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ESTIMATION METHODS FOR THE NEW WEIBULL-PARETO DISTRIBUTION: SIMULATION AND APPLICATION

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ABSTRACT

In this paper, we introduce the alternative methods to estimation for the new weibull-pareto distribution parameters. We discussed of point estimation and interval estimation for parameters of the new weibull-pareto distribution. We have also discussed the method of Maximum Likelihood estimation, the method of Least Squares estimation, the method of Weighted Least Squares estimation and the method of Maximum Product Spacing estimation. In addition, we discussed the raw moment of random variable X and the reliability functions (survival and hazard functions). Further, we compared between the results of the methods that have been discussed using Monte Carlo Simulation method and application study.

Keywords: The new Weibull-Pareto Distribution, Maximum Likelihood Estimation, least-squares Estimation, weighted least-squares Estimation, Maximum Product Spacing method, Interval Estimation, moment, Bootstrap and Reliability Functions.

1. Introduction and motivation

In statistical inference of lifetime data used for statistical analysis follows a particular statistical distributions. Many of researchers uses traditional estimation methods such as the method of moments and maximum likelihood estimation (MLE). Each of them having their own advantages and limitations but the most popular method of estimation is MLE method.

The new Weibull-Pareto distribution (NWPD) was introduced by Nasiru and Luguterah (2015). We are interested in the estimation of the parameters for the new Weibull-Pareto distribution by using different estimation methods. The cumulative distribution function (cdf), the probability density function (pdf) and the quantile function of the NWPD with shape parameters β and α, and scale parameter θ are respectively given by

$$F(x;\theta,\alpha,\beta) = 1 - e^{-\alpha\left(\frac{x}{\theta}\right)^{\beta}}; \quad x \geq 0, \qquad \alpha,\beta,\theta > 0, \tag{1.1}$$

$$f(x;\theta,\alpha,\beta) = \frac{\alpha\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\alpha\left(\frac{x}{\theta}\right)^{\beta}}, \tag{1.2}$$

and

$$x_u = \theta\left(\frac{-1}{\alpha} ln\left(\frac{1}{1-u}\right)\right)^{\frac{1}{\beta}} \quad ; 0 < u < 1. \tag{1.3}$$

The model can be considered as another useful three-parameter of the new Weibull-Pareto distribution (NWPD). It is noted that the NWPD is reduced to the Weibull distribution for $\alpha = 1$. In literature, estimation of parameters in the three parameters NWPD is discussed extensively, but no one has performed comparison of MLE and Maximum Product Spacing (MPS).

Ekström (2006) discussed MPS an alternative to the MLE method. In many situations, the MPS method works better than the MLE method and attractive properties such as consistency and asymptotic efficiency of the MPS estimator closely parallel those of the MLE when the latter works well. For more information of the methods see Ranneby (1984), Hung (2001), Wu et al. (2004), Feng and He (2008) and Torres (2014), see for example Singh et al. (2014), Vani Lakshmi and VaidyaNathan (2016) and Almetwally and Almongy (2019).

In this paper, the methods are Maximum Likelihood (ML), least-square (LS), weighted least-square (WLS) and Maximum Product Spacing (MPS) are applied to estimate the parameters of the NWPD. The purpose of this study is to discuss the best estimation method for parameters of NWPD, which we think would be of deep interest to statisticians. The confidence interval is conducted using ML (asymptotic confidence intervals) and two parametric bootstrap confidence interval (Normal approximation method and percentile bootstrap). Reliability Functions (survival and hazard) are used for comparing between methods. The moment function is counted and through it, we measure skewness and kurtosis for this model under different estimation methods. A simulation study is conducted to compare the preferences between estimation methods. Also, a real data set is introduced and analyzed to investigate the model.

2. Estimation of Models

For the considered distribution, we use the maximum likelihood estimation method, least squares method, weighted least squares method and one which is not very common. Maximum Product Spacing method for estimating the parameters of NWPD.

2.1 Maximum Likelihood Estimation

Casella and Berger (1990) discussed the likelihood function which is the most frequently used method of parameter estimation. The likelihood function of the NWPD is

$$L_{ML} = \left(\frac{\alpha\beta}{\theta}\right)^{n} \prod_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta-1} e^{-\sum_{i=1}^{n} \alpha\left(\frac{x_i}{\theta}\right)^{\beta}}, \tag{2.1}$$

and the log likelihood function is given as

$$ln\, L_{ML} = n(ln\, \alpha + ln\, \beta - ln\, \theta) + (\beta - 1)\sum_{i=1}^{n} ln \frac{x_i}{\theta} - \alpha \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta}. \tag{2.2}$$

To obtain the normal equations for the unknown parameters, we differentiate (2.2) partially with respect to the parameters θ, β and α and equate them to zero. The estimators $\hat{\theta}_{ML}, \beta_{ML}$ and $\hat{\alpha}_{ML}$ can be obtained as the solution of the following equations.

$$\frac{\partial\, ln\, L_{ML}}{\partial\theta} = \frac{-n}{\theta} - \frac{(\beta - 1)n}{\theta} + \frac{\alpha\beta}{\theta^{(\beta+1)}} \sum_{i=1}^{n} {x_i}^{\beta}, \tag{2.3}$$

$$\frac{\partial\, ln\, L_{ML}}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta}, \tag{2.4}$$

$$\frac{\partial\, ln\, L_{ML}}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \left[1 - \alpha\left(\frac{x_i}{\theta}\right)^{\beta}\right] ln \frac{x_i}{\theta}. \tag{2.5}$$

The above equations are not close forms, therefore we can use any iterative procedure techniques such as Newton-Raphson type algorithms, to get the solution.

