

## TYPE II GENERALIZED TOPP–LEONE FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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### ABSTRACT

The Topp-Leone distribution is an attractive model for life testing and reliability studies as it acquires a bathtub shaped hazard function. In this paper, we introduce a new family of distributions, depending on Topp–Leone random variable as a generator, called the *Type II generalized Topp–Leone–G* (TIIGTL-G) family. Its density function can be unimodal, left-skewed, right-skewed, and reversed-J shaped, and has increasing, decreasing, upside-down, J and reversed-J hazard rates. Some special models are presented. Some of its statistical properties are studied. Explicit expressions for the ordinary and incomplete moments, quantile and generating functions, Rényi entropy and order statistics are derived. The method of maximum likelihood is used to estimate the model parameters. The importance of one special model; namely; the Type II generalized Topp–Leone exponential is illustrated through two real data sets.

**Keywords:** Topp–Leone distribution; Order statistics; Rényi entropy; Maximum likelihood method.

## 1. Introduction

The *Topp-Leone* (TL) distribution was originally proposed by Topp and Leone (1955) as an alternative to beta distribution and it has been applied for some failure data. It is a one-parameter distribution with *cumulative distribution function* (cdf) and the *probability density function* (pdf) specified by

$$F(t; \alpha) = t^\alpha (2-t)^\alpha, \quad 0 \leq t \leq 1, \quad \alpha > 0, \quad (1)$$

and,

$$f(t; \alpha) = 2\alpha t^{\alpha-1} (1-t)(2-t)^{\alpha-1}. \quad (2)$$

In recent years, the TL distribution has received a huge attention in the literature; see for example; Nadarajah and Kotz (2003), Ghitany *et al.* (2005), Kotz and Nadarajah (2006), Kotz and Seier (2007), Vicari *et al.* (2008), Zghoul (2011), Genç (2012) and Bayoud (2016).

In the literature, several statistical distributions are very useful in describing and predicting real data in many areas such as economics, engineering and biological studies. However, in many situations, classical distributions do not provide adequate fits to real data. Thus, several generated classes of probability distributions have been suggested by introducing one or more parameters to generate new distributions. The generated families are generalized and extended most of the classical distributions. Some well-established generators and other recently are the following; the beta-generated (Eugene *et al.* (2002)), Kumaraswamy generalized (Cordeiro and de Castro (2011)), gamma-G (Ristic and Balakrishnan (2012)), Transformed-Transformer (Alzaatreh *et al.* (2013)), exponentiated generalized (Cordeiro *et al.* (2013)), Weibull-G (Bourguignon *et al.* (2014)), Kumaraswamy transmuted-G (Afify *et al.* (2016)), Type I half-logistic-G (Cordeiro *et al.* (2016)), Garhy-G (Elgarhy *et al.* (2016)), Kumaraswamy Weibull-G (Hassan and Elgarhy (2016a)), exponentiated Weibull-G (Hassan and Elgarhy (2016b)), additive Weibull-G (Hassan and Hemeda (2016)), Topp-Leone-G (Al-Shomrani *et al.* (2016)), exponentiated extended-G (Elgarhy *et al.* (2017)), Type II half logistic-G (Hassan *et al.* (2017a)), generalized additive Weibull-G (Hassan *et al.* (2017b)), alpha power transformed-G (Mahdavi and Kundu (2017)), transmuted Topp-Leone-G ( Yousof *et al.* (2017)), Kumaraswamy generalized Marshall-Olkin-G (Chakraborty and Handique (2018)), inverse Weibull-G (Hassan and Nassr (2018)) and power Lindley-G (Hassan and Nassr (2019)), among others.

Ristic and Balakrishnan (2012) proposed a gamma generator for any continuous distribution  $G(x)$  with the following cdf

$$F(x; \delta, \zeta) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log G(x; \zeta)} t^{\delta-1} e^{-t} dt. \quad (3)$$

Our objective here, is to introduce a new generated family of continuous distributions with two extra shape parameters based on cdf (3) by taking Topp-Leone distribution as a generator and the upper bound to be  $1 - G(x; \zeta)^\beta$ . We call the new generated family as Type II generalized Topp-Leone which has the cdf specified by

$$F_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = 1 - \int_0^{1-(G(x;\zeta))^\beta} 2\alpha t^{\alpha-1} (1-t)(2-t)^{\alpha-1} dt = 1 - (1-G(x;\zeta)^{2\beta})^\alpha, \quad (4)$$

where,  $\alpha, \beta$  are two shape parameters and  $G(x; \zeta)$  is a baseline cdf, which depends on a parameter vector  $\zeta$ . Therefore, the pdf of the TIIGTL-G family is as follows

$$f_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = 2\alpha\beta g(x; \zeta) [G(x; \zeta)]^{2\beta-1} (1-G(x; \zeta)^{2\beta})^{\alpha-1}. \quad (5)$$

The quantile function of TIIGTL distribution, say  $Q(u)$  of  $X$  which plays an important role in the algebraic developments, is given by inverting (4) as follows

$$Q(u) = G^{-1} \left[ 1 - (1-u)^\alpha \right]^{\frac{1}{2\beta}},$$

where,  $u$  is a uniform distribution on the interval  $(0,1)$  and we require the inverse of  $G(\cdot)$  to obtain the above quantile function of the TIIGTL-G distribution.

A random variable  $X$  has pdf (5) will be denoted by  $X \sim \text{TIIGTL-G}$ . The survival, hazard rate and reversed hazard rate functions are, respectively, given by

$$\bar{F}_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = (1-G(x; \zeta)^{2\beta})^\alpha,$$

$$h_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = 2\alpha\beta g(x; \zeta) [G(x; \zeta)]^{2\beta-1} (1-G(x; \zeta)^{2\beta})^{-1},$$

and,

$$\tau_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = \frac{2\alpha\beta g(x; \zeta) [G(x; \zeta)]^{2\beta-1} (1-G(x; \zeta)^{2\beta})^{\alpha-1}}{1 - (1-G(x; \zeta)^{2\beta})^\alpha}.$$

This paper can be organized as follows. In the next section, two important expansions of the TIIGTL-G for the pdf and cdf of the family are derived. Section 3 provides four new sub-models from the suggested family. Section 4 provides some general mathematical properties of the TIIGTL-G distribution. Section 5 gives the estimation of the model parameters using maximum likelihood method. Simulation study is carried out for one particular case from the family in Section 6. Applications to two real data sets are provided in Section 7. Section 8 ends with some concluding remarks.

