287	1.1469	2.0324
60	$\hat{\alpha}$	0.1338	0.2058	-0.0954	0.1345	-0.0793	0.1142	1.5301	1.8016
	$\hat{\theta}$	-0.2112	0.1742	0.0969	0.1536	0.0345	0.0966	1.1341	1.8039
80	$\hat{\alpha}$	0.1051	0.1497	-0.0837	0.1054	-0.2449	0.0932	1.4203	1.6069
	$\hat{\theta}$	-0.1876	0.1286	0.0643	0.1051	-0.0810	0.0912	1.2236	1.4106
100	$\hat{\alpha}$	0.065	0.1032	-0.0928	0.0843	-0.1740	0.0707	1.2242	1.4592
	$\hat{\theta}$	-0.161	0.1049	0.0544	0.0848	-0.0496	0.0701	1.2370	1.4974
120	$\hat{\alpha}$	0.0674	0.0935	-0.0738	0.0743	-0.0900	0.0389	1.2584	2.4015
	$\hat{\theta}$	-0.129	0.0832	0.0599	0.0742	-0.0290	0.0286	1.1213	2.9064

$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.3783	1.106	-0.1293	0.3805	-0.0880	0.0877	2.9067	12.6182
	$\hat{\theta}$	0.2268	0.4503	-0.3572	0.4456	0.0670	0.0452	1.0105	9.9706
40	$\hat{\alpha}$	0.1897	0.3211	-0.096	0.1815	0.2177	0.0726	1.7691	4.4247
	$\hat{\theta}$	-0.248	0.2184	0.1098	0.1859	0.3143	0.1245	1.1748	1.7540
60	$\hat{\alpha}$	0.107	0.1641	-0.1005	0.1142	-0.2382	0.0847	1.4370	1.9373
	$\hat{\theta}$	-0.1994	0.1491	0.0693	0.1230	0.0299	0.0631	1.2122	2.3617
80	$\hat{\alpha}$	0.0863	0.1131	-0.082	0.0862	-0.1533	0.0409	1.3121	2.7667
	$\hat{\theta}$	-0.1548	0.1037	0.0631	0.0863	-0.3714	0.0790	1.2016	1.3134
100	$\hat{\alpha}$	0.0786	0.0971	-0.068	0.0755	-0.0958	0.0360	1.2861	2.6958
	$\hat{\theta}$	-0.1309	0.0788	0.0579	0.0676	0.1907	0.0436	1.1657	1.8080
120	$\hat{\alpha}$	0.0475	0.0783	-0.0777	0.0668	0.1504	0.0294	1.1722	2.6641
	$\hat{\theta}$	-0.1349	0.0678	0.0272	0.0526	-0.0625	0.0257	1.2890	2.6406

Table 8: Parameter Estimation of GR Distribution in Scheme 3 when $\alpha = 1.5$; $\theta = 3$

