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The Extended Alpha Power Transformed Family of Distributions: Properties and Applications

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Abstract

In this article, a new family of lifetime distributions by adding an additional parameter to the existing distributions is introduced. The new family is called, the extended alpha power transformed family of distributions. For the proposed family, explicit expressions for some mathematical properties along with estimation of parameters through Maximum likelihood Method are discussed. A special sub-model, called the extended alpha power transformed Weibull distribution is considered in detail. The proposed model is very flexible and can be used to model data with increasing, decreasing or bathtub shaped hazard rates. To access the behavior of the model parameters, a small simulation study has also been carried out. For the new family, some useful characterizations are also presented. Finally, the potentiality of the proposed method is showen via analyzing two real data sets taken from reliability engineering and bio-medical fields.

Keywords: Family of distributions; Alpha power transformation; Weibull distribution; Moments; Order statistic; Residual life function; Maximum likelihood estimation.

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1.Introduction

In modeling the lifetime data, classical distributions are widely used in many applied areas such as engineering, medical sciences, actuarial, environmental studies, economics, finance, and insurance. In all of the above mentioned fields, these distributions have been applied quite successfully. However, in many fields such as reliability engineering and medical fields these classical distributions do not provide the best fit when the data follow non-monotonic failure rates. Therefore, to model reliability engineering and bio-medial data, there is a clear need for the generalized versions of these classical distributions. This interest, motivated the researchers to introduce new extensions of the exisitng distributions. These extended distributions provide more fexibility by introducing one "or more" additional parameters to the baseline model. In the recent advances in distribution theory, however, researchers have shown a deep interest in proposing new methods to expand family of lifetime distributions. This has been done through many different approaches via introducing new generators. Some of the well-known generators are: Exponentiated family of Mudholkar and Sarivastava (1993), Marshal-Olkin generated family (MO-G) of Marshall and Olkin (1997), beta-G by Eugene et al. (2002), gamma-G by Zografos and Balakrishanan (2009), Kumaraswamy-G family of Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander et al. (2012), Kumaraswamy Marshal-Olkin family of Alizadeh et al. (2013), exponentiated generalized-G by Cordeiro et al. (2013), Transformed-Transformer (T-X) by Alzaatreh et al. (2013), the exponentiated generalized class of Cordeiro et al. (2013), the exponentiated half-logistic family of Cordeiro et al. (2014), log-gamma-G by Amini et al. (2014), Lomax Generator of Cordeiro et al. (2014), Weibull-G family of Bourguignon et al. (2014), Kumaraswamy odd log logistic-G by Alizadeh et al. (2015), odd generalized exponential-G by Tahir et al. (2015). Logistic-X by Tahir et al. (2015), beta Marshal-OLkin family by Alizadeh et al. (2015), a new neneralized class of distributions of Ahmad (2018). Mahdavi and Kundu (2017) proposed a new method for introducing statistical distributions via the cumulative distribution function (cdf) given by

$$G(x;\alpha,\xi)=\frac{\alpha^{F(x;\xi)}-1}{\alpha-1}, \qquad \alpha,\xi>0,\ \alpha\neq 1,\ x\in\mathbb{R}. \tag{1}$$

Using (1), Mahdavi and Kundu (2017) and Dey et al. (2017) introduced the alpha power exponential (APE) and alpha power transformedWeibull (APTW) distributions, respectively.

Recently, Ahmad (2018) proposed a new family of life distributions, called the Zubair-G family whose cdf is given by

$$G(x;\alpha,\xi)=\frac{e^{\alpha F(x;\xi)^2}-1}{e^{\alpha}-1}, \qquad \alpha,\xi>0,\ x\in\mathbb{R}. \tag{2}$$

In this article, a new family of lifetime distributions, called the extended alpha power transformed (Ex-APT) family of distributions is introduced. The new family is defined by the cdf

$$G(x;\alpha,\xi)=\frac{\alpha^{F(x;\xi)}-e^{F(x;\xi)}}{\alpha-e}, \qquad \xi>0,\ \alpha\neq e,\alpha>e,\ x\in\mathbb{R}, \tag{3}$$

where, $F(x;\xi)$ is cdf of the baseline random variable depending on the vector parameter ξ and α is an additional parameter. The probability density function (pdf), survival function (sf), hazard rate function (hrf), reverse hazard rate function (rhrf) and cumulative hazard rate function (chrf) of the Ex-APT family are given respectively, by

$$g(x;\alpha,\xi)=\frac{f(x;\xi)\left((\log\alpha)\alpha^{F(x;\xi)}-e^{F(x;\xi)}\right)}{\alpha-e}, x\in\mathbb{R}, \tag{4}$$

$$S(x;\alpha,\xi)=\frac{\alpha-e-\alpha^{F(x;\xi)}+e^{F(x;\xi)}}{\alpha-e}, x\in\mathbb{R}, \tag{5}$$

$$h(x;\alpha,\xi)=\frac{f(x;\xi)\left((\log\alpha)\alpha^{F(x;\xi)}-e^{F(x;\xi)}\right)}{\alpha-e-\alpha^{F(x;\xi)}+e^{F(x;\xi)}}, x\in\mathbb{R}, \tag{6}$$

$$H(x;\alpha,\xi)=-\log\left(\frac{\alpha-e-\alpha^{F(x;\xi)}+e^{F(x;\xi)}}{\alpha-e}\right), x\in\mathbb{R}. \tag{7}$$

The new pdf is most tractable when $F(x,\xi)$ and $f(x,\xi)$ have simple analytical expressions. Henceforth, a random variable X with pdf (2) is denoted by $X\sim Ex-APT(x;\alpha,\xi)$. Furthermore, for the sake of simplicity, the dependence on the vector of the parameters is omitted and simply $G(x)=G(x;\alpha,\xi)$ will be used. Moreover, the key motivations for using the Ex-APT family in practice are the following:

- A very simple and convienent method of adding additional parameters to modify the existing distributions.
- To improve the characteristics and flexibility of the existing distributions.
- To introduce the extended version of the baseline distribution whose cdf, sf and hrf, have closed form.
- To provide better fits than the competing modified models.