## 2. Expansion for the Density function

For  $d$  is a positive real non integer and for  $|y| < 1$ , we have the generalized binomial series

$$(1-y)^d = \sum_{k=0}^{\infty} (-1)^k \binom{d}{k} y^k. \quad (6)$$

From the expansion (6) and pdf (5), we can write

$$f_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = \sum_{k=0}^{\infty} \eta_k g(x; \zeta) G(x; \zeta)^{2\beta(k+1)-1}, \quad (7)$$

where,

$$\eta_k = 2\alpha\beta(-1)^k \binom{\alpha-1}{k}.$$

Also, the pdf (7) can be expressed as follows

$$f_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta) = \sum_{k=0}^{\infty} W_k h_{2\beta(k+1)}(x), \quad (8)$$

where,  $W_k = \eta_k / 2\beta(k+1)$ , and  $h_a(x) = a g(x; \zeta) G(x; \zeta)^{a-1}$ . Equation (8) gives exponentiated-generated (exp-G) with power parameter  $2\beta(k+1)$ .

Also, for  $h$  is an integer, an expansion for the  $[F_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta)]^h$  is derived, the binomial expansion (6) is worked out as follows

$$[F_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta)]^h = \sum_{i=0}^h (-1)^i \binom{h}{i} (1 - G(x, \zeta)^{2\beta})^{\alpha i}. \quad (9)$$

Again, the binomial expansion (6) is applied, then (9) becomes

$$[F_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta)]^h = \sum_{i=0}^h \sum_{z=0}^{\infty} (-1)^{i+z} \binom{h}{i} \binom{\alpha i}{z} G(x; \zeta)^{2\beta z}. \quad (10)$$

Another simplification form for (10) is as follows

$$[F_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta)]^h = \sum_{z=0}^{\infty} s_z G(x; \zeta)^{2\beta z}, \quad (11)$$

where,

$$s_z = \sum_{i=0}^h (-1)^{i+z} \binom{h}{i} \binom{\alpha i}{z}.$$

Density functions (7) and (8) can be used to derive several statistical properties of the TIIGTL-G family.

## 3. Special Models of the TIIGTL-G Family

The TIIGTL-G family contains various new generated distributions as special sub-models. Some useful distributions in the TIIGTL-G family are listed in the following subsections.

### 3.1 TIIGTL-Uniform (TIIGTLU) Model

The case  $G(x; \zeta) = \frac{x}{\theta}$ , and  $g(x; \zeta) = \frac{1}{\theta}$ ,  $0 < x < \theta$ , where  $\zeta = (\theta)^T$  in pdf (5), corresponds to the TIIGTLU distribution. Its density function is given by

$$f_{\text{TIIGTLU}}(x; \alpha, \beta, \theta) = \frac{2\alpha\beta}{\theta} \left[ \frac{x}{\theta} \right]^{2\beta-1} \left( 1 - \left( \frac{x}{\theta} \right)^{2\beta} \right)^{\alpha-1}, \quad 0 < x < \theta.$$

The cdf and hazard rate function (hrf) of TIIGTLU distribution are given by

$$F_{\text{TIIGTLU}}(x; \alpha, \beta, \theta) = \left( 1 - \left( \frac{x}{\theta} \right)^{2\beta} \right)^{\alpha},$$

and,

$$h_{\text{TIIGTLU}}(x; \alpha, \beta, \theta) = \frac{2\alpha\beta x^{2\beta-1}}{\theta^{2\beta} - x^{2\beta}}.$$

### 3.2 TIIGTL-BurrXII (TIIGTLBXII) Model

The case  $G(x; \zeta) = 1 - [1 + (\frac{x}{\mu})^c]^{-\sigma}$ , and  $g(x; \zeta) = c\sigma\mu^{-c}x^{c-1}[1 + (\frac{x}{\mu})^c]^{-\sigma-1}$ , where  $c, \sigma, \mu > 0$ ,  $\zeta = (c, \mu, \sigma)^T$ , in pdf (5), corresponds to the TIIGTLBXII distribution. Its density function is given by

$$f_{\text{TIIGTLBXII}}(x; \alpha, \beta, c, \sigma, \mu) = 2\alpha\beta c \sigma \mu^{-c} x^{c-1} [1 + (\frac{x}{\mu})^c]^{-\sigma-1} \left[ 1 - [1 + (\frac{x}{\mu})^c]^{-\sigma} \right]^{2\beta-1} \left( 1 - \left( 1 - [1 + (\frac{x}{\mu})^c]^{-\sigma} \right)^{2\beta} \right)^{\alpha-1}.$$

Furthermore, the cdf and hrf of TIIGTLBXII distribution are given, respectively, as follows

$$F_{\text{TIIGTLBXII}}(x; \alpha, \beta, c, \sigma, \mu) = 1 - \left( 1 - \left( 1 - [1 + (\frac{x}{\mu})^c]^{-\sigma} \right)^{2\beta} \right)^{\alpha},$$

and,

$$h_{\text{TIIGTLBXII}}(x; \alpha, \beta, c, \sigma, \mu) = \frac{2\alpha\beta c \sigma \mu^{-c} x^{c-1} [1 + (\frac{x}{\mu})^c]^{-\sigma-1} \left[ 1 - [1 + (\frac{x}{\mu})^c]^{-\sigma} \right]^{2\beta-1}}{1 - \left( 1 - [1 + (\frac{x}{\mu})^c]^{-\sigma} \right)^{2\beta}}.$$

### 3.3 TIIGTL-Exponential (TIIGTLE) Model

The case  $G(x; \zeta) = 1 - e^{-\delta x}$ , and  $g(x; \zeta) = \delta e^{-\delta x}$ , where  $\zeta = (\delta)^T$  in pdf (5), corresponds to the TIIGTLE distribution. Its density function is given by

$$f_{\text{TIIGTLE}}(x; \alpha, \beta, \delta) = 2\alpha\beta\delta e^{-\delta x} [1 - e^{-\delta x}]^{2\beta-1} \left( 1 - (1 - e^{-\delta x})^{2\beta} \right)^{\alpha-1}.$$

Further, the cdf and hrf of TIIGTLE are given by

$$F_{\text{TIIGTLE}}(x; \alpha, \beta, \delta) = 1 - \left( 1 - (1 - e^{-\delta x})^{2\beta} \right)^{\alpha},$$

and

$$h_{\text{TIIGTLE}}(x; \alpha, \beta, \delta) = \frac{2\alpha\beta\delta e^{-\delta x} [1 - e^{-\delta x}]^{2\beta-1}}{1 - (1 - e^{-\delta x})^{2\beta}}.$$

### 3.4 TIIGTL - Rayleigh (TIIGTLR) Model

The case  $G(x; \delta) = 1 - e^{-\delta x^2}$ , and  $g(x; \zeta) = 2\delta x e^{-\delta x^2}$ , where  $\zeta = (\delta)^T$  in pdf (5), corresponds to the TIIGTLR distribution. Its density function is given by

$$f_{\text{TIIGTLR}}(x; \alpha, \beta, \delta) = 4\alpha\beta\delta x e^{-\delta x^2} [1 - e^{-\delta x^2}]^{2\beta-1} \left(1 - (1 - e^{-\delta x^2})^{2\beta}\right)^{\alpha-1}.$$