$t = 0.5, r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	1.2588	8.0687	0.4816	5.9421	0.2379	0.1222	1.3579	66.0532
	$\hat{\theta}$	0.5016	1.2571	0.0644	0.8575	0.0905	0.0286	1.4660	43.9517
40	$\hat{\alpha}$	0.301	0.5518	0.0975	0.3334	-0.0411	0.0286	1.6551	19.2914
	$\hat{\theta}$	0.2192	0.3853	0.0019	0.3087	-0.2621	0.1041	1.2481	3.7018
60	$\hat{\alpha}$	0.1937	0.2805	0.0664	0.1979	0.1410	0.0473	1.4174	5.9319
	$\hat{\theta}$	0.1561	0.2465	0.0105	0.2091	-0.1713	0.0439	1.1789	5.6186
80	$\hat{\alpha}$	0.1459	0.1882	0.053	0.1438	-0.0470	0.0130	1.3088	14.4872
	$\hat{\theta}$	0.1118	0.1723	0.0024	0.1527	-0.0488	0.0133	1.1284	12.9875
100	$\hat{\alpha}$	0.0943	0.1194	0.0227	0.0974	0.0299	0.0396	1.2259	3.0165
	$\hat{\theta}$	0.0857	0.1342	-0.0021	0.1224	-0.0408	0.0405	1.0964	3.3124
120	$\hat{\alpha}$	0.0886	0.1006	0.0288	0.0839	-0.2357	0.0896	1.1990	1.1228
	$\hat{\theta}$	0.0874	0.1144	0.0138	0.1036	-0.5064	0.1029	1.1042	1.1113
$t = 0.5, r = 0.5$									
		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.5158	1.8434	0.0442	0.6776	-0.1490	0.1025	2.7205	17.9817
	$\hat{\theta}$	0.299	0.6234	-0.1587	0.4763	-0.1807	0.0729	1.3088	8.5549
40	$\hat{\alpha}$	0.2369	0.3987	0.0064	0.2303	-0.1711	0.0484	1.7312	8.2438
	$\hat{\theta}$	0.1528	0.2632	-0.1136	0.2325	-0.4362	0.2311	1.1320	1.1388
60	$\hat{\alpha}$	0.1232	0.1763	-0.0315	0.1225	0.0107	0.0271	1.4392	6.5110
	$\hat{\theta}$	0.0869	0.1656	-0.1065	0.1593	-0.4895	0.1527	1.0395	1.0843
80	$\hat{\alpha}$	0.0979	0.1182	-0.0233	0.0889	-0.3709	0.0816	1.3296	1.4482
	$\hat{\theta}$	0.0776	0.1197	-0.0771	0.1155	-0.8728	0.1085	1.0364	1.1030
100	$\hat{\alpha}$	0.0779	0.0948	-0.024	0.075	0.0304	0.0565	1.2640	1.6781
	$\hat{\theta}$	0.0616	0.0934	-0.0702	0.0912	-0.3890	0.0846	1.0241	1.1046
120	$\hat{\alpha}$	0.0447	0.0741	-0.0403	0.0634	-0.2522	0.0618	1.1688	1.1994
	$\hat{\theta}$	0.0232	0.074	-0.0886	0.0739	-0.7000	0.0654	1.0014	1.1310

Table 9: Parameter Estimation of GR Distribution in Scheme 3 when $\alpha = 1.5$; $\theta = 3$

$t = 1.5, r = 0.4$									
		MLE		MPSE		MCMC		RE1	RE2
		Bias	MSE	Bias	MSE	Bias	MSE		
20	$\hat{\alpha}$	0.4588	2.0687	0.0234	0.4816	0.2379	0.1222	4.2955	16.9351
	$\hat{\theta}$	0.5016	1.2571	0.0644	0.8575	0.0905	0.0286	1.4660	43.9517
40	$\hat{\alpha}$	0.301	0.5518	0.0975	0.3334	-0.0411	0.0286	1.6551	19.2914
	$\hat{\theta}$	0.2192	0.3853	0.0019	0.3087	-0.2621	0.1041	1.2481	3.7018
60	$\hat{\alpha}$	0.1937	0.2805	0.0664	0.1979	0.1410	0.0473	1.4174	5.9319
	$\hat{\theta}$	0.1561	0.2465	0.0105	0.2091	-0.1713	0.0439	1.1789	5.6186
80	$\hat{\alpha}$	0.1459	0.1882	0.053	0.1438	-0.0470	0.0130	1.3088	14.4872
	$\hat{\theta}$	0.1118	0.1723	0.0024	0.1527	-0.0488	0.0133	1.1284	12.9875
100	$\hat{\alpha}$	0.0943	0.1194	0.0227	0.0974	0.0299	0.0396	1.2259	3.0165