The rest of this article is organized as follows. In section 2, a special sub-model of the proposed family is discussed. Some mathematical properties are obtained in section 3. Maximu likelihood estimates of the model parameters are obtained in section 4. A small simulation study is conducted in Section 5. Section 6 contains some useful characterizations of the proposed class. Section 7, is devoted to analyzing two real life applications. Finally, concluding remarks are provided in section 8.

2. Sub-Model Description

In this section, we define a special sub-model of the proposed family, called the extended alpha power transformed Weibull (Ex-APTW) distribution. Let $F(x;\xi)$ be cdf of the Weibull distribution given by $F(x;\xi)=1-e^{-\gamma x^{\theta}}$, $x\geq 0,\ \gamma,\theta>0$, where $\xi=(\gamma,\theta)$. Then, the cdf of the Ex-APTW distribution has the following expression

$$G(x;\alpha,\xi)=\frac{\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}-e^{\left(1-e^{-\gamma x^{\theta}}\right)}}{\alpha-e}, \qquad \xi>0,\ \alpha\neq e,\alpha>e,\ x\geq 0. \tag{8}$$

The pdf, sf and hrf of the Ex-APTW distribution are given, respectively, by

$$g(x;\alpha,\xi)=\frac{\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}}{\alpha-e}\left((\log\alpha)\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}-e^{\left(1-e^{-\gamma x^{\theta}}\right)}\right),\qquad x\geq 0, \tag{9}$$

$$S(x;\alpha,\xi)=\frac{\alpha-e-\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}+e^{\left(1-e^{-\gamma x^{\theta}}\right)}}{\alpha-e},\qquad x\geq 0, \tag{10}$$

$$h(x;\alpha,\xi)=\frac{\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}\left((\log\alpha)\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}-e^{\left(1-e^{-\gamma x^{\theta}}\right)}\right)}{\alpha-e-\alpha^{\left(1-e^{-\gamma x^{\theta}}\right)}+e^{\left(1-e^{-\gamma x^{\theta}}\right)}},\qquad x\geq 0. \tag{11}$$

For different values of the model parameters, plots of the pdf of the Ex-APTW distribution are sketched in Figure 1.

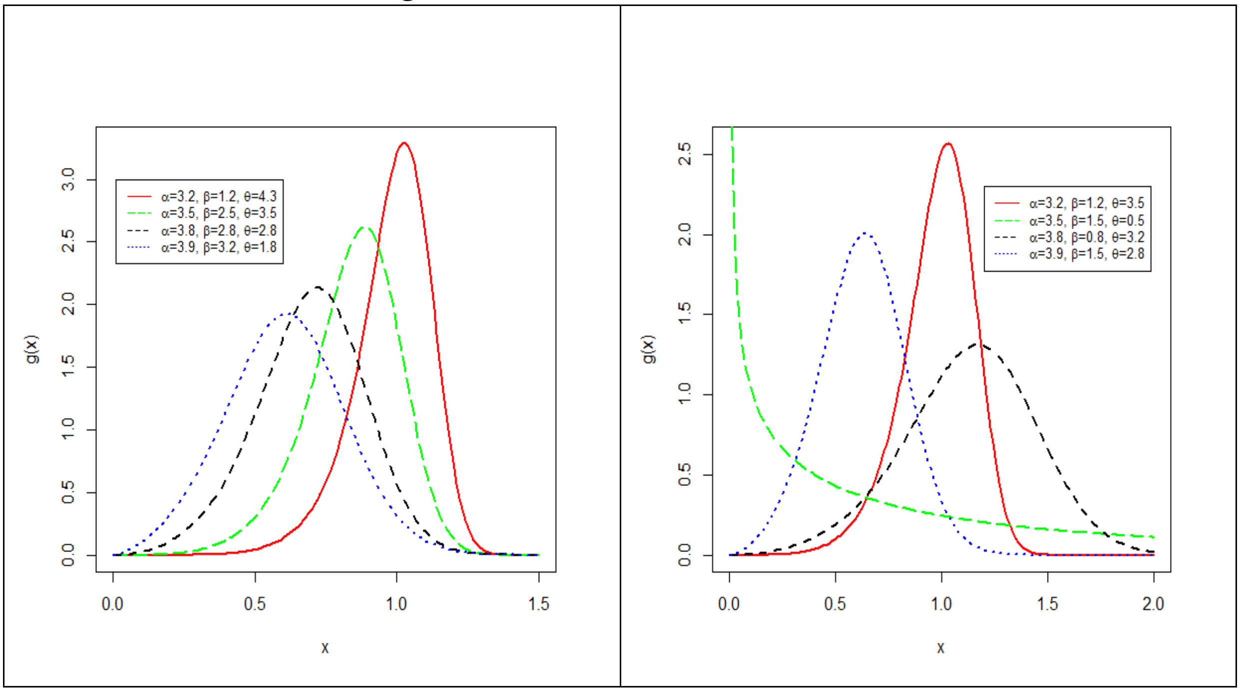

Figure 1. Different plots for the pdf of the Ex-APTW distribution.