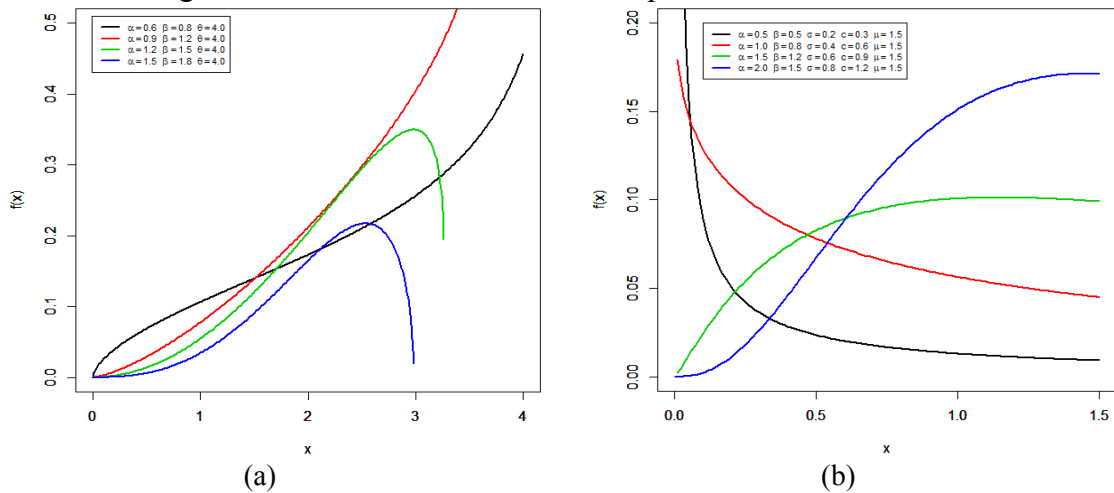
Furthermore, the cdf and hrf of TIIGTLR are given, respectively, as follows

$$F_{\text{TIIGTLE}}(x; \alpha, \beta, \delta) = 1 - \left(1 - (1 - e^{-\delta x^2})^{2\beta}\right)^{\alpha},$$

and,

$$h_{\text{TIIGTLE}}(x; \alpha, \beta, \delta) = \frac{4\alpha\beta\delta x e^{-\delta x^2} [1 - e^{-\delta x^2}]^{2\beta-1}}{1 - (1 - e^{-\delta x^2})^{2\beta}}.$$

Figures 1 and 2 display plots of the pdf and hrf of the TIIGTLU, TIIGTLBXII, TIIGTLE and TIIGTLR distributions for selected parameters values. Figure 1 displays special densities with various shapes such as symmetric, unimodal, left-skewed, right-skewed and reversed J-shaped. Figure 2 displays flexible increasing, decreasing, upside down, J and reversed J hazard rate shapes. So, the TIIGTL-G family can be very useful in fitting different data sets with various shapes.



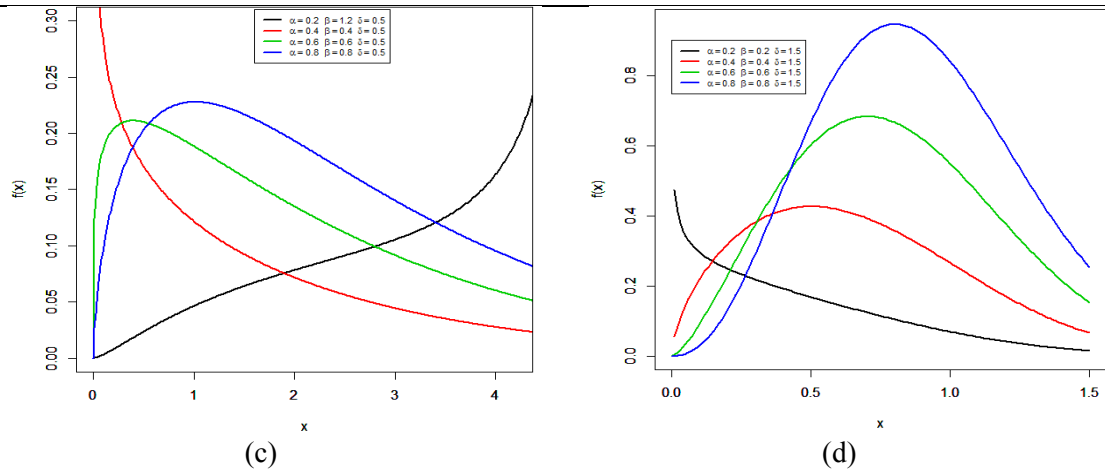


Figure 1: Plots of the TIIGTL -G family density: (a) TIIGTLU, (b) TIIGTLBXII, (c) TIIGTLE, (d) TIIGTLR

