A NEW GENERALIZATION OF LINDLEY DISTRIBUTION

Yassmen, Y. Abdelall

Department of Mathematical Statistics

Institute of Statistical Studies and Research

Cairo University,Egypt

Uyassmenamalek@gmail.com

ABSTRACT

In this study, we introduce a new generalization called the Lomax-Lindley distribution of Lindley distribution constructed by combining the cumulative distribution function (cdf) of Lomax and Lindley distributions. Some mathematical properties of the new distribution are discussed including moments, quantile and moment generating function. Estimation of the model parameters is carried out using maximum likelihood method. Finally, real data examples are presented to illustrate the usefulness and applicability of this new distribution.

Keywords: Lomax-Lindley distribution, moments, quantile, moment generating function, maximum likelihood estimates.

1. Introduction

The Lindley distribution is another life time probability distribution which can be used in modeling data reliability, biology, finance and lifetime analysis. It was introduced by Lindley (1958) to analyze failure time data in applications of reliability with probability density function (pdf):

$$g(x) = \frac{\theta^2}{(1+\theta)}(1+x)e^{-\theta x}, \theta > 0, x > 0$$

where θ is scale parameter. The corresponding cumulative distribution function (cdf) is given by

$$G(x) = 1 - \left(1 + \frac{\theta x}{1+\theta}\right)e^{-\theta x}, \theta > 0, x > 0$$

It can be seen that the Lindley distribution is a mixture of Exponential (θ) and gamma $(2, \theta)$ distributions. Having only one parameter, the Lindley distribution does not provide adequate fits in many types of data. For this reason, many authors suggested new distributions of it in the hope of adding flexibility in the modeling process. Among of these new distributions: generalized Lindley (Nadarajah et al. (2011)), quasi Lindley distributions (Shanker and Mishra (2013)), Kumaraswamy Lindley (Cakmakyapan and Ozel (2014)), beta odd log-logistic Lindley (Cordeiro et al. (2015)), odd burr Lindley distribution (Goken et al. (2017)), exponentiated generalized Lindley distribution (Rodrigues et al. (2017)) and more. These distributions are modification, extension, or combinations of existing one. In this work we will concern with the last idea of combination of existing one. Gupta et al. (2016) suggested a new obtaining family of new distributions from two cdf F and G of known distributions with the following cdf:

$$F_G(x) = \frac{F(G(x))}{F(1)}$$

and the corresponding pdf as

$$f_G(x) = \frac{F'(G(x))}{F(1)}g(x)$$

where $f(.)$ and $F(.)$ be the pdf and cdf of the first distribution having the support $[0, a]$ where $1 \leq a < \infty$, g (.) and G (.) be the pdf and cdf of the second distribution with the support as $[b, c]$, where $b, c \in R \; or \, [0, \infty) \, or \, (-\infty, \infty)$. For different choices of first and second distributions one can construct a large number of distributions.

2. 2 Lomax-Lindley Distribution

In this section we studied the Lomax-Lindley (Lomax-L) distribution. Let F be the cdf

of Lomax distribution defined by $F(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}$ and G be the cdf of Lindley

distribution defined by $G(x) = 1 - \left(1 + \frac{\theta x}{1+\theta}\right)e^{-\theta x}$ .Now using (1) and (2) we have the cdf

of Lomax-L distribution:

$$F_{LL}(x) = K\left[1 - \left\{1 + \frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^{-\alpha}\right], \alpha, \beta, \theta > 0, x \geq 0 \qquad (3)$$

The corresponding pdf of Lomax-Lindley distribution is given by

$$f_{LL}(x) = \frac{K\alpha\theta^2}{\beta(\theta+1)}(x+1)\left[1 + \frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right]^{-(\alpha+1)}e^{-\theta x},$$

$$\alpha, \beta, \theta > 0, x \geq 0 \qquad (4)$$

where $K = [F(1)]^{-1} = \left[1 - \left(1 + \frac{1}{\beta}\right)^{-\alpha}\right]^{-1}$. The hazard function (hf) can be obtained using