For the selected values of parameters, some possible shapes for the hrf of the Ex-APTW model are drawn in Figure 2.

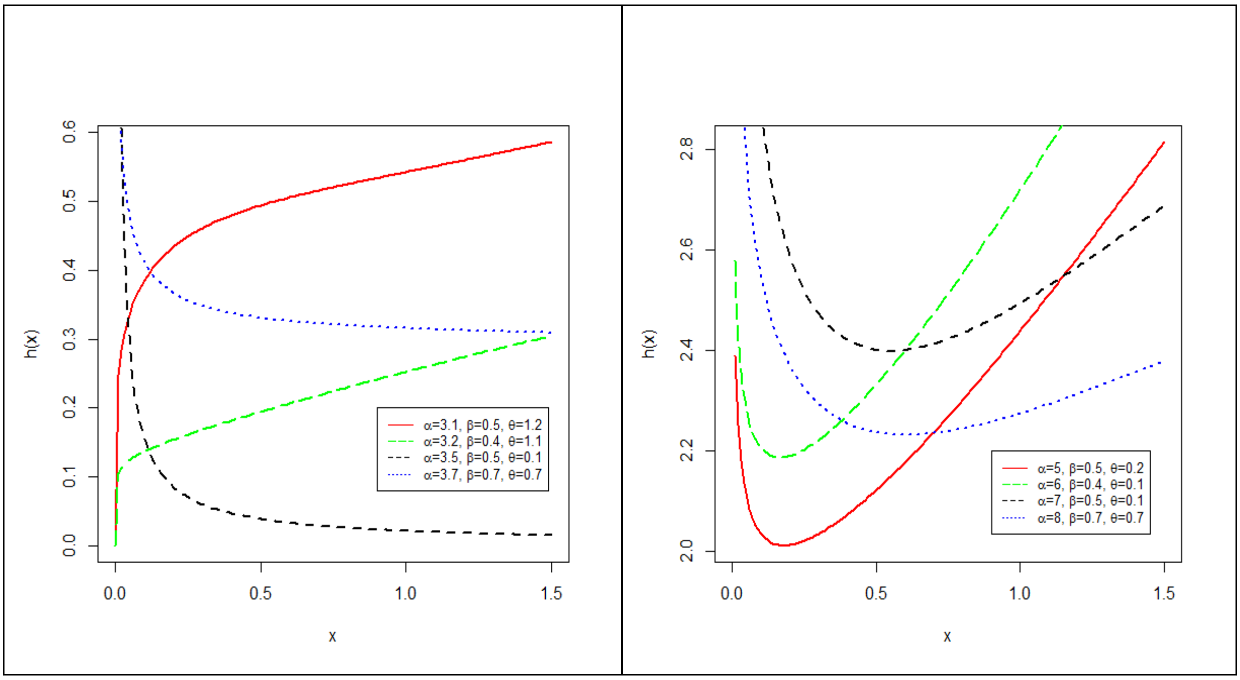

Figure 2. Different plots for the hrf of the Ex-APTW distribution.

3. Basic Mathematical Properties

In this section, some statistical properties of the proposed family are derived.

3.1. Quantile function

Let X be the Ex-APTW random variable with pdf (4), the quantile function of X, say Q(u), is given by

$$Q(u)=F(x;\xi)-\log\left(\alpha^{F(x;\xi)}-u(\alpha-e)\right), \tag{12}$$

where, u has the uniform distribution on the interval [0,1]. From the expression (12), it is clear that the proposed family does not have a closed form solution of its quantile function. Therefore, computer software can be used to obtain solution.

3.2. Moments

Moments are very important and play an essential role in statistical analysis, especially in the applications. It helps to capture the important features and characteristics of the distribution (e.g., central tendency, dispersion, skewness and kurtosis). The rth moment of the Ex-APT family of distributions is derived, using (4) in (12),

$$\mu_r^{/}=\frac{1}{\alpha-e}\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\int_{-\infty}^{\infty}x^r f(x;\xi)F(x;\xi)^i\,dx-\sum_{i=0}^{\infty}\frac{1}{i!}\int_{-\infty}^{\infty}x^r f(x;\xi)F(x;\xi)^i\,dx\right), \tag{13}$$

$$\mu_r^{/}=\frac{1}{\alpha-e}\left(\sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\eta_{r,i}-\sum_{i=0}^{\infty}\frac{\eta_{r,i}}{i!}\right),$$

where,

$$\eta_{r,i} = \int_{-\infty}^{\infty} x^r f(x;\xi) F(x;\xi)^i \, dx.$$

Furthermore, a general expression for moment generating function (mgf) of the Ex-APT random variable X is

$$M_x(t) = \frac{1}{\alpha - e}\left(\sum_{i,r=0}^{\infty} \frac{(\log\alpha)^{i+1} t^r}{r!i!}\eta_{r,i} - \sum_{i=0}^{\infty} \frac{t^r}{r!i!}\eta_{r,i} \right). \tag{14}$$

3.3. Residual and Reverse Residual Life

The residual life offer wider applications in reliability theory and risk management. The residual lifetime of X denoted by $R_{(t)}$ is derived as

$$R_{(t)}(x) = \frac{S(x+t)}{S(t)},$$

$$R_{(t)}(x) = \frac{\alpha - e - \alpha^{F(x+t;\xi)} + e^{F(x+t;\xi)}}{\alpha - e - \alpha^{F(t;\xi)} + e^{F(t;\xi)}}. \tag{15}$$