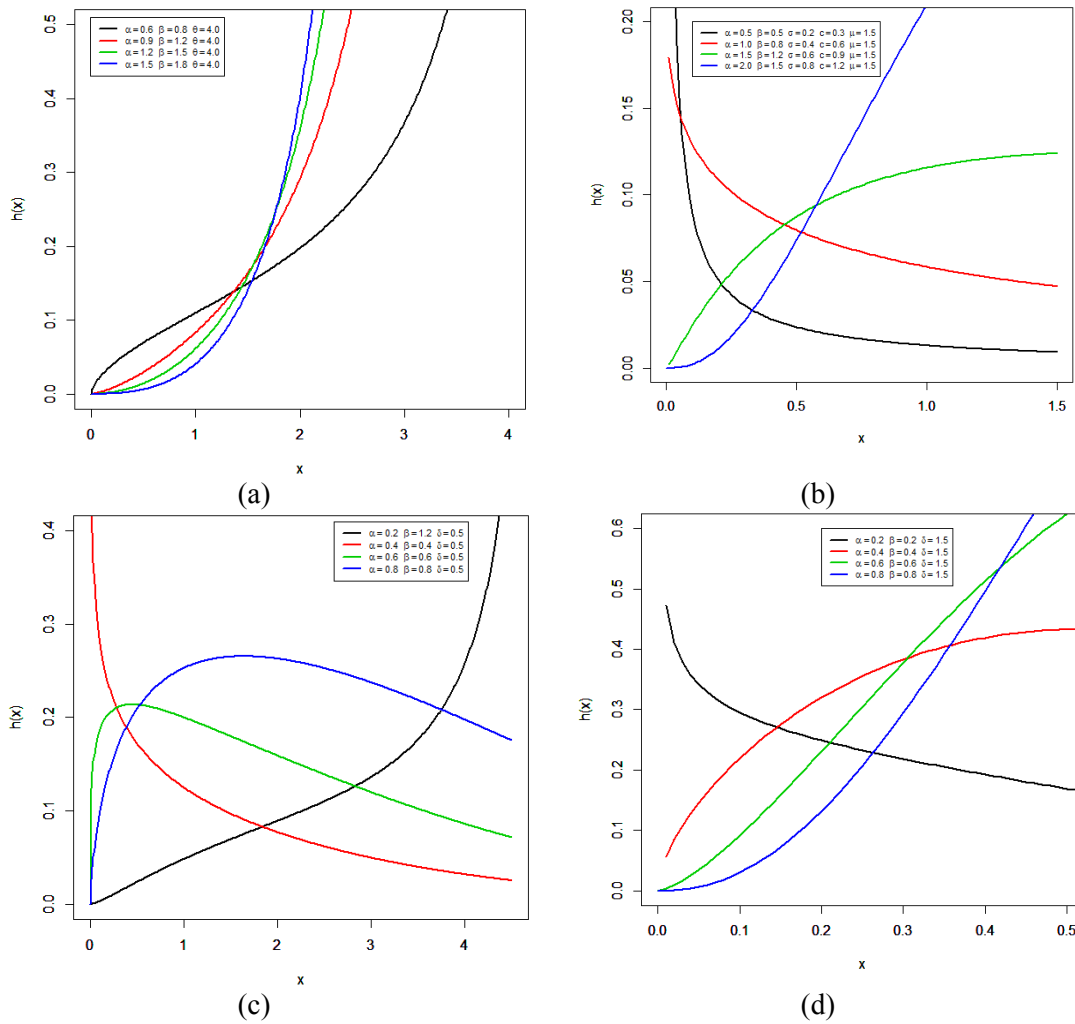


Figure 2: Plots of the TIIGTL -G family hazard: (a) TIIGTLU, (b) TIIGTLBXII, (c) TIIGTLE, (d) TIIGTLR

## 4. Main Properties

In this section, we obtain the probability weighted moments, ordinary moments, incomplete moments, Rényi Entropy and the pdf of the  $s^{\text{th}}$  order statistics.

### 4.1 The probability Weighted Moments

*Probability weighted moments* (PWMs) of  $X$  cover the summarization and description of theoretical probability distributions. The elementary use of these moments is in the estimation of parameters for a distribution whose inverse form cannot be expressed explicitly. For a random variable  $X$ , the PWM denoted by  $\tau_{r,h}$ , is defined by

$$\tau_{r,h} = E[X^r F(x)^h] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^h dx. \quad (12)$$

Inserting (7) and (11) into (12), the PWM of TIIGTL-G family is obtained as follows

$$\tau_{r,h} = \int_{-\infty}^{\infty} \sum_{k,z=0}^{\infty} s_z \eta_k x^r g(x;\zeta)(G(x;\zeta))^{2\beta(k+z+1)-1} dx.$$

Then,

$$\tau_{r,h} = \sum_{k,z=0}^{\infty} s_z \eta_k \tau_{r,2\beta(k+z+1)-1},$$

$$\text{where, } \tau_{r,2\beta(k+z+1)-1} = \int_{-\infty}^{\infty} x^r g(x;\zeta)(G(x;\zeta))^{2\beta(k+z+1)-1} dx.$$

### 4.2 Moments

We hardly need to confirm the necessity and importance of moments in any statistical analysis especially in applied works. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

The  $r^{\text{th}}$  ordinary moment of  $X$  follows from (7) as follows

$$\mu'_r = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \eta_k x^r g(x;\zeta) G(x;\zeta)^{2\beta(k+1)-1} dx.$$

Another formula of  $r^{\text{th}}$  moment of  $X$  is given by

$$\mu'_r = \sum_{k=0}^{\infty} \eta_k \tau_{r,2\beta(k+1)-1},$$

where,  $\tau_{r,2\beta(k+1)-1}$  is the PWMs. Furthermore, the moment generating function of  $X$  can be expressed as



$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r,k=0}^{\infty} \frac{t^r}{r!} \eta_k \tau_{r,2\beta(k+1)-1}$$

Additionally, the  $n^{\text{th}}$  incomplete moment of a distribution play an important role in applications. The  $n^{\text{th}}$  incomplete moment of  $X$  is given by using (7), as follows

$$K_n(y) = \int_{-\infty}^y \sum_{k=0}^{\infty} \eta_k x^n g(x; \zeta) G(x; \zeta)^{2\beta(k+1)-1} dx.$$

### 4.3 Rényi Entropy

An entropy is a concept encountered in physics and engineering. It is a measure of variation or uncertainty of a random variable  $X$ . The Rényi entropy of  $X$  with pdf  $f(x)$  is defined by

$$I_{\varepsilon}(X) = \frac{1}{1-\varepsilon} \log \int_{-\infty}^{\infty} f(x)^{\varepsilon} dx, \quad \varepsilon > 0 \text{ and } \varepsilon \neq 1.$$

Now, we consider the generalized binomial theory in the pdf (5), then the pdf  $f_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta)^{\varepsilon}$  can be expressed as follows

$$f_{\text{TIIGTL-G}}(x; \alpha, \beta, \zeta)^{\varepsilon} = \sum_{k=0}^{\infty} t_k g(x; \zeta)^{\varepsilon} G(x; \zeta)^{2\beta(k+\varepsilon)-\varepsilon},$$

where,

$$t_k = (2\alpha\beta)^{\varepsilon} (-1)^k \binom{\varepsilon(\alpha-1)}{k}.$$

Therefore, the Rényi entropy of TIIGTL-G family is given by

$$I_{\varepsilon}(X) = \frac{1}{1-\varepsilon} \log \sum_{k=0}^{\infty} t_k \int_{-\infty}^{\infty} g(x; \zeta)^{\varepsilon} G(x; \zeta)^{2\beta(k+\varepsilon)-\varepsilon} dx.$$