(3) and (4) as follows:

$$h_{LL}(x) = \frac{\frac{K\alpha\theta^2}{\beta(\theta+1)}(x+1)\left[1 + \frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right]^{-(\alpha+1)}e^{-\theta x}}{1 - K\left[1 - \left\{1 + \frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^{-\alpha}\right]}$$

Figures 1, 2 and 3 illustrates the plots of the pdf, cdf and hazard function of Lomax-L
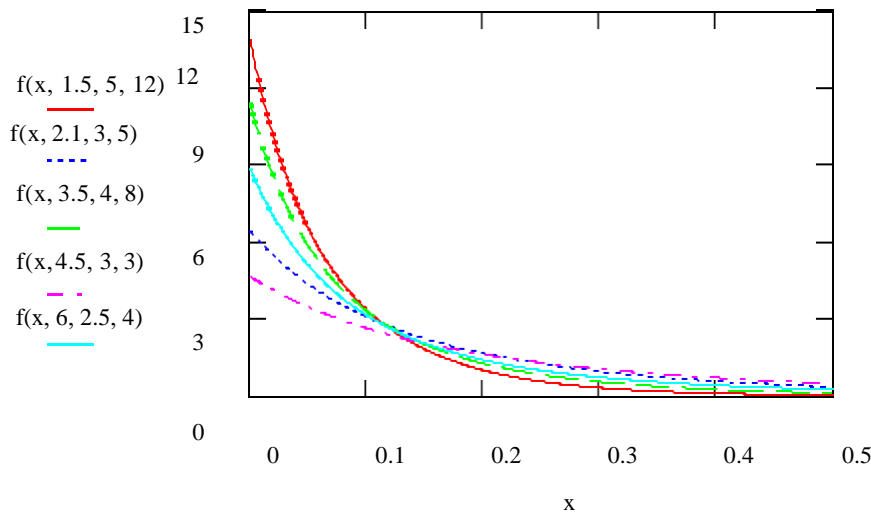distribution for various parameter values.

Fig 1. Pdf of Lomax-L distribution for various values of parameters

The quantile xq of the Lomax-L distribution has no closed form so one can solve the
following equation numerically to obtain

$$x_q = \theta x_q + \log\left[\left(\frac{\theta+1}{\theta+\theta x_q+1}\right)\left(1 - \beta\left[\left(1 - \frac{q}{K}\right)^{-\frac{1}{\alpha}} - 1\right]\right)\right] = 0$$
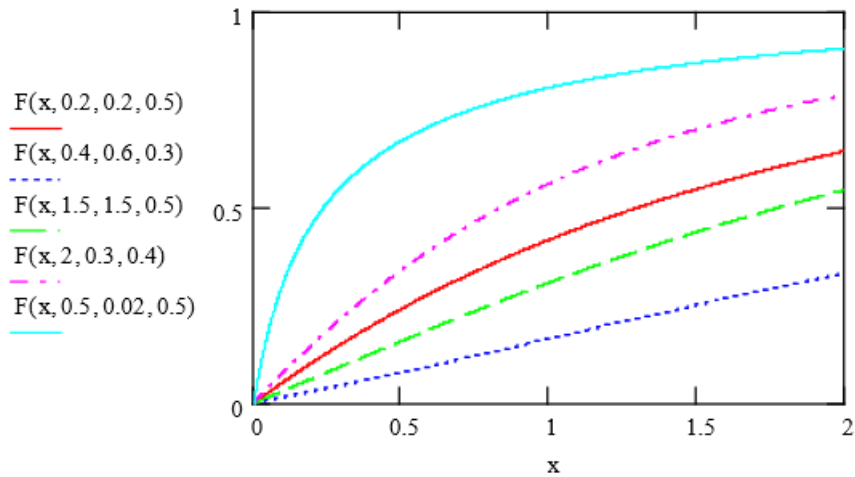
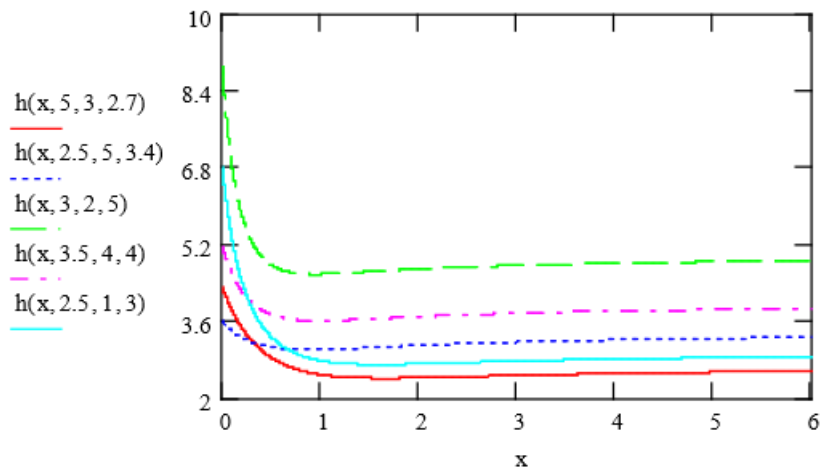Fig 2. cdf of Lomax-L distribution for various values of parameters