	$\hat{\theta}$	0.0857	0.1342	-0.0021	0.1224	-0.0408	0.0405	1.0964	3.3124
120	$\hat{\alpha}$	0.0886	0.1006	0.0288	0.0839	-0.2357	0.0896	1.1990	1.1228
	$\hat{\theta}$	0.0874	0.1144	0.0138	0.1036	-0.5064	0.1029	1.1042	1.1113
$t = 1.5, r = 0.5$									
		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\alpha}$	0.5172	1.8437	0.0451	0.6772	0.0541	0.0494	2.7225	37.2990
	$\hat{\theta}$	0.3007	0.6205	-0.1574	0.4733	0.2188	0.0658	1.3110	9.4312
40	$\hat{\alpha}$	0.2393	0.3984	0.0083	0.2294	0.4258	0.2497	1.7367	1.5953
	$\hat{\theta}$	0.1564	0.2594	-0.1107	0.2279	0.5146	0.2191	1.1382	1.1838
60	$\hat{\alpha}$	0.129	0.1761	-0.027	0.1209	-0.0213	0.1196	1.4566	1.4724
	$\hat{\theta}$	0.0947	0.1599	-0.0998	0.1517	-0.2644	0.1206	1.0541	1.3257
80	$\hat{\alpha}$	0.1038	0.1173	-0.0185	0.087	-0.2087	0.0626	1.3483	1.8732
	$\hat{\theta}$	0.0861	0.1142	-0.0696	0.1081	-0.2516	0.1028	1.0564	1.1104
100	$\hat{\alpha}$	0.0862	0.0942	-0.0169	0.073	-0.1268	0.0433	1.2904	2.1738
	$\hat{\theta}$	0.0735	0.0884	-0.0595	0.0835	0.3951	0.0834	1.0587	1.0601
120	$\hat{\alpha}$	0.0552	0.0732	-0.0311	0.061	-0.0628	0.0546	1.2000	1.3404
	$\hat{\theta}$	0.0383	0.0676	-0.0747	0.0664	-0.2183	0.0666	1.0181	1.0149

From simulation results of parameter estimation for model (GR distribution under adaptive censoring scheme), we note that: the Bias, MSE and RE for all the estimators for parameters decrease when the sample size n and effective sample size m increase in all cases. The efficiency of the model (Bias, MSE and RE for all the estimators when parameters decrease) increases when the ratio of the effective sample size (r) increases. Three different samples schemes were applied on adaptive type-II progressive censoring scheme and that to get to the most effective scheme, the efficiency is the best for scheme 2 followed by scheme 3 then scheme 1. The schemes 1 and 2 not affected by the changes in time t, whatever the changes in time t, there is a stability in the numerical and practical results of schemes 1 and 2. See table (1), where scheme 1 is $R_1 = R_2 = \cdots = R_{\mathrm{m}-1} = 0, and\ R_{\mathrm{m}} = n - m$, it is type-II scheme and scheme 2 is $R_1 = n - m\ and\ R_2 = R_3 = \cdots = R_{\mathrm{m}-1} = 0$ so not affected by the time, while scheme 3 is $R_1 = R_2 = \cdots = R_{\mathrm{m}-1} = 1, and\ R_{\mathrm{m}} = n - 2m - 1$ affected by the time. The parameters of the model were estimated with various values of parameters, we note that when θ increases, the estimated efficiency decreases with constancy of the α. The results prove that, the efficiency of the MPSE over MLE, and MCMC over MLE, then MPSE and MCMC are good alternative estimator of MLE, where alternative estimator has lesser in the Bias and MSE for MLE estimators for parameters. The MPSE has more relative efficiency than MLE for all parameters, but Bayesian estimates is the best method in this study.

6.Application

We present the numerical results of the parameter estimation of GR distribution under adaptive type-II progressive censoring scheme of a two real data.

6.1.Real Data I

Badar and Priest (1982) discussed the real data set of sample size 69 observed failure times, the data set is represented the data measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows. Mahmoud et al (2016) discussed the real data to inferences for new Weibull-Pareto distribution (NWPD) based on progressively type-II censored Data. Almetwally and Almongy (2018) used this data to fit of the Marshall–Olkin Extended Weibull (MOEW) distribution under adaptive type-II progressive censoring scheme.

We computed the Kolmogorov-Smirnov (KS) distance between the empirical and the fitted distributions.

Table 10: Goodness of Fit Data

	GR		
	MLE	MPSE	MCMC
D	0.0683	0.0626	0.0679
p-value	0.9036	0.9493	0.9073

A fitted GR distribution curve to data have been shown in Figure (5).