### 4.4 Order Statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables with their corresponding continuous distribution function  $F(\cdot)$ . Let  $X_{1:n}, X_{2:n}, \dots, X_{n:n}$  be the corresponding ordered random sample from a population of size  $n$ . The pdf of the  $s^{\text{th}}$  order statistic, is defined as

$$f_{s:n}(x) = \frac{f(x)}{B(s, n-s+1)} \sum_{v=0}^{n-s} (-1)^v \binom{n-s}{v} F(x)^{v+s-1}, \quad (13)$$

where,  $B(\cdot, \cdot)$  stands for beta function. Inserting (7) and (11) in (13) replacing  $h$  with  $v+s-1$ , then the pdf of the  $s^{\text{th}}$  order statistic for TIIGTL-G distributions is derived by

$$f_{s:n}(x; \alpha, \beta, \zeta) = \frac{g(x; \zeta)}{B(s, n-s+1)} \sum_{v=0}^{n-s} \sum_{z=0}^{\infty} \eta_k P_{z,v} G(x; \zeta)^{2\beta(k+z+1)-1}, \tag{14}$$

where,

$$p_{z,v} = (-1)^v \binom{n-s}{v} s_z.$$

Further, the  $r$ th moment of  $s$ th order statistics for TIIGTL-G family is defined by:

$$E(X_{s:n}^r) = \int_{-\infty}^{\infty} x^r f_{s:n}(x; \alpha, \beta, \zeta) dx. \tag{15}$$

Inserting (14) in (15), leads to

$$E(X_{s:n}^r) = \frac{1}{B(s, n-s+1)} \sum_{v=0}^{n-s} \sum_{z=0}^{\infty} \eta_k P_{z,v} \int_{-\infty}^{\infty} x^r g(x; \zeta) G(x; \zeta)^{2\beta(k+z+1)-1} dx.$$

Then,

$$E(X_{s:n}^r) = \frac{1}{B(s, n-s+1)} \sum_{v=0}^{n-s} \sum_{z=0}^{\infty} \eta_k P_{z,v} \tau_{r, 2\beta(k+z+1)-1}.$$

### 5. Maximum Likelihood Estimation

We consider the estimation of the unknown parameters of TIIGTL-G family from complete samples only by the method of maximum likelihood. Let  $X_1, X_2, \dots, X_n$  be the observed values from the TIIGTL-G family with set of parameters  $\Phi = (\alpha, \beta, \zeta)^T$ .

The log-likelihood function of  $\Phi$  is given by

$$\begin{aligned} \ln L(\Phi) = & n \ln 2\alpha + n \ln \beta + \sum_{i=1}^n \ln [g(x_i; \zeta)] + (2\beta - 1) \sum_{i=1}^n \ln [G(x_i; \zeta)] \\ & + (\alpha - 1) \sum_{i=1}^n \ln [1 - (G(x_i; \zeta))^{2\beta}]. \end{aligned}$$

The elements of the score function  $U(\Phi) = (U_\alpha, U_\beta, U_{\zeta_k})$  are given by

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^n \ln [1 - (G(x_i; \zeta))^{2\beta}],$$

$$U_\beta = \frac{n}{\beta} + 2 \sum_{i=1}^n \ln [G(x_i; \zeta)] - 2(\alpha - 1) \sum_{i=1}^n \frac{(G(x_i; \zeta))^{2\beta} \ln(G(x_i; \zeta))}{1 - (G(x_i; \zeta))^{2\beta}},$$

and,

$$\begin{aligned} U_{\zeta_k} = & \sum_{i=1}^n \frac{\partial g(x_i; \zeta) / \partial \zeta_k}{g(x_i; \zeta)} + (2\beta - 1) \sum_{i=1}^n \frac{\partial G(x_i; \zeta) / \partial \zeta_k}{G(x_i; \zeta)} \\ & - 2\beta(\alpha - 1) \sum_{i=1}^n \left[ \frac{(G(x_i; \zeta))^{2\beta-1} \partial G(x_i; \zeta) / \partial \zeta_k}{1 - (G(x_i; \zeta))^{2\beta}} \right]. \end{aligned}$$

Setting  $U_\alpha, U_\beta$  and  $U_{\zeta_k}$  equal to zero and solving these equations simultaneously yield the maximum likelihood estimators  $\hat{\Phi} = (\hat{\alpha}, \hat{\beta}, \hat{\zeta})$  of  $\Phi = (\alpha, \beta, \zeta)^T$ . These

equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

### 6. Simulation Study

In this section, we assess the performance of the maximum likelihood estimators in terms of the sample size  $n$ . A numerical evaluation is carried out to examine the performance of maximum likelihood estimators for TIIGTLE model (as particular case from the family). The evaluation of estimates is performed based on the following quantities for each sample size; the biases and the empirical *mean square errors* (MSEs) using the Mathematica (9) package. The numerical steps are listed as follows:

**Step (1):** A random sample  $X_1, \dots, X_n$  of sizes;  $n = 50, 75, 100, 200$  and  $300$  are considered, these random samples are generated from the TIIGTLE distribution by using inversion method.

**Step (2):** Six sets of the parameters are considered as; Set1( $\alpha = 0.5, \beta = 0.5, \delta = 0.5$ ), Set2( $\alpha = 0.75, \beta = 0.5, \delta = 1$ ), Set3( $\alpha = 1.25, \beta = 0.5, \delta = 1.5$ ), Set4( $\alpha = 0.5, \beta = 0.75, \delta = 1.5$ ), Set5( $\alpha = 0.5, \beta = 1.25, \delta = 2$ ), and Set6( $\alpha = 0.5, \beta = 1.5, \delta = 2$ ). Then, the *maximum Likelihood estimates* (MLEs) of TIIGTLE model are evaluated for each parameters value and for each sample size.

**Step (3):** Repeat this process 10000 times and then obtain the means, biases and MSEs of the MLEs for different values of parameters and at each sample size. Empirical results are reported in Tables 1 to 3. We can detect from these tables that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase.

Table 1: Simulation Results: MLE, Bias and MSE

$n$	Parameter	Set1( $\alpha = 0.5, \beta = 0.5, \delta = 0.5$ )			Set2( $\alpha = 0.75, \beta = 0.5, \delta = 1$ )		
		MLE	Bias	MSE	MLE	Bias	MSE
50	$\alpha$	0.5112	0.0112	0.0058	0.7650	0.0150	0.0124
	$\beta$	0.5051	0.0051	0.0027	0.5078	0.0078	0.0041
	$\delta$	0.5112	0.0112	0.0058	1.0199	0.0199	0.0220
75	$\alpha$	0.5069	0.0069	0.0035	0.7600	0.0100	0.0081
	$\beta$	0.5036	0.0036	0.0017	0.5060	0.0060	0.0027
	$\delta$	0.5069	0.0069	0.0035	1.0133	0.0133	0.0144
100	$\alpha$	0.5057	0.0057	0.0027	0.7574	0.0074	0.0061
	$\beta$	0.5031	0.0031	0.0013	0.5042	0.0042	0.0021
	$\delta$	0.5057	0.0057	0.0027	1.0099	0.0099	0.0108

