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SOME THEORETICAL AND COMPUTATIONAL ASPECTS OF THE INVERSE GENERALIZED POWER WEIBULL DISTRIBUTION

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ABSTRACT

This paper introduces a new three-parameter distribution called inverse generalized power Weibull distribution. This distribution can be regarded as a reciprocal of the generalized power Weibull distribution. The new distribution is characterized by being a general formula for some well-known distributions, namely inverse Weibull, inverse exponential, inverse Rayleigh and inverse Nadarajah-Haghighi distributions. Some of the mathematical properties of the new distribution including the quantile, density, cumulative distribution functions, moments, moments generating function and order statistics are derived. The model parameters are estimated using the maximum likelihood method. The Monte Carlo simulation study is used to assess the performance of the maximum likelihood estimators in terms of mean squared errors. Two real datasets are used to demonstrate the flexibility of the new distribution as well as to demonstrate its applicability.

Keywords: Inverse Weibull distribution, Nadarajah-Haghighi distribution, Maximum likelihood estimation, Order statistics, Monte Carlo simulation

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1. Introduction

Weibull distribution is one of the most important distributions used in reliability engineering and other disciplines. Also, it adequately describes the observed failure times of many different types of components and phenomena. Therefore, the Weibull distribution was more widely used as a basis for several generalizations. See, for example, the exponentiated Weibull distribution by Mudholkar and Srivastava (1993), beta-Weibull distribution by Lee et al. (2007) and Kumaraswamy Weibull distribution by Cordeiro et al. (2010). For more detail on the generalizations of Weibull distribution, refer to the books by Murthy et al. (2004), Rinne (2008) and Lai (2014). Also, Haghighi and Nikulin (2006) proposed a new extension of Weibull distribution called the generalized power Weibull distribution. The cumulative density function (cdf) and probability density function (pdf) of the generalized power Weibull distribution, respectively are

$$F(y) = 1 - exp\{1 - \left(1 + \lambda y^{\theta}\right)^{\alpha}\},\ \lambda, \alpha, \theta > 0,\ \ y > 0 \tag{1}$$

and

$$f(y) = \lambda\alpha\theta y^{\theta-1}\left(1 + \lambda y^{\theta}\right)^{\alpha-1} exp\{1 - \left(1 + \lambda y^{\theta}\right)^{\alpha}\},\ \lambda, \alpha, \theta > 0,\ \ y > 0 \tag{2}$$

where α and θ are two shape parameters and λ is a scale parameter. The Weibull distribution is a special case of (1) when $\alpha = 1$. Its hazard rate function according to Nikulin and Haghighi (2009) can be constant, monotone, unimodal, bathtub-shaped. In the literature, some extensions of the generalized power Weibull distribution proposed by many authors, such as Selim and Badr (2016) proposed the Kumaraswamy generalized power Weibull distribution, Selim (2018) proposed the generalized power generalized Weibull distribution, Khan (2018) proposed the transmuted generalized power Weibull distribution and Pena-Ramirez et al. (2018) proposed The exponentiated power generalized Weibull distribution.

This paper aims to introduce a reciprocal of the generalized power Weibull distribution named inverse generalized power Weibull (IGPW) distribution and studies its mathematical properties. The motivations for deriving the inverse generalized power Weibull distribution are to provide more usefulness and flexibility of the ordinary distribution and to improve its goodness-of-fit in comparison with the well-known distributions in lifetime data analysis.

The rest of this paper is organized as follows. The inverse generalized power Weibull distribution and the special cases thereof are introduced in Section 2. Some of the mathematical properties of IGPW distribution are derived in Section 3, including the quantile function, skewness, kurtosis, ordinary moments, moment generating function and order statistics. The maximum likelihood estimation of the model parameters is introduced in Section 4. In Section 5, the Monte Carlo simulation study is used to assess the performance of the maximum likelihood estimators in terms of mean squared errors. Two real data sets are used to illustrate the usefulness of the IGPW distribution in Section 6. The final Section is devoted to the conclusion.

2. Inverse Generalized Power Weibull Distribution

The inverse generalized power Weibull distribution can be derived using the transformation $X = 1/Y$, whereupon if the random variable Y follows the GPW distribution, the random variable X follows the IGPW distribution. The cdf and pdf of IGPW distribution are given, respectively, by

$$\mathrm{F}(\mathrm{x}) = \exp\{1 - (1 + \lambda x^{-\theta})^{\alpha}\}, \ \alpha, \theta, \lambda > 0, \ \ x > 0 \tag{3}$$

and

$$f(x) = \alpha\theta\lambda x^{-\theta-1}(1 + \lambda x^{-\theta})^{\alpha-1} \exp\{1 - (1 + \lambda x^{-\theta})^{\alpha}\} \tag{4}$$

where λ is scale parameter and α, θ are shape parameters. This model has inverse Weibull (IW) distribution as a special case when $\alpha = 1$. Hence, it can also be considered as an extension of the inverse exponential distribution which is developed by Keller et al. (1982) when $\alpha = \theta = 1$. The graphs of the pdf and cdf for selected values of the model parameters are plotted in Fig. 1.