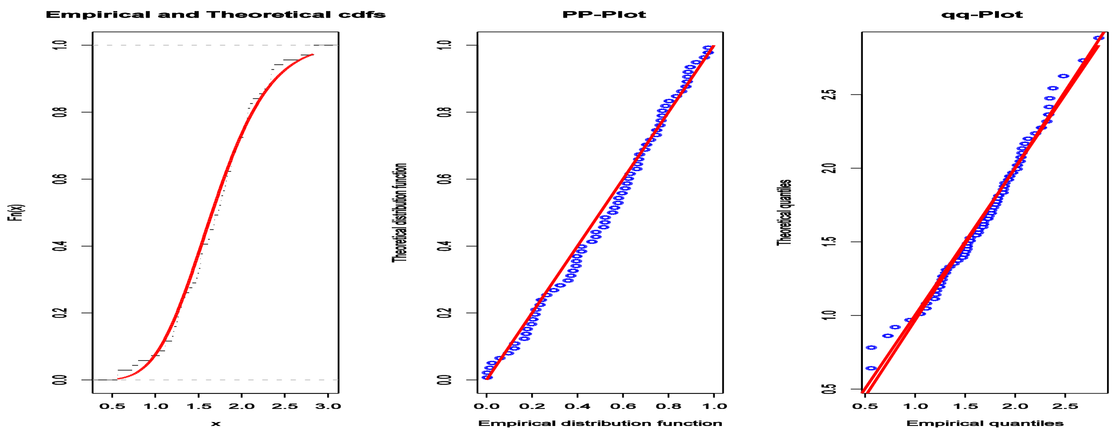


Figure (5): Plot of Empirical cdf for the GR distribution

Table 11: Estimation of coefficient, stander error and reliability estimation for Complete Data

	$\hat{\alpha}$	$\hat{\theta}$		$t = 0.7$	$t = 1.5$
MLE	3.2446 (0.6290)	0.7749 (0.0461)	$\hat{S}$ $\hat{h}$	0.9881 0.0957	0.6218 1.2424
MPSE	2.9838 (0.5793)	0.7497 (0.0459)	$\hat{S}$ $\hat{h}$	0.9857 0.1072	0.6284 1.1704
MCMC	3.1270 (0.5689)	0.7653 (0.0445)	$\hat{S}$ $\hat{h}$	0.9889 0.1017	0.6225 1.2179

The convergence of MCMC estimation of α and θ can be showed in Figure (6) in cases of complete censoring for Real Data I.

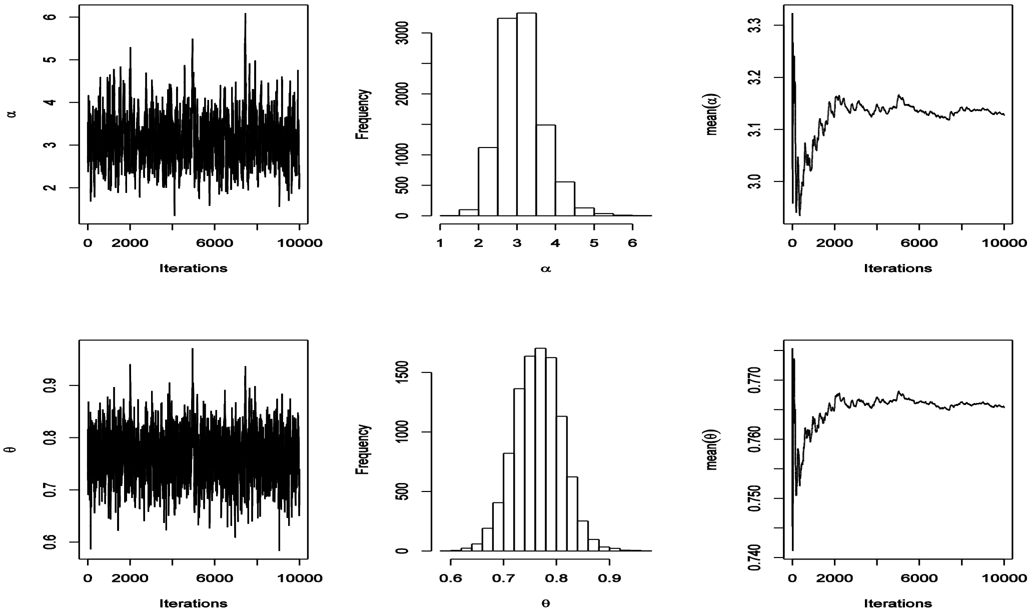


Figure (6): Convergence of MCMC Estimation of α and θ for Real Data I

Table 12: The Sample Order of Adaptive type-II Progressive Censoring Schemes for Real Data I