200	$\beta$	0.5023	0.0023	0.0013	0.7545	0.0045	0.0029
	$\delta$	0.5019	0.0019	0.0007	0.5022	0.0022	0.0010
	$\alpha$	0.5023	0.0023	0.0013	1.0060	0.0060	0.0051
300	$\beta$	0.5017	0.0017	0.0008	0.7532	0.0032	0.0019
	$\delta$	0.5009	0.0009	0.0004	0.5010	0.0010	0.0006
	$\beta$	0.5017	0.0017	0.0008	1.0042	0.0042	0.0034

Table 2: Simulation Results: MLE, Bias and MSE

$n$	Parameter	Set3( $\alpha=1.25, \beta=0.5, \delta=1.5$ )			Set4( $\alpha=0.5, \beta=0.75, \delta=1.5$ )		
		MLE	Bias	MSE	MLE	Bias	MSE
50	$\alpha$	1.2666	0.0166	0.0339	0.5049	0.0049	0.0052
	$\beta$	0.5104	0.0104	0.0065	0.7574	0.0074	0.0056
	$\delta$	1.5200	0.0200	0.0489	1.5220	0.0220	0.0625
75	$\alpha$	1.2693	0.0193	0.0228	0.5072	0.0072	0.0036
	$\beta$	0.5064	0.0064	0.0043	0.7559	0.0059	0.0041
	$\delta$	1.5231	0.0231	0.0328	1.5280	0.0280	0.0429
100	$\alpha$	1.2653	0.0153	0.0170	0.5032	0.0032	0.0026
	$\beta$	0.5059	0.0059	0.0034	0.7553	0.0053	0.0028
	$\delta$	1.5184	0.0184	0.0245	1.5134	0.0134	0.0308
200	$\beta$	1.2589	0.0089	0.0077	0.5028	0.0028	0.0014
	$\delta$	0.5031	0.0031	0.0015	0.7516	0.0016	0.0015
	$\alpha$	1.5106	0.0106	0.0111	1.5109	0.0109	0.0163
300	$\beta$	1.2547	0.0047	0.0057	0.5015	0.0015	0.0009
	$\delta$	0.5035	0.0035	0.0011	0.7514	0.0014	0.0010
	$\beta$	1.5056	0.0056	0.0082	1.5058	0.0057	0.0098

Table 3: Simulation Results: MLE, Bias and MSE

$n$	Parameter	Set5( $\alpha=0.5, \beta=1.25, \delta=2$ )			Set6( $\alpha=0.5, \beta=1.5, \delta=2$ )		
		MLE	Bias	MSE	MLE	Bias	MSE
50	$\alpha$	0.5142	0.0142	0.0057	0.5077	0.0077	0.0051
	$\beta$	1.2588	0.0088	0.0168	1.5155	0.0155	0.0244

	$\delta$	2.1026	0.1026	0.2099	2.0745	0.0745	0.2004
75	$\alpha$	0.5052	0.0052	0.0033	0.5086	0.0086	0.0036
	$\beta$	1.2623	0.0123	0.0108	1.4996	-0.0004	0.0153
	$\delta$	2.0390	0.0390	0.1048	2.0724	0.0724	0.1372
100	$\alpha$	0.5027	0.0027	0.0027	0.5062	0.0062	0.0026
	$\beta$	1.2623	0.0123	0.0090	1.5074	0.0074	0.0114
	$\delta$	2.0233	0.0233	0.0827	2.0460	0.0460	0.0923
200	$\beta$	0.5028	0.0028	0.0013	0.5034	0.0034	0.0013
	$\delta$	1.2546	0.0046	0.0043	1.5025	0.0025	0.0060
	$\alpha$	2.0178	0.0178	0.0376	2.0255	0.0254	0.0440
300	$\beta$	0.5010	0.0010	0.0008	0.5031	0.0031	0.0008
	$\delta$	1.2519	0.0019	0.0026	1.5023	0.0023	0.0040
	$\beta$	2.0085	0.0085	0.0232	2.0200	0.0200	0.0258

## 7. Applications

In this section, we analyze two data sets on account to illustrate how the proposed model works well. The first data are taken from Badar and Priest (1982) which represent the strength of a single carbon fibers. The second data are taken from Murthy *et al.* (2004) which represent the time between failures of 30 repairable items. The two data sets are listed in Table 4 as follows

Table 4: Carbon fibers and failure data sets

<b>Data 1 Carbon fibers data</b>	1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020
<b>Data 2 Failure data</b>	1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17

We fit the above data sets using the proposed model along with the other well-known lifetime distributions namely; *Kumaraswamy Weibull* (Ku-W), *transmuted Weibull* (TW), *exponential Lomax* (EL), *exponential flexible Weibull extension* (EFWEx) and *alpha power transformed Weibull* (APTW) distributions. The pdfs of the competing models are

- Kumaraswamy Weibull (Cordeiro *et al.* (2010))

$$g(x; a, b, \beta, \theta) = \frac{ab\beta\theta x^{\beta-1} e^{-\theta x^\beta} \left(-e^{-\theta x^\beta}\right)^{a-1} \left(1 - \left(1 - e^{-\theta x^\beta}\right)^a\right)^{b-1}}{\beta}; \quad a, b, \beta, \theta > 0.$$

- Transmuted Weibull (Aryal and Tsokos (2011))

$$g(x; \beta, \theta, \lambda) = \beta\theta x^{\beta-1} e^{-\theta x^\beta} \left\{1 - \lambda + 2\lambda e^{-\theta x^\beta}\right\}; \quad \beta, \theta > 0, \quad |\lambda| < 1.$$

- Exponential Lomax (El-Bassiouny *et al.* (2015))

$$g(x; \alpha, \beta, \theta) = \frac{\alpha\theta}{\beta} \left(\frac{\beta}{x + \beta}\right)^{-\alpha+1} e^{-\theta\left(\frac{\beta}{x + \beta}\right)^\alpha}; \quad x \geq -\beta, \alpha, \beta, \theta > 0.$$

- Exponential flexible Weibull extension (El-Desouky *et al.* (2016))

$$g(x; \alpha, \beta, \theta) = \theta \left(\alpha + \frac{\beta}{x^2}\right) e^{\left(\alpha x - \frac{\beta}{x}\right)} e^{e^{\left(\alpha x - \frac{\beta}{x}\right)}} \exp\left[-\theta e^{e^{\left(\alpha x - \frac{\beta}{x}\right)}}\right]; \quad x, \alpha, \beta, \theta > 0.$$

- Alpha power transformed Weibull (Nassr *et al.* (2017))

$$g(x; \alpha, \beta, \theta) = \frac{\log \alpha}{\alpha - 1} \beta\theta x^{\beta-1} e^{-\theta x^\beta} \alpha^{\left(1 - e^{-\theta x^\beta}\right)} \quad \alpha, \beta, \theta > 0, \quad \alpha \neq 1.$$

For deciding about the flexibility among these models, we consider the analytical measures including *Kolmogorov–Smirnov* (KS) statistic, *Anderson Darling* (AD) statistic, *Cramer-von-Misses* (CM) statistic, *log-likelihood* (LL), *Akaike Information Criterion* (AIC), *Bayesian information criterion* (BIC), *corrected Akaike information criterion* (CAIC) and *Hannan-Quinn information criterion* (HQIC). Tables 5 and 7 give the MLEs and their corresponding standard errors (in parentheses) of the model parameters The analytical results provided in Tables 6 and 8 show that the proposed model works much better than the other fitted distributions.