Fig 3. hf of Lomax-L distribution for various values of parameters

3. Moments and Moment Generating Function

3.1 Moments

The rth non-central moments of the Lomax-L distribution, denoted by μ_r , is given by the following theorem.

Theorm 1. If X is a continuous random variable has the Lomax-Lindley distribution, then the rth non-central moments is given by

$$\mu_r = \omega_{i,j,s} \left[\frac{\Gamma(r+s+2)}{[\theta(j+1)]^{r+s+2}} + \frac{\Gamma(r+s+1)}{[\theta(j+1)]^{r+s+1}} \right]$$

where $\omega_{i,j,s} = \frac{K\alpha\theta^2}{\beta(\theta+1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} (-1)^j \beta^{-i} \left(\frac{\theta}{\theta+1}\right)^s \binom{-(\alpha+1)}{i} \binom{i}{j} \binom{j}{s}$

Proof: Let X be a random variable with density function (4). The rth non-central moment of Lomax-L distribution is given by

$$\mu_r' = \int_0^{\infty} x^r f_{LL}(x)dx$$

$$= \frac{K\alpha\theta^2}{\beta(\theta+1)} \int_0^{\infty} x^r (x+1) \left(1 + \frac{1}{\beta}\left[1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]\right)^{-(\alpha+1)} e^{-\theta x} dx$$

Using binomial expansion,

$$\left(1 + \frac{1}{\beta}\left[1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]\right)^{-(\alpha+1)} = \sum_{i=0}^{\infty} \binom{-(\alpha+1)}{i}\left[\frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right]^i$$

Therefore

$$\mu_r' = \frac{K\alpha\theta^2}{\beta(\theta+1)} \sum_{i=0}^{\infty} \binom{-(\alpha+1)}{i} \beta^{-i} \int_0^{\infty} x^r (x+1) \left[1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^i e^{-\theta x} dx$$

For any real non-integer b>0, and |z|<1, Gradshteyn and Ryzhik (2007) defined the power series:

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} z^j$$

Using this fact,

$$\left[1 - \left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^i = \sum_{j=0}^{\infty} (-1)^j \binom{i}{j}\left(1 + \frac{\theta x}{\theta+1}\right)^j e^{-\theta j x}$$