m	scheme	$x_{i:m:n}$
28	1	0.562, 0.564, 0.729, 0.802, 1.216, 1.247, 1.256, 1.271, 1.474, 1.490, 1.503, 1.520, 1.632, 1.676, 1.684, 1.685, 1.816, 1.824, 1.836, 1.879, 2.020, 2.023, 2.050, 2.059, 2.317, 2.334, 2.340, 2.346
	2	0.562, 0.729, 1.053, 1.208, 1.271, 1.277, 1.348, 1.503, 1.524, 1.551, 1.632, 1.685, 1.740, 1.764, 1.804, 1.824, 1.898, 2.020, 2.050, 2.071, 2.098, 2.130, 2.262, 2.317, 2.340, 2.346, 2.378, 2.683
	3	0.562, 0.564, 0.729, 0.802, 0.950, 1.053, 1.216, 1.247, 1.256, 1.271, 1.277, 1.474, 1.490, 1.520, 1.522, 1.524, 1.632, 1.676, 1.684, 1.685, 1.816, 1.824, 1.836, 2.020, 2.023, 2.317, 2.334, 2.378
35	1	0.562, 0.564, 0.729, 0.802, 0.950, 1.216, 1.247, 1.256, 1.271, 1.277, 1.474, 1.490, 1.503, 1.520, 1.522, 1.632, 1.676, 1.684, 1.685, 1.728, 1.816, 1.824, 1.836, 1.879, 1.883, 2.020, 2.023, 2.050, 2.059, 2.068, 2.317, 2.334, 2.340, 2.346, 2.378
	2	0.562, 0.729, 1.053, 1.208, 1.271, 1.277, 1.313, 1.348, 1.429, 1.503, 1.524, 1.551, 1.632, 1.632, 1.685, 1.740, 1.764, 1.804, 1.824, 1.883, 1.892, 1.898, 2.020, 2.050, 2.071, 2.098, 2.130, 2.204, 2.262, 2.317, 2.340, 2.346, 2.378, 2.683, 2.835
	3	0.562, 0.564, 0.729, 1.208, 1.216, 1.256, 1.271, 1.305, 1.390, 1.474, 1.490, 1.520, 1.522, 1.632, 1.676, 1.684, 1.685, 1.761, 1.816, 1.824, 1.836, 1.879, 1.934, 1.976, 2.020, 2.023, 2.050, 2.059, 2.068, 2.130, 2.204, 2.317, 2.334, 2.340, 2.346

Table 13: Estimation for GR Model |Under Adaptive Type-II Progressive Censoring Scheme of Real Data I