Additionally, the reverse residual life of the Ex-APT random variable denoted by $\bar{R}_{(t)}$ is

$$\bar{R}_{(t)} = \frac{S(x-t)}{S(t)},$$

$$\bar{R}_{(t)}(x) = \frac{\alpha - e - \alpha^{F(x-t;\xi)} + e^{F(x-t;\xi)}}{\alpha - e - \alpha^{F(t;\xi)} + e^{F(t;\xi)}}. \tag{16}$$

3.4. Order statistics

Order statistics are among the essential tools in inferencial and non-parametric statistics. The applications of these statistics appear in the study of reliability and life testing. Let $X_1, X_2, \ldots, X_k$ be a random sample of size k taken from the Ex-APT distribution with parameters α and ξ. Let $X_{1:k}, X_{2:k}, \ldots, X_{k;k}$ be the corresponding order statistics. Then, from David (1981), the density of $X_{r:k}$ for ($r=1, 2, \ldots, k$) is given by

$$g_{r:k}(x) = \frac{g(x;\alpha,\beta,\xi)}{B(r,\ k-r+1)} \sum_{i=0}^{k-r} \binom{k-r}{i} (-1)^i \left[G(x;\alpha,\beta,\xi)\right]^{i+r-1}. \tag{17}$$

4. Estimation

In this section, the estimation of the unknown parameters of the Ex-APT family via the method of maximum likelihood is discussed. Let $x_1, x_2, \ldots, x_k$ be observed values of a random sample from Ex-APT family with parameters α and ξ. The log-likelihood function of this sample is

$$\log L(x;\alpha,\xi) = -k\log(\alpha - e) + \sum_{i=1}^{k} \log\left[f(x_i;\xi)\right] + \sum_{i=1}^{k} \log\left[(\log\alpha)\alpha^{F(x_i;\xi)} - e^{F(x_i;\xi)}\right]. \tag{18}$$

The partial derivatives of (18) are

$$\frac{\partial}{\partial\alpha}\log L(x;\alpha,\xi)=-\frac{k}{\alpha-e}+\sum_{i=1}^{k}\frac{(\log\alpha)F(x_i;\xi)\alpha^{F(x_i;\xi)-1}+\alpha^{F(x_i;\xi)-1}}{(\log\alpha)\alpha^{F(x_i;\xi)}-e^{F(x_i;\xi)}}, \tag{19}$$

$$\frac{\partial}{\partial\xi}\log L(x;\alpha,\xi)=\sum_{i=1}^{k}\frac{\partial f(x_i;\xi)/\partial\xi}{f(x_i;\xi)}+\sum_{i=1}^{k}\frac{(\log\alpha)^2\alpha^{F(x_i;\xi)}\partial F(x_i;\xi)/\partial\xi-e^{F(x_i;\xi)}\partial F(x_i;\xi)/\partial\xi}{(\log\alpha)\alpha^{F(x_i;\xi)}-e^{F(x_i;\xi)}}. \tag{20}$$

Setting $\frac{\partial}{\partial\alpha}\log L(x;\alpha,\xi)$ and $\frac{\partial}{\partial\xi}\log L(x;\alpha,\xi)$ equal to zero and solving numerically these expressions simultaneously, yields the maximum likelihood estimates of (α,ξ).

5. Simulation Study

In order to assess the performances of the maximum likelihood parameters of the proposed distribution, a small simulation study is carried out. The process is carried out as follow:

- The number of Monte Carlo replications was made 1000 times each with sample sizes n = 30, 50 and 100.
- Initial values for the parameters are selected as given in Table 1.
- Formulas used for calculating Bias and MSE are given by $Bias(\hat{\alpha})=\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)$ and $MSE(\hat{\alpha})=\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)^2$, respectively.
- Step (iii) is also repeated for the other parameters (β,θ).

The empirical results are given in Table 1.

Table 1: The parameter estimation from the Ex-APTW distribution using MLE.

n	Par	Init	MLE	Bias	MSE	Init	MLE	Bias	MSE
30	α	3	3.219	0.0132	0.0073	3	0.7711	0.0213	0.0165
	β	0.5	0.5151	0.0149	0.0079	0.5	0.5154	0.0152	0.0084
	θ	0.5	0.5331	0.0262	0.0181	0.5	0.5212	0.0231	0.0162
50	α	3	3.187	0.0071	0.0042	3	0.7584	0.0112	0.0084
	β	0.5	0.5078	0.0080	0.0052	0.5	0.5075	0.0078	0.0048
	θ	0.5	0.5213	0.0243	0.0106	0.5	0.5092	0.0109	0.0087
100	α	3	3.087	0.0030	0.0021	3	0.7593	0.0097	0.0039
	β	0.5	0.5021	0.0029	0.0037	0.5	0.5070	0.0071	0.0025
	θ	0.5	0.5187	0.0136	0.0053	0.5	0.5040	0.0097	0.0044

Continued of Table (1)

n	Par	Init	MLE	Bias	MSE	Init	MLE	Bias	MSE
30	α	3.5	3.5489	0.0485	0.0669	3.5	1.5589	0.0586	0.1212
	β	0.5	0.5183	0.0185	0.0089	1.5	1.5083	0.0088	0.0206
	θ	0.5	0.5736	0.0769	0.0411	0.5	0.5120	0.0126	0.0147
50	α	3.5	3.5272	0.0273	0.0437	3.5	1.5221	0.0222	0.0615
	β	0.5	0.5109	0.0106	0.0056	1.5	1.4996	0.0065	0.0109
	θ	0.5	0.5674	0.0675	0.0243	0.5	0.5039	0.0039	0.0081
	α	3.5	3.5174	0.0172	0.0181	3.5	1.5166	0.0169	0.0293

100	β	0.5	0.5063	0.0057	0.0028	1.5	1.5021	0.0029	0.0059
	θ	0.5	0.5527	0.0530	0.0239	0.5	0.4924	0.0031	0.0032

6. Characterizations of Ex-APT Distribution

This section is devoted to various characterizations of Ex-APT distribution. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function and (ii) the reverse (or reversed) hazard function. It should be mentioned that for characterization (i) the cdf may not have a closed form. We present our characterizations (i) (iii) in three subsections.