The survival $s(x)$ and the hazard rate $h(x)$ functions of the $IGPW$ distribution are given, respectively by

$$s(x) = 1 - F(x) = 1 - \exp\{1 - (1 + \lambda x^{-\theta})^{\alpha}\} , \quad x > 0 \tag{5}$$

and

$$h(x) = \frac{f(x)}{s(x)} = \frac{\alpha\theta\lambda x^{-\theta-1}(1+\lambda x^{-\theta})^{\alpha-1} \, exp\{1-(1+\lambda x^{-\theta})^{\alpha}\}}{1-exp\{1-(1+\lambda x^{-\theta})^{\alpha}\}}, \quad x > 0 \tag{6}$$

Fig. 2, shows some possible shapes of the IGPW hazard rate function.

The reversed hazard $r(x)$ and the cumulative failure rate $H(x)$ functions of the IGPW distribution are given, respectively by

$$r(x) = \alpha\theta\lambda x^{-\theta-1}(1 + \lambda x^{-\theta})^{\alpha-1}, \quad x > 0, \tag{7}$$

and

$$H(x) = (1 + \lambda x^{-\theta})^{\alpha} - 1 , \quad x > 0. \tag{8}$$

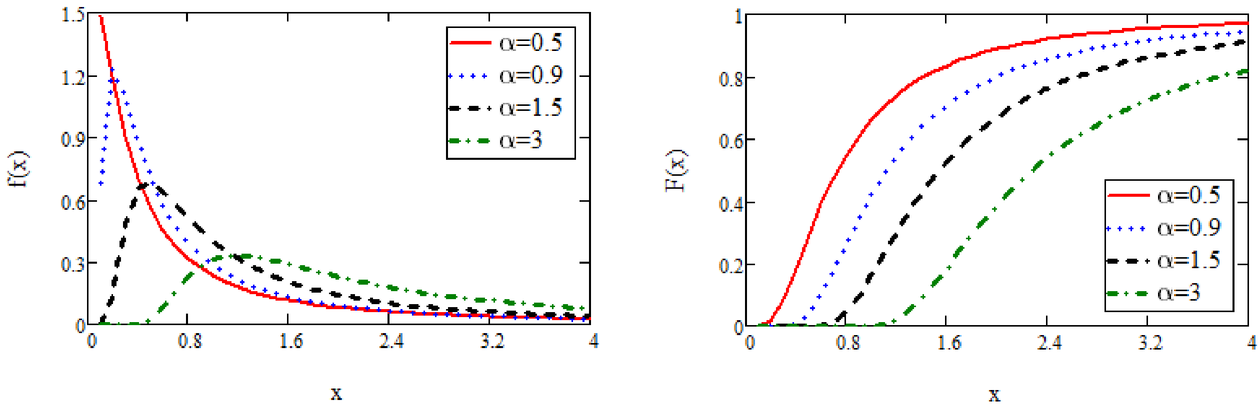

Fig. 1: Some possible shapes of the IGPW density function (left panel) and the IGPW cumulative density function (right panel)

2.1 Special cases of the $IGPW$ distribution

A number of the important distributions can be obtained as special cases of the $IGPW$ distribution, are specifically inverse Weibull (IW), inverse exponential (IE), inverse Nadarajah-Haghighi (INH) and inverse Rayleigh (IR) distributions. The special cases of $IGPW$ distribution for selected values of the parameters (α, θ) are listed in Table 1.

Table 1. Special cases of the $IGPW$ distribution

Model	λ	α	θ	*Author(s)*
IW	-	*1*	-	*Keller et al. (1982); (Keller and Kamath 1982)*
IE	-	*1*	*1*	*(Keller et al. 1982)*
INH	-	-	*1*	*(Tahir et al. 2018)*
IR	-	*1*	*2*	*(Voda 1972)*

3. The Statistical Properties

In this section, some of the statistical properties of IGPW distribution including the quantile function, random variables generation function, moments, moment generating function, skewness, kurtosis and order statistics are derived.

3.1 Quantile function and simulation

The quantile function has a number of important applications, for example, it can be used to obtain the median, skewness, kurtosis and can be also used to generate random variables. The q-th quantile is a solution of the following equation $F(x_q) = q, \quad 0 \leq q \leq 1$.