		MLE	MPSE	MCMC	MLE	MPSE	MCMC
		$m = 28$			$m = 35$		
		Scheme 1					
$\hat{\alpha}$		1.3879 (0.3279)	1.3090 (0.3125)	1.5977 (0.3095)	1.5188 (0.3348)	1.4404 (0.3201)	1.6058 (0.3302)
$\hat{\theta}$		0.3661 (0.0536)	0.3521 (0.0540)	0.3885 (0.0527)	0.4250 (0.0506)	0.4117 (0.0508)	0.4327 (0.0488)
$t = 0.7$	$\hat{S}$	0.9782	0.9754	0.9853	0.9764	0.9738	0.9799
	$\hat{h}$	0.0856	0.0914	0.0656	0.1001	0.1060	0.0897
$t = 1.5$	$\hat{S}$	0.8456	0.8427	0.9853	0.8108	0.8087	0.8199
	$\hat{h}$	0.2895	0.2824	0.0656	0.3828	0.3730	0.3781
		Scheme 2					
$\hat{\alpha}$		3.1044 (0.8300)	2.7209 (0.7159)	2.9492 (0.5515)	3.3145 (0.8456)	2.9034 (0.7318)	3.4212 (0.8271)
$\hat{\theta}$		0.7414 (0.0663)	0.6983 (0.0652)	0.7327 (0.0507)	0.7441 (0.0599)	0.7040 (0.0592)	0.7449 (0.0593)
$t = 0.7$	$\hat{S}$	0.98867	0.9852	0.9866	0.9914	0.9883	0.9926
	$\hat{h}$	0.0885	0.1033	0.0995	0.0710	0.0862	0.0632
$t = 1.5$	$\hat{S}$	0.6550	0.6688	0.6489	0.6750	0.6844	0.6855
	$\hat{h}$	1.1025	0.9876	1.0951	1.0702	0.9709	1.0512
		Scheme 3					
		$m = 28$			$m = 35$		
$t = 0.7$	$\hat{\alpha}$	1.5634 (0.3763)	1.4709 (0.3576)	1.5946 (0.3618)	1.8490 (0.4205)	1.7473 (0.4006)	1.8958 (0.4174)
	$\hat{\theta}$	0.3917 (0.0539)	0.3773 (0.0534)	0.39031 (0.0531)	0.4672 (0.0510)	0.4534 (0.0513)	0.4666 (0.0503)
	$\hat{S}$	0.9834	0.9810	0.9849	0.9854	0.9833	0.9869
	$\hat{h}$	0.0721	0.0782	0.0669	0.0738	0.0801	0.0676
$t = 1.5$	$\hat{\alpha}$	1.6841 (0.4028)	1.5828 (0.3823)	1.7436 (0.3961)	2.0891 (0.4732)	1.9680 (0.4490)	2.1073 (0.4534)
	$\hat{\theta}$	0.4230 (0.0549)	0.4077 (0.0553)	0.4256 (0.0515)	0.5177 (0.0523)	0.5022 (0.0526)	0.5146 (0.0509)
	$\hat{S}$	0.8443	0.8416	0.8516	0.8089	0.8073	0.8150
	$\hat{h}$	0.3362	0.3273	0.3280	0.4794	0.4652	0.4664

6.2.Real Data Set II: Economic Data

The economic data set, which is reproduced in table 13 consists of 31 yearly time series observations [1980:2010] on response variable: GDP growth (% per year) of Egypt, Source: The employed of economics data are collected by World Bank National Accounts data and OECD National Accounts data. The data set are 10.01132, 3.756100, 9.907171, 7.401136, 6.091518, 6.602036, 2.646586, 2.519411, 7.930073, 4.972375, 5.701749, 1.078837, 4.431994, 2.900787, 3.973172, 4.642467, 4.988731, 5.491124, 4.036373, 6.105463, 5.367998, 3.535252, 2.370460, 3.192285, 4.089940, 4.478960, 6.853908, 7.090271, 7.157617, 4.673845, 5.145106.

Table 14: Goodness of Fit to Data II

	GR		
	MLE	MPSE	MCMC
D	0.05417	0.0833	0.05174
p-value	0.9999	0.9702	1

A fitted GR distribution curve to data have been shown in Figure (5).

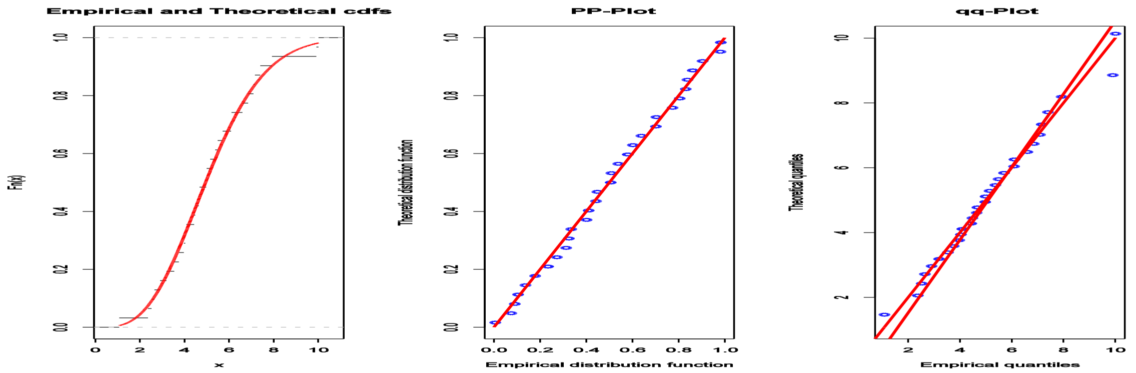


Figure (7): Plot of Empirical cdf for the GR distribution

Table 15: Estimation of coefficient, stander error and reliability estimation for Complete Data II