Then

$$\mu_r' = \frac{K\alpha\theta^2}{\beta(\theta+1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{-(\alpha+1)}{i} \binom{i}{j} \beta^{-i} \int_0^{\infty} x^r (x+1) \left[1 + \frac{\theta x}{\theta+1}\right]^j e^{-\theta(j+1)x} dx$$

also by using binomial expansion, we get

$$\left[1 + \frac{\theta x}{\theta+1}\right]^j = \sum_{s=0}^{\infty} \binom{j}{s}\left(\frac{\theta}{\theta+1}\right)^s x^s \quad \text{Now}$$

$$\mu_r = \omega_{i,j,s} \int_0^{\infty} x^{r+s} (x+1)e^{-\theta(j+1)x} dx$$

$$= \omega_{i,j,s} \left[\int_0^\infty x^{r+s+1} e^{-\theta(j+1)x} dx + \int_0^\infty x^{r+s} e^{-\theta(j+1)x} dx \right]$$

$$= \omega_{i,j,s} \left[\frac{\Gamma(r+s+2)}{[\theta(j+1)]^{r+s+2}} + \frac{\Gamma(r+s+1)}{[\theta(j+1)]^{rs+1}} \right]$$

where $\omega_{i,j,s} = \frac{K\alpha\theta^2}{\beta(\theta+1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} (-1)^j \beta^{-i} \left(\frac{\theta}{\theta+1} \right)^s \binom{-(\alpha+1)}{i} \binom{i}{j} \binom{j}{s}$

Based on first four moments of Lomax-L distribution, the measures of skewness and kurtosis of the Lomax-L distribution respectively can obtained as:

$$S = \frac{\mu_3 - 3\mu_2'\mu_1 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}}, and \quad K = \frac{\mu_4' - 4\mu_3\mu_1 + 6\mu_2\mu_1'^2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}$$

3.2 Moment Generating Function

The moment generating function of the Lomax-L distribution, denoted by given by the following theorem.

Theorm 2. If X is a continuous random variable has the Lomax-Lindley distribution, then the moment generating function has the following form

$$M_X(t) = \omega_{i,j,s} \left[\frac{\Gamma(s+2)}{[\theta(j+1)-t]^{s+2}} + \frac{\Gamma(s+1)}{[\theta(j+1)-t]^{s+1}} \right]$$

where $\omega_{i,j,s} = \frac{K\alpha\theta^2}{\beta(\theta+1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} (-1)^j \beta^{-i} \left(\frac{\theta}{\theta+1} \right)^s \binom{-(\alpha+1)}{i} \binom{i}{j} \binom{j}{s}$

Proof: The moment generating function of Lomax-L distribution is given by

$$M_X(t) = \int_0^\infty e^{tx} f_{LL}(x) dx = \omega_{t,j,s} \int_0^\infty x^s (x+1) e^{-[\theta(j+1)-t]x} dx$$

$$= \omega_{t,j,s} \left[\int_0^\infty x^{s+1} e^{-[\theta(j+1)-t]x} dx + \int_0^\infty x^s e^{-[\theta(j+1)-t]x} dx \right]$$

$$= \omega_{t,j,s} \left[\frac{\Gamma(s+2)}{[\theta(j+1)-t]^{s+2}} + \frac{\Gamma(s+1)}{[\theta(j+1)-t]^{s+1}} \right]$$

Which completes the proof.

4. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the Lomax-L distribution with cdf and pdf given by (3) and (4), respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(j)}, j = 1, 2, \ldots, n$ is given by

$$f_{X_{(j)}}(x) = \frac{1}{B(j, n-j+1)} [F_{LL}(x)]^{j-1} [1 - F_{LL}(x)]^{n-j} f_{LL}(x)$$

Using binomial series expansion of $[1 - F_{LL}(x)]^{n-j}$:

$$[1 - F_{LL}(x)]^{n-j} = \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} [F_{LL}(x)]^r$$

we have

$$f_{X_{(j)}}(x) = \sum_{r=0}^{n-j} \frac{(-1)^r}{B(j, n-j+1)} \binom{n-j}{r} [F_{LL}(x)]^{j+r-1} f_{LL}(x) \qquad (5)$$

Substituting from (3) and (4) into (5), we can get the pdf of $X_{(j)}$. The moments of the order statistics of the Lomax-L distribution can be easily written in terms of moments of the Lomax-L distribution.

5. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n follows the Lomax-L distribution. To determine the maximum likelihood estimates (MLEs) of the three unknown parameters (α, β, θ) of the Lomax-L distribution we first obtain the likelihood function as follows:

$$L = \left(\frac{K\alpha\theta^2}{\beta(\theta+1)}\right)^n \prod_{i=1}^n (x_i$$

$$+ 1) \prod_{i=1}^n \left[1 + \frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right)\right]^{-(\alpha+1)} e^{-\theta \sum_{i=1}^N x_i}$$

The log-likelihood function becomes:

$$\ln L = n\ln K + n\ln\alpha + n\ln\theta^2 - n\ln\beta - n\ln(\theta+1) + \sum_{i=1}^n \ln(x_i+1) - \theta \sum_{i=1}^n x_i$$

$$- (\alpha+1) \sum_{i=1}^n \ln\left[1 + \frac{1}{\beta}\left(1 - \left(1 + \frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right)\right]$$

The first partial derivatives of the log-likelihood function with respect to, β α and θ are

$$\frac{\partial \ln L}{\partial \alpha} = -nK(1 + 1/\beta)^{-\alpha}\ln(1 + 1/\beta) + \frac{n}{\alpha} - \sum_{i=1}^n \ln[1 + \beta^{-1}V_i]$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n\alpha K}{\beta^2}(1 + 1/\beta)^{-(\alpha+1)} - \frac{n}{\beta} + \frac{(\alpha+1)}{\beta^2} \sum_{i=1}^n \frac{V_i}{1 + \beta^{-1}V_i}$$

and

$$\frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{(\theta+1)} - \sum_{i=1}^n x_i + \frac{(\alpha+1)}{\beta(1+\theta)} \sum_{i=1}^n \frac{[(\theta+1)^{-1} - (1+\theta+\theta x_i)]x_i e^{-\theta x_i}}{[1 + \beta^{-1}V_i]}$$

$$where\ K = [1 - (1 + 1/\beta)^{-\alpha}]^{-1}, and\ V_i = \left[1 - \left(1 + \frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right]$$

Setting these equations to zero and solving the resulting system of non-linear equations to obtain the MLEs of the three unknown parameters, α β and θ of the Lomax-L distribution, $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$. The negative second partial derivatives of the loglikelihood function are:

$$I_{\alpha\alpha} = -\frac{\partial^2 \ln L}{\partial \alpha^2} = nK(\ln(1 + 1/\beta))^2 (1 + 1/\beta)^{-\alpha}[1 + K(1 + 1/\beta)^{-\alpha}] + \frac{n}{\alpha^2}$$

$$I_{\alpha\beta} = -\frac{\partial^2 \ln L}{\partial \alpha\, \partial \beta} = \frac{nK}{\beta^2}(^{1+1/\beta)-(\alpha+1)}[\alpha \ln(1 + 1/\beta) - 1/\beta) - \frac{1}{\beta^2} \sum_{i=1}^{n} \frac{V_i}{(1 + \beta^{-1}V_i)}$$

$$I_{\alpha\theta} = -\frac{\partial^2 \ln L}{\partial \alpha\, \partial \theta} = \frac{1}{\beta(\theta + 1)} \sum_{i=1}^{n} \frac{((\theta + \theta x_i + 1) - (\theta + 1)^{-1})x_i e^{-\theta x_i}}{(1 + \beta^{-1}V_i)}$$

$$I_{\beta\beta} = -\frac{\partial^2 \ln L}{\partial^2 \beta} = \frac{n\alpha K}{\beta^3}(^{1+1/\beta)-(\alpha+1)}\left[2 - \frac{1}{(\beta + 1)} - \frac{\alpha K}{\beta}(1 + 1/\beta)^{-(\alpha+1)}\right] - \frac{n}{\beta^2}$$

$$+ \frac{(\alpha + 1)}{\beta^3} \sum_{i=1}^{n} \frac{V_i}{(1 + \beta^{-1}V_i)}\left[2 - \frac{V_i}{(1 + \beta^{-1}V_i)}\right]$$