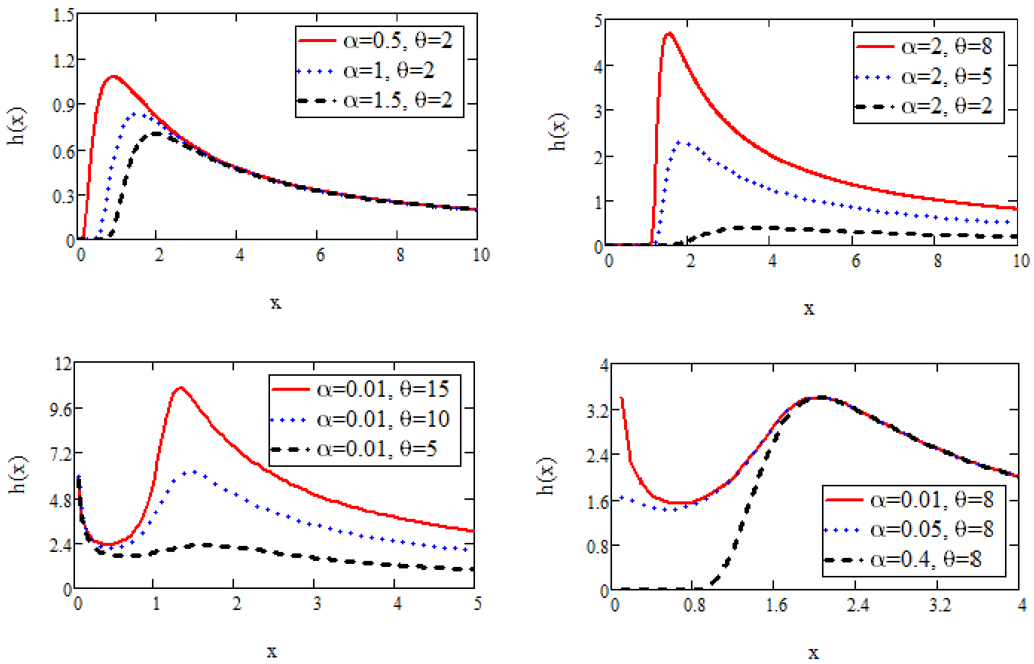

Fig. 2: Some possible shapes of the $IGPW$ hazard rate function

Thus, the quantile function $Q(q)$ corresponding of the IGPW distribution is

$$Q(q) = \lambda^{1/\theta}\left[\sqrt[\alpha]{1 - ln(q)} - 1\right]^{-1/\theta} \tag{9}$$

Setting $q = 0.5$, in equation (9) we obtain the median of the IGPW distribution, as follows

$$Q(0.5) = \lambda^{1/\theta}\left[\sqrt[\alpha]{1 - ln(0.5)} - 1\right]^{-1/\theta} \tag{10}$$

Fig. 3, shows the median of the IGPW distribution as a function of the parameters α and θ.

The random variables X of IGPW distribution can be simulated using equation (9) as following

$$X = \lambda^{1/\theta}\left[\sqrt[\alpha]{1 - ln(u)} - 1\right]^{-1/\theta} \tag{11}$$

where $u \sim$ the uniform $(0, 1)$ distribution and $X \sim IGPW(\lambda, \alpha, \theta)$.

3.2 Skewness and kurtosis

The shortcomings of the classical skewness and kurtosis measures can be avoided by using the skewness and kurtosis measures based on quantiles like Bowley's skewness and Moors' kurtosis. The Bowley's skewness measure based on quartiles ((Kenney and Keeping 1962)) is given by

$$Sk = \frac{Q_{3/4} - 2Q_{1/2} + Q_{1/4}}{Q_{3/4} - Q_{1/4}} \tag{12}$$

and the Moors' kurtosis measure based on octiles (Moors (1988)) is given by

$$Ku = \frac{Q_{7/8} - Q_{5/8} + Q_{3/8} - Q_{1/8}}{Q_{6/8} - Q_{2/8}} \tag{13}$$

The Fig. 3, shows the behaviors of median, skewness and kurtosis of the IGPW distribution as a function of the parameters α and θ.