6.1. Characterizations based on two truncated moments

In this subsection we characterize Ex-APT distribution in terms of the ratio of two truncated moments. This characterization result employs a theorem due to Glänzel (1987); see Theorem 1 in Appendix A. Note that the result holds also when the interval H is not closed. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 6.1. Let X: $\Omega \rightarrow \mathbb{R}$ be a continuous random variable and let $h_1(x) = \left((\log\alpha)\alpha^{F(x;\xi)} - e^{F(x;\xi)}\right)$ and $h_2(x) = h(x;\xi)F(x;\xi)$ for $x \in \mathbb{R}$. The random variable X belong has pdf (4) if and only if the function $\eta(x)$ defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(1 + F(x;\xi)\right), \qquad x \in \mathbb{R}.$$

Proof: If X *has* pdf (14), then

$$\left(1 - F(x)\right)E\left(h_1(X)/X \geq x\right) = \frac{1}{\alpha - e}\left(1 - F(x;\xi)\right), \qquad x \in \mathbb{R},$$

and

$$\left(1 - F(x)\right)E\left(h_2(X)/X \geq x\right) = \frac{1}{2(\alpha - e)}\left(1 - F(x;\xi)^2\right), \qquad x \in \mathbb{R},$$

and finally

$$\eta(x)h_1(x) - h_2(x) = \frac{1}{2}h_1(x;\xi)\left(1 - F(x;\xi)\right) > 0, \qquad \text{for} \quad x \in \mathbb{R}.$$

Conversely, if $\eta(x)$ is given as above, then

$$s'(x) = \frac{\eta'(x)h_1(x)}{\eta(x)h_1(x) - h_2(x)} = \frac{f(x;\xi)}{1 - F(x;\xi)}, \qquad x \in \mathbb{R},$$

and hence

$$s(x) = -\log\left(1 - F(x;\xi)\right), \qquad x \in \mathbb{R},$$

Now, in view of Theorem 1, X has density (4).

Corollary 6.1. Let X: $\Omega \to \mathbb{R}$ be a continuous random variable and let $h(x)$ be as in Proposition 6.1. Then, X has pdf (4) if and only if there exist functions $\eta(x)$ and $\xi(x)$ defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)h_1(x)}{\eta(x)h_1(x)-h_2(x)} = \frac{f(x;\xi)}{1-F(x;\xi)}, \qquad x \in \mathbb{R}.$$

The general solution of the differential equation in Corollary 6.1 is

$$\eta(x) = \left(1-F(x;\xi)\right)^{-1} = \left[-\int f(x;\xi)h_1(x;\xi)^{-1}h_2(x;\xi)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 6.1 with $D=1/2$: However, it should be also noted that there are other triplets $(h_1(x), h_2(x), \eta(x))$ satisfying the conditions of Theorem 1.

6.2. Characterization based on hazard function

It is known that the hazard function, $h_F(x)$, of a twice differentiable distribution function, $F(x)$, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of Ex-APT, in terms of the hazard function, which is not of the above trivial form.

Proposition 6.2. Let $X: \Omega \to \mathbb{R}$ be a continuous random variable. The random variable X has pdf (4) if and only if its hazard function $h_F(x)$ satisfies the following differential equation

$$h_F'(x) - \frac{f'(x;\xi)}{f(x;\xi)}h_F(x;\xi) = \frac{(\log\alpha) f^2(x;\xi)\alpha^{F(x;\xi)}\left\{(\alpha-2)(\log\alpha)-\left(2-(\log\alpha)\right)e^{F(x;\xi)}\right\}}{\left(\alpha-e-\alpha^{F(x;\xi)}+e^{F(x;\xi)}\right)^2}, \quad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to-\infty} h_F(x) = \frac{(\log\alpha)-1}{\alpha-e}\lim_{x\to-\infty} f(x;\xi).$

Proof. If X has pdf (4), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left(f^{-1}(x;\xi)h_F(x;\xi)\right) = \frac{d}{dx}\frac{\left((\log\alpha)\alpha^{F(x;\xi)}-e^{F(x;\xi)}\right)}{\left(\alpha-e-\alpha^{F(x;\xi)}+e^{F(x;\xi)}\right)^2}, \quad x \in \mathbb{R},$$

or

$$h_F(x;\xi) = \frac{f(x;\xi)\left((\log\alpha)\alpha^{F(x;\xi)}-e^{F(x;\xi)}\right)}{\alpha-e-\alpha^{F(x;\xi)}+e^{F(x;\xi)}},$$

which is the hazard function of the Ex-APT distribution.