$$I_{\beta\theta} = -\frac{\partial^2 \ln L}{\partial \beta\, \partial \theta} = \frac{(\alpha + 1)}{\beta^2(\theta + 1)} \sum_{i=1}^{n} \frac{((\theta + 1)^{-1} - (\theta + \theta x_i + 1))x_i e^{-\theta x_i}}{(1 + \beta^{-1}V_i)}\Bigg[1$$

$$- \frac{V_i}{\beta(1 + \beta^{-1}V_i)}\Bigg]$$

and

$$I_{\theta\theta} = -\frac{\partial^2 \ln L}{\partial^2 \theta} = \frac{2n}{\theta^2} - \frac{n}{(\theta + 1)^2} - \frac{(\alpha + 1)}{\beta(\theta + 1)} \times$$

$$\sum_{i=1}^{n} \frac{x_i e^{-\theta x_i}}{(1 + \beta^{-1}V_i)}\left[(\theta + \theta x_i + 1)x_i - \frac{2[x_i(\theta + 1) - 1]}{(\theta + 1)^2} + \frac{((\theta + 1)^{-1} - (\theta + \theta x_i + 1))^2 x_i e^{-\theta x_i}}{\beta(\theta + 1)(1 + \beta^{-1}V_i)}\right]$$

The observed 3×3 information matrix I is given by

$$I = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\theta} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\theta} \\ I_{\theta\alpha} & I_{\theta\beta} & I_{\theta\theta} \end{pmatrix}$$

The asymptotic variance-covariance matrix of $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ of the Lomax-L distribution obtained by inverting the information matrix and replace the unknown parameters by their MLEs of parameters. The $100(1 - \gamma)\%$ approximate confidence intervals for the parameters, α β and θ of the Lomax-L distribution are respectively given by

$$\hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{\text{var}(\hat{\alpha})}, \hat{\beta} \pm z_{\frac{z}{2}}\sqrt{\text{var}(\hat{\beta})}, and\ \hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\text{var}(\hat{\theta})}$$

6. Applications to Real Data

Now we use real data set to fit the Lomax-L distribution with some sub-models of

Log Generalized Lindley-Weibull distribution LGLW $(\alpha, \beta, \theta, \gamma, c)$ (Oluyede et al. (2015)) are LGLW (θ, γ, c) and LGLW (α, θ, c) with corresponding densities:

$$\text{LGLW}(\theta, \gamma, c): f_{LGLW}(x; \theta, \gamma, c) = \frac{c\theta^2}{\gamma(1+\theta)} \left(\frac{x}{\gamma}\right)^{c-1} \left\{1 + \left(\frac{x}{\gamma}\right)^c\right\} e^{-\theta\left(\frac{x}{\gamma}\right)^c}, x > 0$$

$$\text{LGLW}(\alpha, \theta, c): f_{LGLW}(x; \alpha, \theta, c) = \frac{c\theta^{\alpha+1}}{(1+\theta)\Gamma(\alpha+1)} x^{c\alpha-1}(\alpha + x^c) e^{-\theta x^e}, x > 0$$

$where\ \alpha > 0, \theta > 0, \gamma > 0,\ and\ c > 0$

The data set is given by Murthy et al. (2004) consists of the failure times of 20 mechanical components. The data are:

0.067 0.068 0.076 0.081 0.084 0.085 0.085 0.086 0.089 0.098 0.098 0.114 0.114

0.115 0.121 0.125 0.131 0.149 0.160 0.485. The LGLW (θ, γ, c) , LGLW (α, θ, c), and Lomax-L distributions are fitted to the data and MLEs of the parameters are given in Table (1).