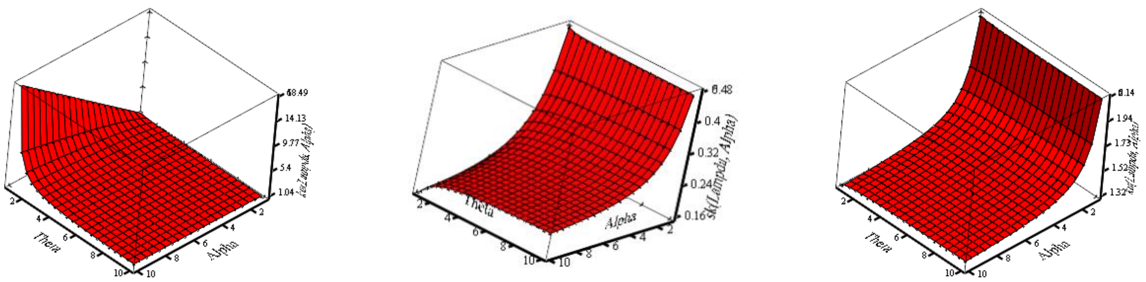

Fig. 3: The median (left panel), Skewness (middle panel) and kurtosis (right panel) of the IGPW distribution as a function of the parameters α and θ.

3.3 Moments and moment generating function

The moments and moment generating function of the IGPW distribution are given by the following theorems:

Theorem 1. If X has the IGPW distribution, then the rth moments of X for integer value of $r\theta^{-1}$ is

$$\mu_r' = e\lambda^{r\theta^{-1}} \sum_{i=0}^{-r\theta^{-1}} (-1)^{i - r\theta^{-1}} \binom{-r\theta^{-1}}{i} \Gamma(i\alpha^{-1} + 1, 1), \quad r < 1 \tag{14}$$

where $\Gamma(a, b)$ denotes the upper incomplete gamma function and e is Euler's number.

Proof. The $r\ th$ moment of X is defined as follows

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x) dx \tag{15}$$

Inserting Eq.(4) into Eq. (15), yields

$$\mu_r' = e\alpha\lambda\theta \int_0^\infty x^{r-\theta-1}\left(1+\frac{\lambda}{x^\theta}\right)^{\alpha-1} exp\left\{-\left(1+\frac{\lambda}{x^\theta}\right)^\alpha\right\}dx \tag{16}$$

Let $v = \left(1+\frac{\lambda}{x^\theta}\right)^\alpha$, the above expression reduce to

$$\mu_r' = e\lambda^{\frac{r}{\theta}} \int_1^\infty \left(v^{\frac{1}{\alpha}}-1\right)^{-r\theta^{-1}} e^{-v}dv \tag{17}$$

Then, by applying the binomial expansion of

$$\left(v^{\frac{1}{\alpha}}-1\right)^{-r\theta^{-1}} = \sum_{i=0}^\infty (-1)^{i-r\theta^{-1}}\binom{-r\theta^{-1}}{i} v^{\frac{i}{\alpha}}$$

we get

$$\mu_r' = e\lambda^{r\theta^{-1}} \sum_{i=0}^{-r\theta^{-1}} (-1)^{i-r\theta^{-1}}\binom{-r\theta^{-1}}{i}\int_1^\infty v^{\frac{i}{\alpha}}e^{-v}dv \tag{18}$$

By integrating the incomplete gamma function in (18) we get the r *th* moment of X as follows

$$\mu_r' = e\lambda^{r\theta^{-1}} \sum_{i=0}^{-r\theta^{-1}} (-1)^{i-r\theta^{-1}}\binom{-r\theta^{-1}}{\mathrm{i}}\Gamma\left[\frac{i}{\alpha}+1,1\right] \quad \blacksquare$$

If $\alpha = \theta = 1$, we get the moments of inverse exponential distribution as follows

$$\mu_r' = e\lambda^r \sum_{\mathrm{i}=0}^{-\mathrm{r}} (-1)^{\mathrm{i}-\mathrm{r}}\binom{-\mathrm{r}}{\mathrm{i}}\Gamma[\mathrm{i}+1,1]$$

And if $\theta = 1$, we get the moments of inverse Weibull distribution as follows

$$\mu_r' = e\lambda^{r\theta^{-1}} \sum_{i=0}^{-r\theta^{-1}} (-1)^{i-r\theta^{-1}}\binom{-r\theta^{-1}}{i}\Gamma(i+1,1), \quad r \lhd 1$$

Theorem 2. If $X \sim IGPW$ distribution, then for any integer value of $r\theta^{-1}$, the moment generating function is

$$M_x(t) = e\lambda^{r\theta^{-1}} \sum_{r=0}^\infty \sum_{i=0}^{-r\theta^{-1}} (-1)^{i-r\theta^{-1}}\binom{-r\theta^{-1}}{\mathrm{i}}\Gamma\left[\frac{i}{\alpha}+1,1\right] \tag{19}$$

Proof. The moment generating function is defined as follows

$$M_x(t) = \int_0^\infty e^{tx}\, f(x)dx$$

Using exponential function formula $e^{tx} = \sum_{r=0}^\infty \frac{(tx)^r}{r!}$, we get

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!}E(X^r) \tag{20}$$

By inserting Eq. (14) in Eq. (20), yields the moment generating function of *IGPW* distribution as in (19).