6.3. Characterization in terms of the reverse hazard function

The reverse hazard function, $r_F(x)$, of a twice differentiable distribution function, $F(x)$, is defined

$$r_F(x) = \frac{f(x)}{F(x)}, \qquad x \in \text{support of } F(x).$$

Proposition 6.3. Let $X:\Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is (4) if and only if its reverse hazard function $r_F(x)$ satisfies the differential equation

$$r_F^{/}(x;\xi) = \frac{f^{/}(x;\xi)}{f(x;\xi)} r_F(x;\xi) = -\frac{\left(1-(\log\alpha)\right)^2 f(x;\xi)^2 (\alpha e)^{F(x;\xi)}}{\left(\alpha^{F(x;\xi)} - e^{F(x;\xi)}\right)^2}, \quad x \in \mathbb{R},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = \left(\frac{(\log\alpha)-e}{\alpha - e}\right) \lim_{x\to\infty} f(x;\xi)$.

Proof. Is similar to that of Proposition 6.2.

7. Applications

In the this section, we provide two applications of the proposed mode to the real data sets. We compare the fits of the proposed distribution to those of the three-parameter exponentiated Weibull of Mudholkar and Sarivastava (1993), Marshall-Olkin Weibull (MOW) of Marshall and Olkin (1997) and beta distribution of Fayomi et al. (2007). The goodness-of-fit measures such as Anderson-Darling (AD) statistic, Cramer–von Mises (CM), Kolmogorov-Smirnov (KS) and the corresponding p-value are considered to compare the proposed method with the fitted models. In general, a model with smaller values of these analytical measure and high p-value indicate better fit to the data. All the required computations have been carried out in the R-language using “BFGS” algorithm.

Data 1: The first data set representing the remission times (in months) of a random sample of 128 bladder cancer patients taken from Lee and Wang (2003). The data are listed as: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05. Coressonpong to data 1, the maximum likelihood estimates of the fitted models are provided in Table 2. While, the goodness of fit measure are given in Table 3.

Table 2. Maximum likelihood estimates of the fitted distributions using data 1.

Dist.	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$
Ex-APTW	2.933	2.645	0.510	
MOW	11.829		0.564	0.877
EW	4.332		0.541	0.720
BW	3.196	1.143	0.609	0.486

Table 3. The statistics of the fitted models using data 1.

Dist.	KS	CM	AD	P-value
Ex-APTW	0.034	0.022	0.144	0.978
MOW	0.075	0.150	0.884	0.451
EW	0.046	0.046	0.324	0.940
BW	0.945	1.592	1.576	2.2e-16

From the results given in Table 3, it is clear that the proposed model provide best fit to the data. Furthermore, for data 1, the estimated pdf and cdf are skected in Figure 3, while, the Kaplan-Meier survival and pp-plots are provided in Figure 4. These Figures show that the proposed model fit the data very closely.

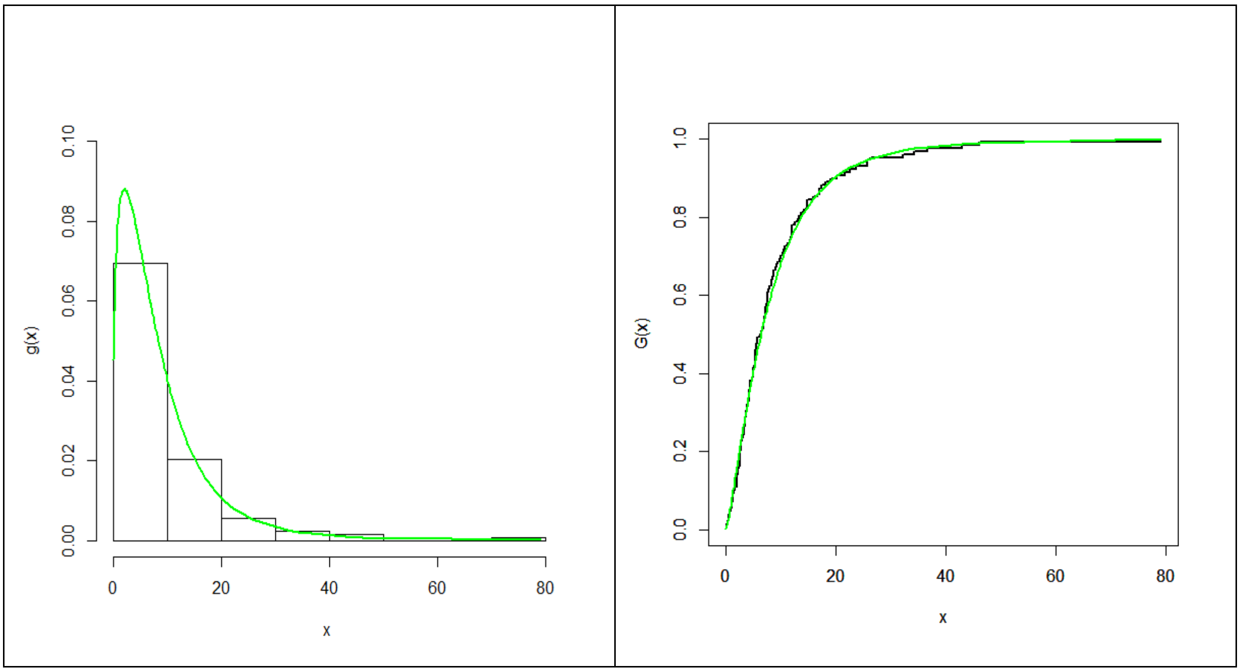

Figure 3. Plots of the estimated pdf and cdf of the Ex-APTW distribution for data 1.