Table 5: MLEs of the fitted distributions for carbon fibers data

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{a}$	$\hat{b}$
<b>TIIGTLE</b>	<b>3.664</b> <b>(0.9120)</b>	<b>1.217</b> <b>(0.1263)</b>			<b>0.849</b> <b>(0.3912)</b>		
Ku-W		1.807 (0.4565)	0.206 (0.1565)			6.606 (3.6118)	3.077 (1.0118)
TW		4.679 (0.4702)	0.703 (0.5985)	0.006 (0.1985)			
APTW	0.780 (2.5746)	3.843 (0.2334)	0.074 (0.0974)				
EL	0.304 (3.450)	0.518 (1.927)	1.026 (1.4303)				
EWFE <sub>x</sub>	0.718 (0.3219)	0.359 (0.5706)	0.060 (0.1706)				

Table 6: The analytical measures of the fitted models for carbon fibers data

Model	KS	AD	CM	LL	AIC	BIC	CAIC	HQIC
<b>TIIGTLE</b>	<b>0.091</b>	<b>0.362</b>	<b>0.060</b>	<b>57.10</b>	<b>120.20</b>	<b>126.63</b>	<b>120.61</b>	<b>122.73</b>
Ku-W	0.087	0.443	0.067	57.68	123.37	131.94	124.06	126.74
TW	0.095	0.450	0.074	62.29	130.59	137.02	131.00	133.12

APTW	0.104	0.9746	0.141	62.39	130.78	137.21	131.18	133.30
EL	0.140	1.222	0.179	63.24	135.98	139.02	134.89	136.78
EFWEx	0.229	1.316	0.526	65.79	139.59	140.02	137.99	139.11

Figure 3 displays the fitted pdfs and cdfs of the proposed and other competing distributions for carbon fibers data. Further, Figure 4 gives the PP-plots of the fitted models for carbon fibers data.

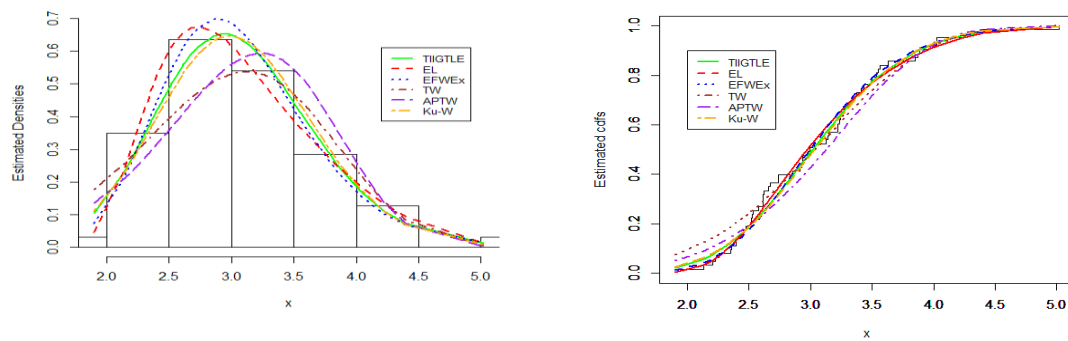


Figure 3: Estimated densities and cdfs of the fitted models for carbon fibers data

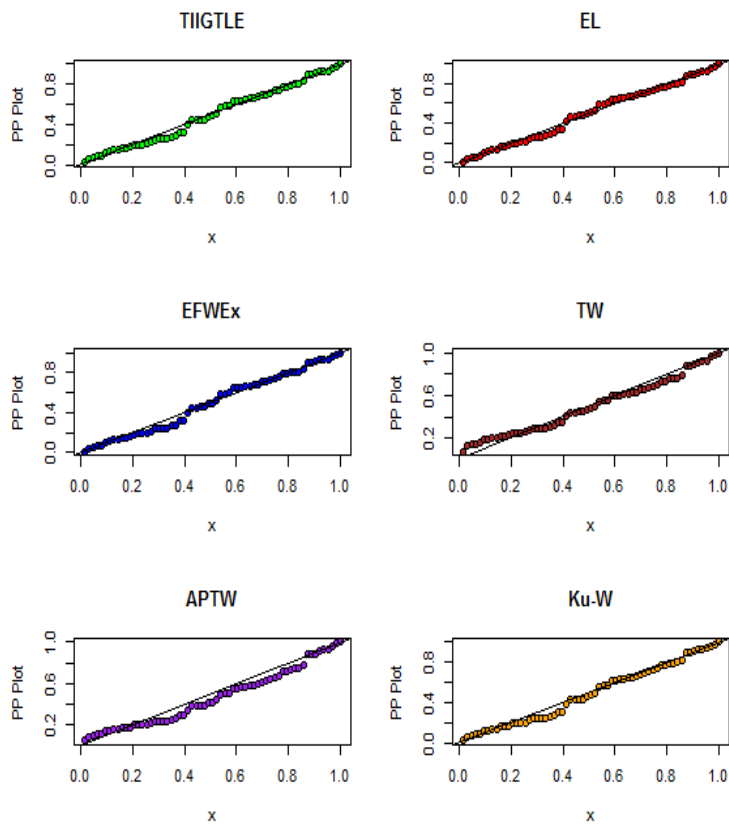


Figure 4: PP plots of the fitted models for carbon fibers data

Table 7: MLEs of the fitted distributions for failure data

Mode l	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{a}$	$\hat{b}$
TIIG	<b>0.88</b>	<b>1.06</b>			<b>1.08</b>		
TLE	<b>2</b> <b>(0.87</b> <b>03)</b>	<b>5</b> <b>(0.37</b> <b>81)</b>			<b>9</b> <b>(0.91</b> <b>33)</b>		
Ku- W		1.07 7 (1.32 29)	0.91 7 (0.78 79)			1.93 4 (2.14 05)	1.30 8 (3.82 70)
TW		1.09 7 (0.93 02)	0.90 9 (1.34 90)	0.90 3 (1.09 89)			
APT W	0.97 0 (0.90 39)	1.23 8 (0.41 45)	1.69 3 (0.31 19)				
EL	1.09 4 (1.90 76)	0.81 8 (1.01 97)	0.90 6 (0.90 83)				
EWf Ex	0.90 9 (0.92 19)	1.05 9 (1.90 06)	0.89 0 (1.68 96)				