3.4 Order statistics

Assuming that $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ are the order statistics of a random sample follows a continuous distribution with cdf $F(x)$ and pdf $f(x)$, then the pdf of $X_{(k)}$ is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}, \quad k = 1,2,\dots,n \tag{21}$$

Let X is a random variable of *IGPW* distribution, then the density function of the k-th order statistics of the *IGPW* distribution is

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\alpha\theta\lambda x^{-\theta-1}\left(1+\lambda x^{-\theta}\right)^{\alpha-1}\left[e^{1-\left(1+\lambda x^{-\theta}\right)^\alpha}\right]^k\left[1-e^{1-\left(1+\lambda x^{-\theta}\right)^\alpha}\right]^{n-k} \tag{22}$$

If $k = 1$, the pdf of order statistics is

$$f_{1:n}(x) = n\alpha\theta\lambda x^{-\theta-1}\left(1+\lambda x^{-\theta}\right)^{\alpha-1} e^{1-\left(1+\lambda x^{-\theta}\right)^{\alpha}}\left[1-e^{1-\left(1+\lambda x^{-\theta}\right)^{\alpha}}\right]^{n-1} \quad (23)$$

and if $k = \text{n}$, the pdf of order statistics is

$$f_{n:n}(x) = n\alpha\theta\lambda x^{-\theta-1}\left(1+\lambda x^{-\theta}\right)^{\alpha-1}\left[e^{1-\left(1+\lambda x^{-\theta}\right)^{\alpha}}\right]^{n} \quad (24)$$

4. Maximum Likelihood Estimation

This section is devoted to discussing the maximum likelihood estimation (MLE) and the approximate confidence intervals for the unknown parameters of $IGPW$ distribution. Let $x_1, x_1, \dots, x_n$ is a complete random sample of size n from the $IGPW$ distribution. Then the likelihood function (LF) is

$$L(\alpha,\theta,\lambda|\boldsymbol{x}) = (\alpha\theta\lambda)^n \prod_{i=1}^{n} x^{-\theta-1}\left(1+\lambda x^{-\theta}\right)^{\alpha-1} exp\left\{1-\left(1+\lambda x^{-\theta}\right)^{\alpha}\right\} \quad (25)$$

and the log-likelihood function (lnL) is

$$lnL = n(\ln(\alpha\theta\lambda)+1) - (\theta+1)\sum_{i=1}^{n} ln\, x_i + (\alpha-1)\sum_{i=1}^{n} ln\left(1+\lambda x_i{}^{-\theta}\right) - \sum_{i=1}^{n}\left(1+\lambda x_i{}^{-\theta}\right)^{\alpha} \quad (26)$$

The maximum likelihood estimators of α, θ and λ are the solution of the following three equations

$$\frac{\partial\, ln\, L}{\partial\lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{1}{x_i{}^{\theta}\left(1+\lambda x_i{}^{-\theta}\right)} - \alpha\sum_{i=1}^{n}\frac{\left(1+\lambda x_i{}^{-\theta}\right)^{\alpha-1}}{x_i{}^{\theta}} = 0, \quad (27)$$

$$\frac{\partial\, ln\, L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln\left(1+\lambda x_i{}^{-\theta}\right) - \sum_{i=1}^{n} ln\left(1+\lambda x_i{}^{-\theta}\right)\left(1+\lambda x_i{}^{-\theta}\right)^{\alpha} = 0, \quad (28)$$

$$\frac{\partial\, ln\, L}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} ln\, x_i - (\alpha-1)\lambda\sum_{i=1}^{n}\frac{ln\, x_i x_i{}^{-\theta}}{\left(1+\lambda x_i{}^{-\theta}\right)} + \sum_{i=1}^{n}\frac{\alpha\lambda\, ln\, x_i}{x_i{}^{\theta}\left(1+\lambda x_i{}^{-\theta}\right)^{1-\alpha}} = 0 \quad (29)$$

These nonlinear equations cannot be analytically solved, but the statistical software like R program (Team (2015)) can be used to solve them numerically using iterative techniques.