	$\hat{\alpha}$	$\hat{\theta}$		$t = 5$	$t = 10$
MLE	1.7457 (0.4558)	0.2136 (0.0219)	rt ht	0.4894 0.3902	0.0184 0.9089
MPSE	1.4737 (0.3841)	0.1944 (0.0213)	rt ht	0.5157 0.3326	0.0334 0.7521
MCMC	1.8467 (0.4016)	0.2117 (0.0211)	rt ht	0.4976 0.3799	0.0200 0.8860

The convergence of MCMC estimation of α and θ can be showed in Figure (8) in cases of complete censoring for Real Data II.

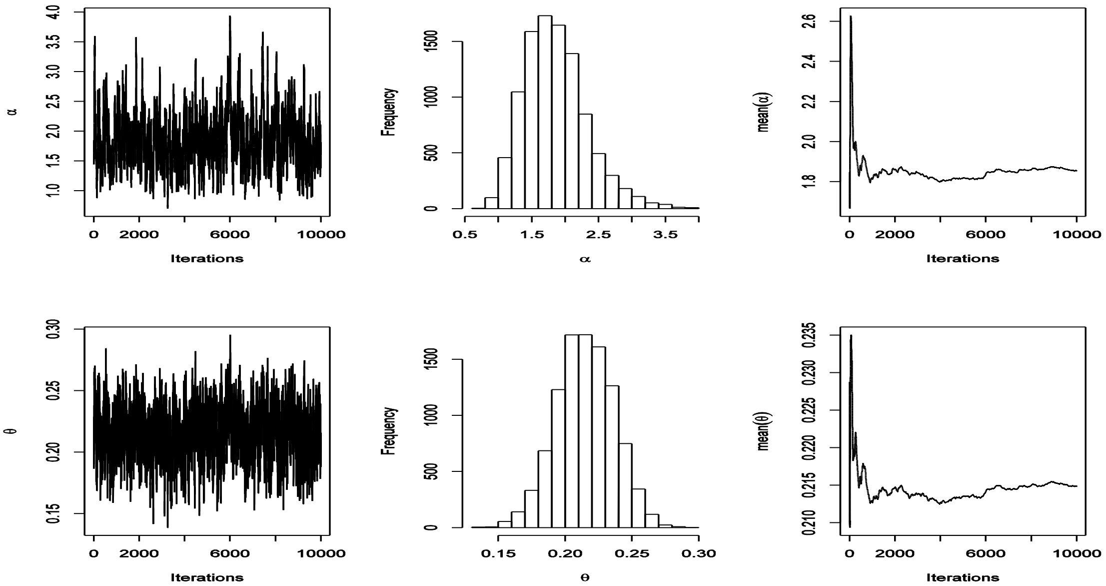


Figure (8): Convergence of MCMC Estimation of α and θ for Real Data I

Table 16: The Sample Order of Adaptive type-II Progressive Censoring Schemes for Real Data II

m	scheme	$x_{i:m:n}$
10	1	2.519411, 2.646586, 3.756100, 4.972375, 6.091518, 6.602036, 7.401136, 7.930073, 9.907171, 10.011320
	2	1.078837, 4.642467, 5.367998, 5.491124, 5.701749, 7.090271, 7.401136, 7.930073, 9.907171, 10.011320
	3	1.078837, 2.519411, 2.646586, 3.756100, 6.091518, 6.602036, 7.401136, 7.930073, 9.907171, 10.011320
16	1	1.078837, 2.519411, 2.646586, 4.036373, 4.431994, 4.642467, 4.673845, 5.367998, 5.491124, 5.701749, 6.853908, 7.090271, 7.40113, 7.930073, 9.907171, 10.011320
	2	1.078837, 2.519411, 2.646586, 4.036373, 4.431994, 4.642467, 4.673845, 5.367998, 5.491124, 5.701749, 6.853908, 7.090271, 7.40113, 7.930073, 9.907171, 10.011320
	3	1.078837, 2.519411, 2.646586, 2.900787, 3.756100, 4.036373, 4.431994, 4.988731, 5.145106, 5.491124, 6.091518, 6.602036, 7.401136 7.930073, 9.907171, 10.011320