Table 8: The analytical measures of the fitted models for failure data

Model	KS	A D	C M	LL	AI C	BI C	CA IC	HQI C
<b>TIIGT LE</b>	<b>0.0</b> <b>63</b>	<b>0.1</b> <b>32</b>	<b>0.0</b> <b>17</b>	<b>39.</b> <b>62</b>	<b>85.</b> <b>24</b>	<b>89.</b> <b>44</b>	<b>86.</b> <b>16</b>	<b>86.5</b> <b>90</b>
Ku-W	0.0 67	0.1 37	0.0 19	42. 62	87. 24	92. 84	88. 84	89.0 3
TW	0.0 85	0.1 49	0.0 21	47. 72	91. 96	95. 09	90. 19	91.9 8
APT W	0.0 65	0.1 35	0.0 18	41. 58	86. 16	90. 36	87. 08	86.5 0
EL	0.0 89	0.1 50	0.0 23	48. 09	94. 91	98. 92	93. 98	93.9 1
EWf x	0.0 93	0.1 52	0.0 25	49. 79	95. 06	99. 32	94. 06	94.1 2



Based on data set 2; Figure 5 gives the fitted pdfs and cdfs of the TIIGTLE model and the other competing distributions. Also, Figure 6 sketches the PP-plots of the fitted models for the same data.

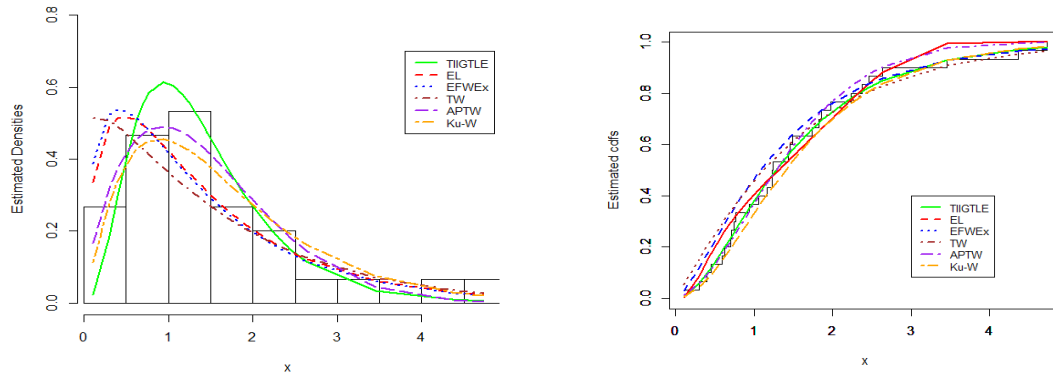


Figure 5: Estimated densities and cdfs of the models for failure data

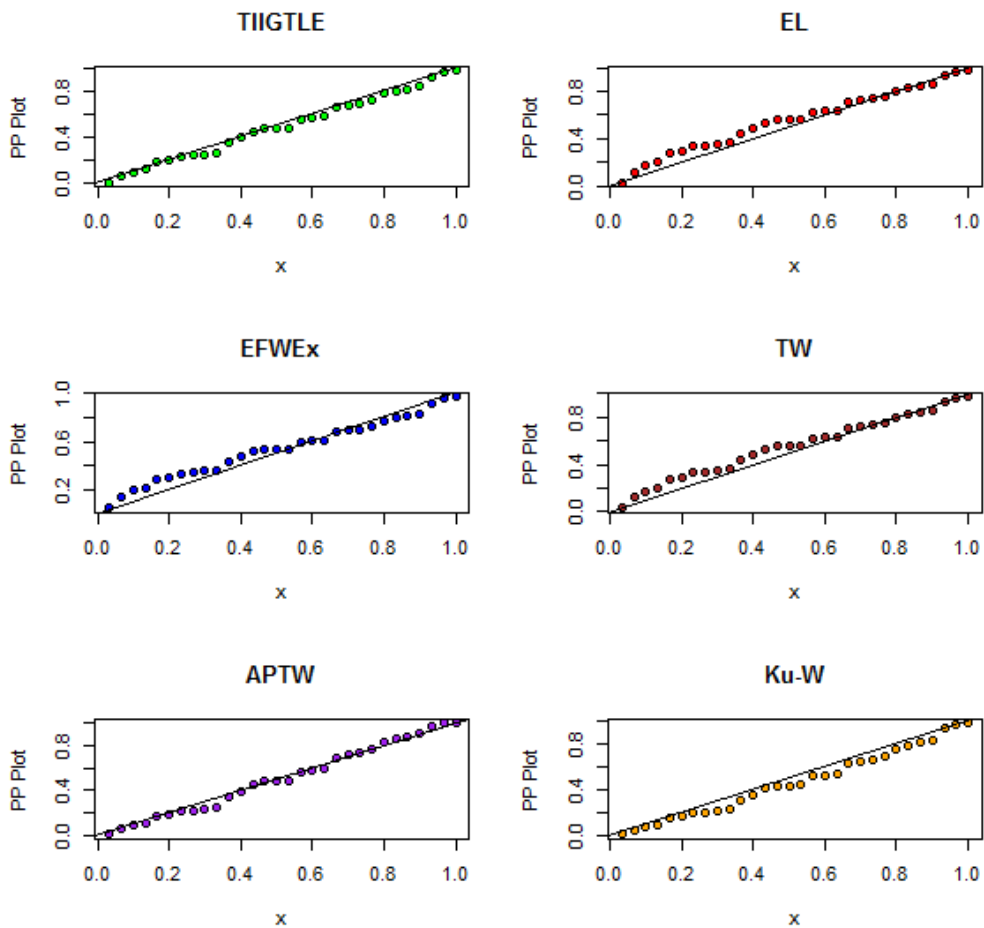


Figure 6: PP plots of the fitted models for failure data.

Overall, the plots in Figures 3, 4, 5 and 6 reveal that this model gives a close fit to both data sets. So, the TIIGTLE distribution could be chosen as the best model to fit both data sets.

## **8. Concluding Remarks**

In this paper we propose and study a new class of distributions called the Type II generalized Topp–Leone-G family. Some special distributions are presented. We investigate several of its structural properties such as an expansion for the density function and explicit expressions for the quantile function, ordinary and incomplete moments, generating function, Rényi entropy, and distribution of order statistics. We estimate the parameters using maximum likelihood method. We perform a Monte Carlo simulation study, for one particular case, to assess the finite sample behavior of the maximum likelihood estimates. Examples to real data prove empirically the importance and potentiality of the suggested family.

## **Acknowledgements**

The authors would like to thank the reviewers and the editor for their comments which helped improve the paper.

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