The asymptotic variance-covariance matrix of the MLEs for the three parameters α, θ and λ is the inverse of the observed Fisher information matrix as follows

$$\hat{\sigma} = \begin{bmatrix} -\frac{\partial^2 \ln \mathrm{L}}{\partial\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} \\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} & -\frac{\partial^2 \ln L}{\partial\lambda^2} & -\frac{\partial^2 \ln L}{\partial\lambda\partial\theta} \\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & -\frac{\partial^2 \ln L}{\partial\lambda\partial\theta} & -\frac{\partial^2 \ln L}{\partial\theta^2} \end{bmatrix}^{-1}_{\alpha=\hat{\alpha},\lambda=\hat{\lambda},\theta=\hat{\theta}} = \begin{bmatrix} \hat{\sigma}^2_{\alpha} & \hat{\sigma}_{\alpha,\lambda} & \hat{\sigma}_{\alpha,\theta} \\ \hat{\sigma}_{\alpha,\lambda} & \hat{\sigma}^2_{\lambda} & \hat{\sigma}_{\lambda,\theta} \\ \hat{\sigma}_{\alpha,\theta} & \hat{\sigma}_{\lambda,\theta} & \hat{\sigma}^2_{\theta} \end{bmatrix} \quad (30)$$

The elements of the sample Fisher information matrix can be obtained by deriving the second derivatives of the log-likelihood function (26) and evaluating them at the MLEs ((Cohen 1965)). These elements can be derived as follow

$$-\frac{\partial^2 \ln \mathrm{L}}{\partial\alpha^2} = \frac{n}{\alpha^2} + \sum_{i=1}^{n} ln\left(1+\lambda x_i{}^{-\theta}\right)^2\left(1+\lambda x_i{}^{-\theta}\right)^{\alpha}, \quad (31)$$

$$-\frac{\partial^2\, ln\, L}{\partial\lambda^2} = \frac{n}{\lambda^2} + (\alpha-1)\sum_{i=1}^{n}\frac{\left(1+\lambda x_i{}^{-\theta}\right)^{-2}}{x_i^{2\theta}} + \alpha(\alpha-1)\sum_{i=1}^{n}\frac{\left(1+\lambda x_i{}^{-\theta}\right)^{\alpha-2}}{x_i^{2\theta}} \quad (32)$$

$$-\frac{\partial^2\, ln\, L}{\partial\theta^2} = \frac{n}{\theta^2} - (\alpha-1)\lambda\sum_{i=1}^{n}\frac{ln(x_i)^2 x_i^{-\theta}}{\left(1+\lambda x_i{}^{-\theta}\right)} + \alpha\lambda\sum_{i=1}^{n}\frac{ln(x_i)^2 x_i^{-\theta}}{\left(1+\lambda x_i{}^{-\theta}\right)^{1-\alpha}} + \lambda^2(\alpha-1)\sum_{i=1}^{n}\frac{ln(x_i)^2 x_i^{-2\theta}}{\left(1+\lambda x_i{}^{-\theta}\right)^2} + \lambda^2\alpha(\alpha-1)\sum_{i=1}^{n}\frac{ln(x_i)^2 x_i^{-2\theta}}{\left(1+\lambda x_i{}^{-\theta}\right)^{2-\alpha}} \quad (33)$$

$$-\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} = -\sum_{i=1}^{n}\left(1+\frac{\lambda}{x_i^{\theta}}\right)^{\alpha-1} x_i^{-\theta}\left[-1-\alpha\ln\left(1+\frac{\lambda}{x_i^{\theta}}\right)+\left(1+\frac{\lambda}{x_i^{\theta}}\right)^{-\alpha}\right], \tag{34}$$

$$-\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n}\lambda\ln x_i\left(1+\frac{\lambda}{x_i^{\theta}}\right)^{\alpha-1} x_i^{-\theta}\left[1+\alpha\ln\left(1+\frac{\lambda}{x_i^{\theta}}\right)-\left(1+\frac{\lambda}{x_i^{\theta}}\right)^{-\alpha}\right], \tag{35}$$

$$-\frac{\partial^2 \ln L}{\partial\lambda\partial\theta} = -\sum_{i=1}^{n}(\alpha-1)\ln(x_i)x_i^{-\theta}\left[\alpha(\alpha-1)^{-1}\left(1+\lambda x_i^{-\theta}\right)^{\alpha-1}-\left(1+\lambda x_i^{-\theta}\right)^{-1}+\lambda x_i^{-\theta}\left(1+\lambda x_i^{-\theta}\right)^{-2}+\alpha\lambda x_i^{-\theta}\left(1+\lambda x_i^{-\theta}\right)^{\alpha-2}\right]. \tag{36}$$

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for the parameters α, λ and θ as follow

$$\hat{\alpha}_{ML} \pm z_{\tau/2}\sqrt{\hat{\sigma}_{\alpha}^2}\ , \qquad \hat{\lambda}_{ML} \pm z_{\tau/2}\sqrt{\hat{\sigma}_{\lambda}^2} \quad and \qquad \hat{\theta}_{ML} \pm z_{\tau/2}\sqrt{\hat{\sigma}_{\theta}^2} \tag{37}$$

where $z_{\tau/2}$ is an upper (τ /2)100% of the standard normal distribution.