Table 17: Estimation for GR Model |Under Adaptive Type-II Progressive Censoring Scheme of Real Data II

		MLE	MPSE	MCMC	MLE	MPSE	MCMC
		$m = 10$			$m = 16$		
		Scheme 1					
$\hat{\alpha}$		0.9321 (0.3356)	0.8143 (0.3033)	1.1100 (0.2431)	0.7424 (0.2147)	0.6769 (0.1997)	0.7892 (0.2091)
$\hat{\theta}$		0.0593 (0.0191)	0.0521 (0.0191)	0.0707 (0.0139)	0.0737 (0.0179)	0.0673 (0.0178)	0.0748 (0.0167)
$t = 5$	$\hat{S}$	0.9002	0.9093	0.9093	0.7839	0.7946	0.7946
	$\hat{h}$	0.0395	0.0375	0.0375	0.0764	0.0761	0.0761
$t = 10$	$\hat{S}$	0.6774	0.6880	0.6880	0.4755	0.4767	0.4767
	$\hat{h}$	0.0740	0.0739	0.0739	0.1233	0.1281	0.1281
		Scheme 2					
$\hat{\alpha}$		1.48323 (0.5353)	1.2188 (0.4301)	1.52348 (0.4714)	1.4168 (0.4351)	1.1857 (0.3588)	1.5112 (0.4216)
$\hat{\theta}$		0.16109 (0.0279)	0.13878 (0.0267)	0.16146 (0.0265)	0.1809 (0.0261)	0.1589 (0.0248)	0.1832 (0.0256)
$t = 5$	$\hat{S}$	0.6661	0.6743	0.6743	0.5616	0.5749	0.5749
	$\hat{h}$	0.2113	0.2088	0.2088	0.2859	0.2854	0.2854
$t = 10$	$\hat{S}$	0.1087	0.1101	0.1101	0.0533	0.0523	0.0523
	$\hat{h}$	0.5092	0.5109	0.5109	0.6493	0.6649	0.6649
		Scheme 3					
$t = 5$	$\hat{\alpha}$	0.7388 (0.2471)	0.6545 (0.2265)	0.8600 (0.2177)	1.4168 (0.4351)	1.1858 (0.3588)	1.1857 (0.2006)
	$\hat{\theta}$	0.0539 (0.0187)	0.0469 (0.0185)	0.0608 (0.0181)	0.1809 (0.0260)	0.1589 (0.0248)	0.1589 (0.0149)
	$\hat{S}$	0.8596	0.8758	0.8758	0.5616	0.5931	0.5931
	$\hat{h}$	0.0465	0.0465	0.0465	0.2859	0.2334	0.2334
$t = 10$	$\hat{\alpha}$	0.7861 (0.2617)	0.6957 (0.2395)	0.6989 (0.2493)	1.2539 (0.3634)	1.0850 (0.3176)	1.2674 (0.3353)
	$\hat{\theta}$	0.0598 (0.0196)	0.0660 (0.0195)	0.0575 (0.0190)	0.1403 (0.0237)	0.1254 (0.0232)	0.1394 (0.0227)
	$\hat{S}$	0.6110	0.6300	0.6300	0.1717	0.1781	0.1781
	$\hat{h}$	0.0832	0.0710	0.0710	0.3862	0.3803	0.3803

7.Conclusion

In this article, parameters estimation for the GR distribution are discussed under adaptive type-II progressively censoring sample with different schemes based on the MPSE, MLE and the Bayesian methods. In this model, the estimators based on MPSE method behave quite better than the estimators based on the MLE but the Bayesian estimation is the best one, where the bias, MSE and RE are less than from the other methods. In generally, The MPSE and Bayesian methods can be used as alternative methods for the MLE method. The parameters of the model were estimated by using various values of parameters, we note that when θ increases, the model efficiency decreases with constancy of α. In case of reliability estimation, we note that when θ increases, the estimated model efficiency decreases (reliability decrease and hazard increase) with constancy of α. By checking the previous results, we note that MPSE is better than MLE. We can conclude that the MPSE method is a good alternative method to the usual MLE method in many situation. We hope that the finding in this paper will be useful for researchers and statistician, where such types of things were required and also in cases where we have small sample size for analysis of lifetime data.

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