Table 1: MLEs for the different fitted distributions

Model	α	β	θ	γ	c
LGLW(α, θ, c)	0.790	1	3.518	1	0.924
	(0.329)	-	(0.967)	-	(0.245)
LGLW(θ, γ, c)	1	1	6.712	1.605	0.684
			(2.104)	(0.274)	(0.015)
Lomax-L(α, β, θ)	4.453	9.22	6.431		
	(1.032)	(5.073)	(0.836)		

The estimated variance-covariance matrix for the Lomax-L distribution is given by:

$$\begin{pmatrix} 1.064 & 3.802 & 0.081 \\ 3.802 & 25.734 & 0.802 \\ 0.081 & 0.802 & 0.699 \end{pmatrix}$$

The 95% asymptotic confidence intervals of the Lomax-L distribution are $\alpha \in (2.43, 6.476)$, $\beta \in (-0.723, 19.163), and\ \theta \in (4.792, 8.07)$. The following statistics-2log-likelihood function (-2lnL) evaluated at the parameter estimates, Akaike information criterion (AIC), Bayesian Information Criterion (BIC), corrected Akaike information Criteria (CAIC), and Kolmogorove-Smirnov statistic (K-S) of distributions are listed in Table (2).

Table 2: Model selection criteria for the data set

Model	-2lnL	AIC	BIC	CAIC	K-S
LGLW(α, θ, c)	-30.762	-24.762	-21.775	-23.262	0.348

| LGLW(θ, γ, c) | -32.236 | -26.236 | -23.249 | -24.736 | 0.561 |
| Lomax-L(α, β, θ) | -42.266 | -36.266 | -33.279 | -34.766 | 0.375 |

Results in Table 2 , indicates that the Lomax-L model gives the smallest -2lnL, AIC, BIC and CAIC and gives the second smallest KS value when compare to other distributions. We conclude that the lomax-L distribution is a better model than the other competitive models.

7. Conclusion

In this paper, we concerned with one of the new family of distributions suggested by Gupta et al. (2016) to introduce a new generalization of Lindley distribution which called the Lomax-Lindley (Lomax-L) distribution. Some mathematical properties of the new distribution are derived and maximum likelihood estimation of the three unknown parameters are obtained. We fit the Lomax-L distribution to real data set in order to explain the flexibility of the new distribution in life time analysis.

References

[1]   Cakmakyapan, S., and Ozel, G. (2014). A new Customer Lifetime duration Distribution: The Kumaraswamy Lindley Distribution. International Journal of Trade, Economics and Finance. 5(5). 441-444.

[2]   Cordeiro, G. M., Alizadeh, M., Tahir, M. H., Mansoor, M., Bourguignon, M., and Hamedani, G. G. (2015). The Beta Odd Log-Logistic Generalized Family of Burr Distribution. Hacettepe Journal of Mathematics and Statistics. 45(73).

[3]   Gokcen, A., Morad, A., Emrah, A., and Gamze, O. (2017). Odd Burr Lindley Distribution with Properties and Application. Hacettepe Journal of Mathematics and Statistics. 46(2). 255-276.

[4]   Gradshteyn, I. S., and Ryzhik, I. M. (2007). Table of Integrals, Series, and Products. San Diego: Elsevier.

[5]   Gupta, J., Garg, M., and Gupta, M. (2016). The Lomax-Gumbel Distribution. Palestine Journal of Mathematics. 5(1). 35-42.

[6]   Lindley, D. V. (1958) . Fiducial Distributions and Bayes Theorem. Journal of the Royal Statistical Society. 20(1). 102-107.

[7]   Murthy, D. N. P., Xie, M., and Jiang, R. (2004). Weibull Models. John Weily and Sons, New Jersey.

[8]   Nadarajah, S., Bakouch, H. S., and Tahmasbi, R. (2011). A Generalized Lindley Distribution.

Sankhya B. 73. 331-359.

[9]   Oluyede, B. O., Mutiso, F., and Huang, S. (2015). The Log Generalized Lindley- Weibull Distribution with Application. Journal of Data Science. 13. 281-310.

[10] Shanker, R., and Mishra, A. (2013). A Quasi Lindley Distribution. African Journal of Mathematices and Computer Science Research. 6(4). 64-71.