5. Simulation Study

In this section, the simulation study is executed to assess the performance of the proposed MLE method for estimating the parameters of IGPW distribution. Monte Carlo experiments were carried out based on generated data from IGPW distribution. By using the inversion method in Section 3.1, We generated 1000 samples of size $n = 20, 50, 100$ from IGPW distribution for different combinations of parameters α, λ and θ. The mean square errors (MSE) of the MLEs were computed using the “CG” optimization' method in R program. The simulation results were displayed in Table 2. The main conclusion from the figures in Table 2, is that the mean square errors of MLEs decrease with increasing the sample size. This indicates that the MLE method is suitable for estimating the unknown parameters of IGPW distribution.

Table 2: Mean square errors of the MLEs' $\hat{\alpha}, \hat{\lambda}, \hat{\theta}$

Parameters	$n = 25$			$n = 50$			$n = 100$		
$\alpha,\ \lambda,\ \theta$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$
0.5, 0.5, 0.5	0.2708	0.0169	0.3979	0.0834	0.0162	0.2890	0.0706	0.0151	0.2621
1, 0.5, 0.5	0.9698	0.3349	0.1144	0.7509	0.2783	0.0868	0.4699	0.1865	0.0705
1.5, 0.5, 0.5	0.6715	0.7987	0.1129	0.2145	0.3303	0.1036	0.2217	0.3096	0.0410
0.5, 1, 0.5	0.0797	0.2063	0.5093	0.0445	0.1355	0.4508	0.0523	0.0930	0.4220
0.5, 1.5, 0.5	0.0772	0.7335	0.4802	0.0453	0.5282	0.4562	0.0403	0.3793	0.3218
0.5, 0.5, 1	0.1727	0.0142	0.3131	0.1145	0.0129	0.3044	0.0836	0.0102	0.1863
0.5, 0.5, 1.5	0.3400	0.0198	0.4910	0.2845	0.0168	0.3666	0.1592	0.0139	0.2545

6. Real Data Illustration

This section illustrates the usefulness of the IGPW distribution using two real datasets. These datasets are described as follows:

The data set (I): Stress-rupture life data

The first data set consists of 76 observations of the strengths of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. For previous studies with the data sets see Andrews and Herzberg (1985), Barlow et al. (1984) and Oluyede et al. (2016). These data are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

The data set (II): Remission times data

The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (see Lee and Wang (2003)). The data are: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. The data has been used by Kumar et al. (2015), El-Gohary et al. (2015), Chandra (2017) and De Andrade and Zea (2018).

We fitted the above-mentioned datasets using MLE to the inverse generalized power Weibull (IGPW), inverse Nadarajah-Haghighi (INH), inverse Weibull (IW) and inverse exponential (IE) distributions. The MLEs and their standard errors for IGPW, INH, IW, and IE distributions are displayed in Table 3. The fitted models were compared by using Cramérvon Mises (W^*), Anderson Darling (A^*), Kolmogorov-Smirnov $(K-S)$, -Log-likelihood $(-\ln L)$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion $(CAIC)$, Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC). Based on these criteria, the best model is the one that achieves the lowest values for the information criteria and goodness-of-fit statistics. Hence, it is clear from the numerical results in Table 4, that the IGPW model provides a better fit than the other competing models. The Figures 4 and 5 display the graphical comparison of the fitted models for datasets I and II, respectively. Also, these figures graphically illustrate that IGPW distribution provides the best fit to our data sets, as compared to the other considered models. Therefore, the IGPW model can be used as a possible alternative to the well- known models like inverse exponential and inverse Weibull models.

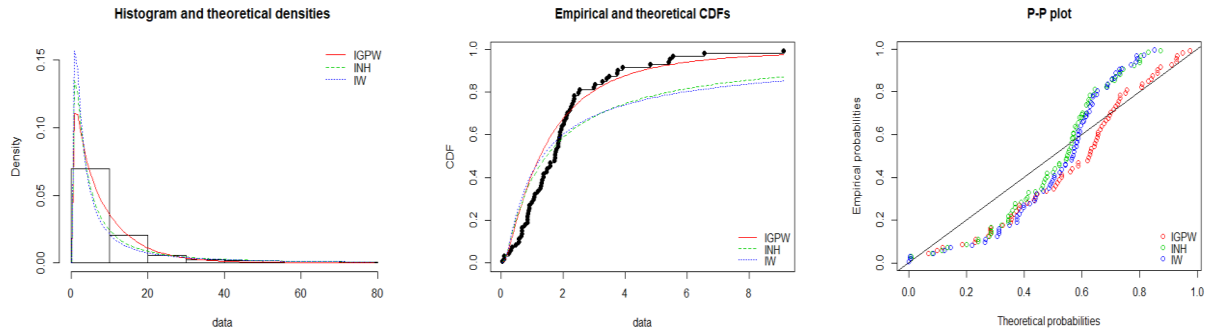

Fig. 4: Histogram and estimated densities (left panel); Theoretical and estimated CDFs (middle panel); P-P plots (right panel) for stress-rupture life data