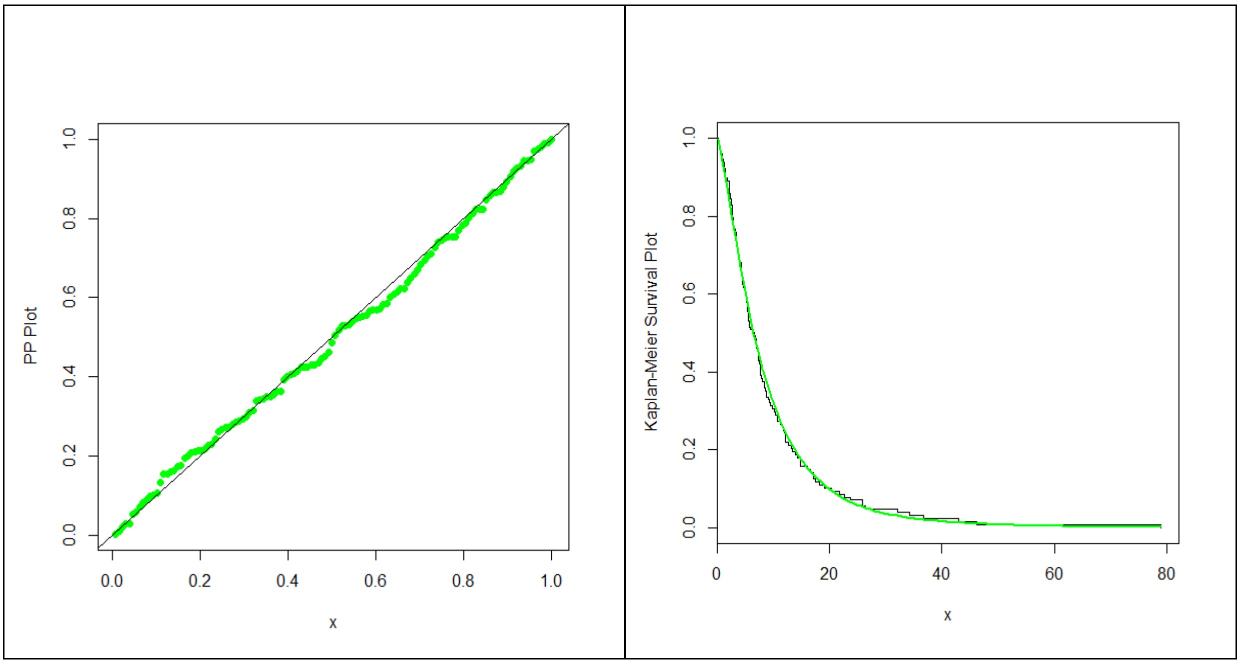

Figure 4. PP and Kaplan-Meir survival plots of the Ex-APTW distribution for data 1.

Data 2: The second data set representing the time between failures for 30 repairable items taken from Murthy et al. (2004). The data are given as: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17. The maximum likelihood estimates and the considered statistics are provided in Tables 4 and 5, respectively. Corresponding to data 2, the estimated pdf and cdf of the proposed model are ploted in Figure 5, while, the Kaplan-Meier survival and pp-plots are presented in Figure 6. These figures show that how the proposed model fit the data closely.

Table 4. Maximum likelihood estimates of the fitted distributions using data 2.

Dist.	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$
Ex-APTW	2.907	1.364	0.726	
MOW	0.336		1.807	0.200
EW	1.950		0.937	1.040
BW	1.810	0.433	1.120	1.758

Table 5. The statistics of the fitted models using data 2.

Dist.	KS	CM	AD	P-value
Ex-APTW	0.073	0.016	0.117	0.997
MOW	0.075	0.022	0.151	0.915
EW	0.083	0.027	0.165	0.899
BW	0.633	0.955	5.367	7.063e-11