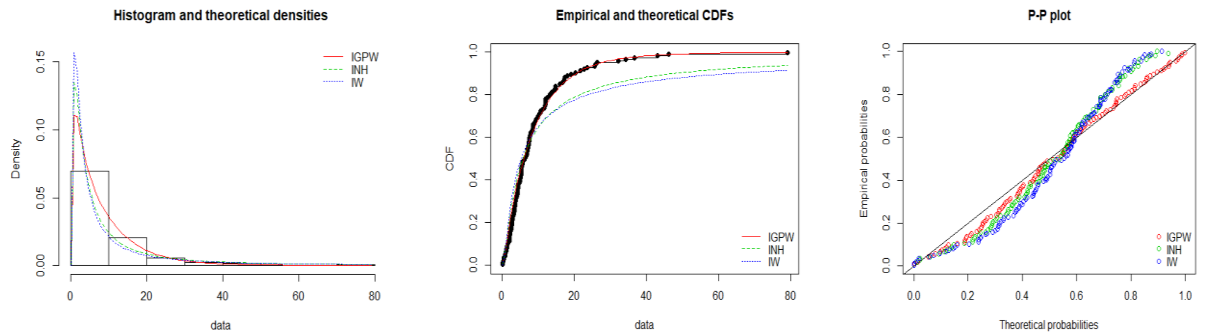

Fig. 5: Histogram and estimated densities (left panel); Theoretical and estimated CDFs (middle panel); P-P plots (right panel) for Remission times data

Table 3. The estimates and the standard errors (in parentheses) for data set I and II

Data set	Model	α	λ	θ
Data set I	IE	-	0.6248 (0.0716)	-
	IR	-	0.0395 (0.0045)	-
	IW	-	0.8625 (0.1089)	0.7585 (0.0540)
	INH	0.5130 (0.0594)	2.6070 (0.7020)	-
	IGPW	0.2356 (0.0327)	12.3717 (4.6809)	2.0876 (0.2692)
Data set II	IE	-	2.4847 (0.2200)	-
	IR	-	0.6174 (0.0545)	-
	IW	-	2.4311 (0.2192)	0.7520 (0.0424)
	INH	0.5064 (0.0480)	10.5947 (2.3220)	-
	IGPW	0.4435 (0.0482)	15.7165 (4.8326)	1.2110 (0.1023)

Table 4. The estimates of the goodness-of-fit test for data set I and II

	Model	K-S	W^*	A^*	-L	AIC	CAIC	BIC	HQIC
Data set I	IE	0.2899	1.2059	6.8516	163.1015	328.203	328.257	330.5337	329.1344
	IR	0.7940	2.0476	10.9515	345.9147	693.8294	693.8834	696.1601	694.7608
	IW	0.1886	0.9166	5.3388	153.5393	311.0787	311.2431	315.7401	312.9416
	INH	0.1798	0.5652	3.3981	144.5465	293.0930	293.2574	297.7545	294.9560
	IGPW	0.1841	0.3082	1.8968	132.0617	270.1234	270.4568	277.1156	272.9179
Data set II	IE	0.2311	1.1139	6.6074	460.382	922.765	922.796	925.617	923.923
	IR	0.7502	2.3754	13.2264	774.342	1550.683	1550.715	1553.535	1551.842
	IW	0.1408	0.7443	4.5464	444.001	892.002	892.098	897.706	894.319
	INH	0.1636	0.3565	2.2844	431.059	866.118	866.214	871.822	868.436
	IGPW	0.1364	0.3368	2.1713	426.910	859.819	860.013	868.375	863.296

7. Conclusion

This paper introduces a new three-parameter distribution, called the inverse generalized power Weibull distribution. This distribution is considered as a reciprocal of the generalized power Weibull distribution and a generalization of inverse Weibull distribution. Some of the statistical properties of the inverted generalized power Weibull distribution, including the moments, hazard rate function, quantile function and order statistics are derived. The maximum likelihood method is used to estimate the model parameters. The performances of the maximum likelihood estimators are assessed in terms of mean squared errors using Monte Carlo simulation. The practical applications have established that the proposed distribution is quite useful for dealing with reliability data and behaves better than its four special cases (inverse Weibull, inverse exponential, inverse Rayleigh and inverse Nadarajah-Haghighi distributions).

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