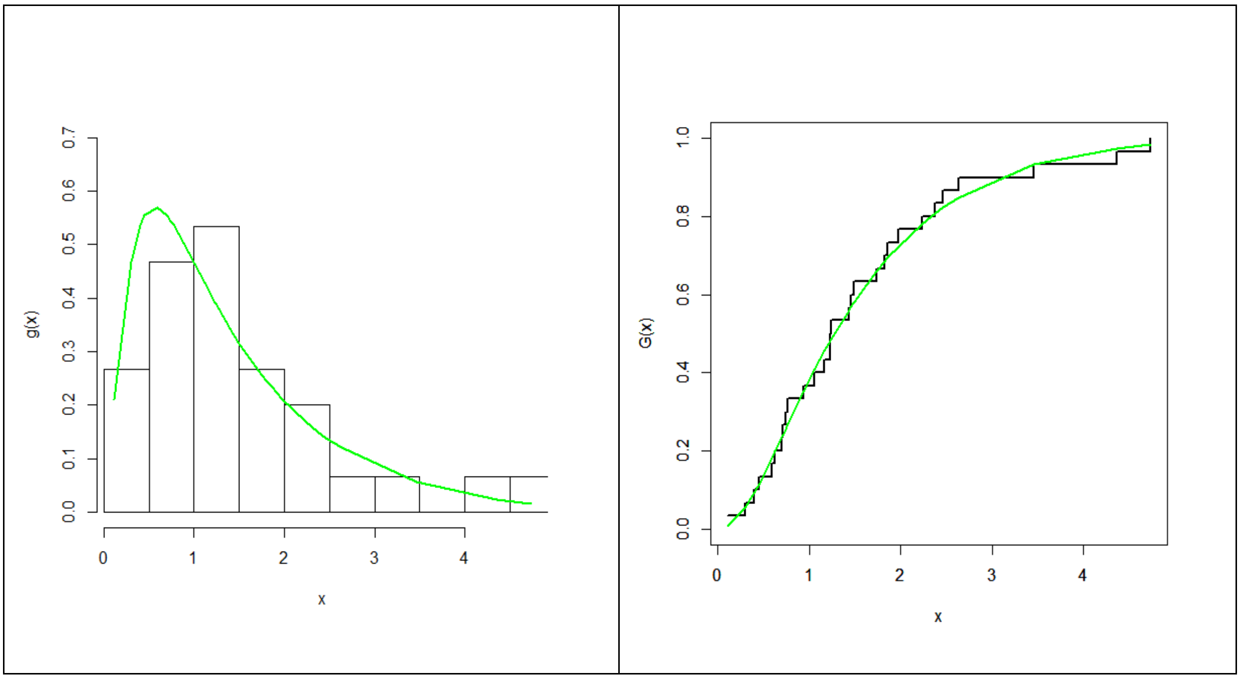

Figure 5. Plots of the estimated pdf and cdf of the Ex-APTW distribution corresponding to data 2.

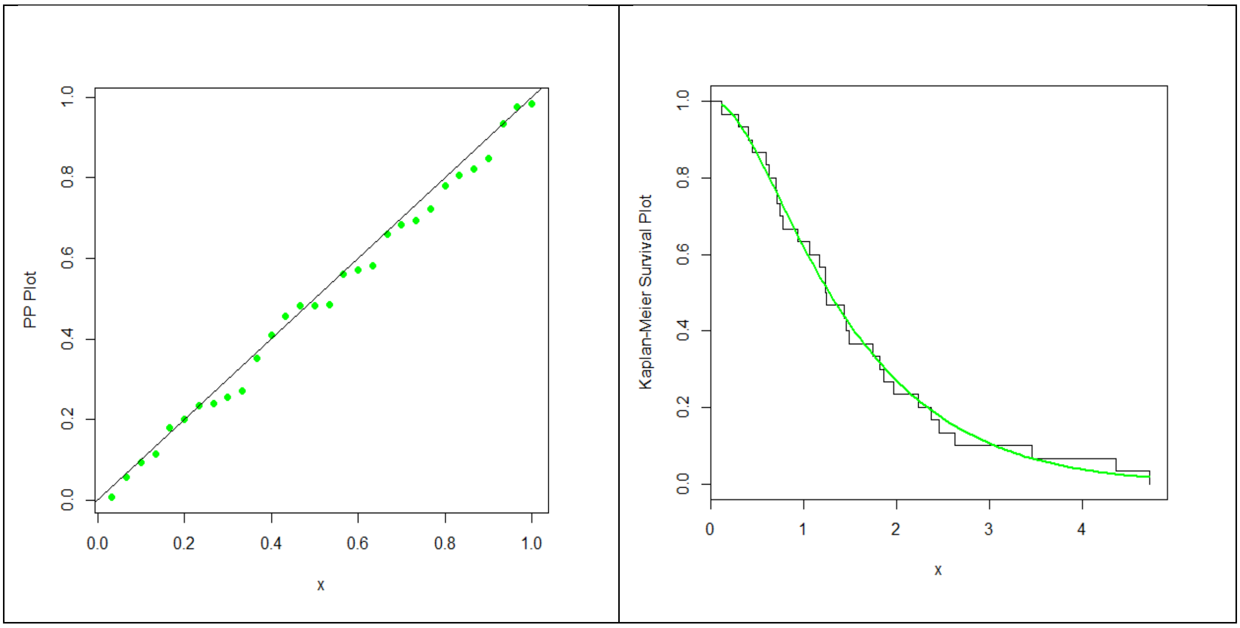

Figure 6. PP and Kaplan-Meir survival plots of the Ex-APTW distribution corresponding to data 2.

8.Conclusions

In this article, a new method is adopted to extend the existing distributions. This effort leads to a new family of lifetime distributions, called the extended alpha power transformed family of distributions. General expressions for some of the mathematical properties of the new family are investigated. Maximum likelihood estimates are also obtained. There are certain advantages of using the proposed method like its cdf has a closed form and facilitating data modeling with monotonic and non-monotonic failure rates. A special sub-model of the new family, called the extended alpha power transformed Weibull distribution is considered and two real applications are analyzed. In simulation study, the consistency and proficiency of the maximum likelihood estimators of the proposed model are also illustrated. The practical applications of the proposed model reveal better fit to real-life data than the other well-known competitors. It is hoped, that the proposed method will attract wider applications from reliability engineering and bio-medical analysis.

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Appendix A

Theorem 1. Let (Ω, F, P) be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $\left(d=-\infty,\ e=\infty \text{ might as well be allowed}\right)$. Let $X\colon \Omega \rightarrow H$ be a continuous

random variable with the distribution function $F(x)$ and let $h_1(x)$ and $h_2(x)$ be two real functions defined on H such that

$$E\left(h_2(X)/X \geq x\right) = E\left(h_1(X)/X \geq x\right)\eta(x), \qquad x \in H,$$

is defined with some real function $\eta(x)$. Assume that $h_1(x), h_2(x) \in C^1(H)$, $\eta(x) \in C^2(x)$ and $F(x)$ is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta(x)h_1(x) = h_2(x)$ has no real solution in the interior of H. Then $F(x)$ is uniquely determined by the functions $h_1(x)$, $h_2(x)$ and $\eta(x)$ particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\xi(u)q_1(u) - q_2(u)}\right| \exp\left(-s(u)\right)du,$$

where the function $s(u)$ is a solution of the differential equation $s'(u) = \frac{\eta'(u)h_1(u)}{\eta(u)h_1(u) - h_2(u)}$ and C is the normalization constant, such that $\int_H dF = 1$.