2.2 Least-Square Method

Swain et al. (1988) introduced the LS method, it based on the observed sample $x_1 < \cdots < x_n$ from be n ordered random sample of any distribution with CDF, where $F(.)$ denotes the

CDF , we get

$$E\big(F(x_i)\big) = \frac{i}{(n+1)}. \tag{2.6}$$

The least squares method are obtained by minimizing

$$P(\theta,\alpha,\beta) = \sum_{i=1}^{n}\left(F(x_i) - \frac{i}{(n+1)}\right)^2. \tag{2.7}$$

Putting the CDF of NWPD in equation (2.7) we get

$$P_{LS} = P(\theta,\alpha,\beta) = \sum_{i=1}^{n}\left(1 - e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}} - \frac{i}{(n+1)}\right)^2. \tag{2.8}$$

After differentiating equation (2.8) with respect to parameters θ,β and α and then equating them to zero, we will get the following:

$$\frac{\partial P_{LS}}{\partial\theta} = 2\sum_{i=1}^{n}\left[\left(1 - e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}} - \frac{i}{(n+1)}\right)e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\frac{-\alpha\beta}{\theta}\left(\frac{x_i}{\theta}\right)^{\beta}\right], \tag{2.9}$$

$$\frac{\partial P_{LS}}{\partial\alpha} = 2\sum_{i=1}^{n}\left[\left(1 - e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}} - \frac{i}{(n+1)}\right)e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}\right], \tag{2.10}$$

$$\frac{\partial P_{LS}}{\partial\beta} = 2\sum_{i=1}^{n}\left[\left(1 - e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}} - \frac{i}{(n+1)}\right)e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\alpha\left(\frac{x_i}{\theta}\right)^{\beta}\ln\frac{x_i}{\theta}\right]. \tag{2.11}$$

The above nonlinear equations can't be solved analytically so, the $\hat{\theta}_{LS},\beta_{LS}$ and $\hat{\alpha}_{LS}$ of θ,β and α respectively, we can use any iterative procedure techniques such as Newton-Raphson type algorithms, to obtained as the numerical solution.

2.3 Weighted Least-Squares Method

Swain et al. (1988) introduced the WLS method. We use the WLS procedure for estimating the parameters θ,β and α of the NWPD distribution. Let

$$W(\theta,\alpha,\beta) = \sum_{i=1}^{n} w_i\left(F(x_i) - \frac{i}{(n+1)}\right)^2. \tag{2.12}$$

Hence, we suggest the weights

$$w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Putting the CDF of NWPD in equation (2.12) we get

$$W(\theta,\alpha,\beta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1 - e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}} - \frac{i}{(n+1)}\right)^2. \tag{2.13}$$

After differentiating equation (2.13) with respect to parameters θ,β and α and then equating them to zero we got the normal equation as follows:

$$\frac{\partial W}{\partial \theta}=2\sum_{i=1}^{n}\left[\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}-\frac{i}{(n+1)}\right)e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\frac{-\alpha\beta}{\theta^{(\beta+1)}}x_i{}^{\beta}\right], \quad (2.14)$$

$$\frac{\partial W}{\partial \alpha}=2\sum_{i=1}^{n}\left[\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}-\frac{i}{(n+1)}\right)e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}\right], \quad (2.15)$$

$$\frac{\partial W}{\partial \beta}=2\sum_{i=1}^{n}\left[\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}-\frac{i}{(n+1)}\right)e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\alpha\left(\frac{x_i}{\theta}\right)^{\beta}\ln\frac{x_i}{\theta}\right]. \quad (2.16)$$

The above nonlinear equations can't be solved analytically so, the $\hat{\theta}_{WLS}, \hat{\beta}_{WLS}$ and $\hat{\alpha}_{WLS}$ of θ, β and α respectively, we can use any iterative procedure techniques such as Newton-Raphson type algorithms, to obtained as the numerical solution.

2.4 Maximum Product Spacing

According to Cheng and Amin (1983) introduced Maximum Product Spacing as following:

$$G=\left(\prod_{i=1}^{n+1}D_i\right)^{\frac{1}{n+1}}, \quad (2.17)$$

where G is defined as the geometric mean of the product spacing function and where

$$D_i=\begin{cases} D_1=F(x_1) \\ D_i=F(x_i)-F(x_{i-1});\ i=2\ldots n\ , \\ D_{n+1}=1-F(x_n) \end{cases} \quad (2.18)$$

such that $\sum D_i=1$, then the product spacing function is

$$G=\left(\left(1-e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\right)\left(e^{-\alpha\left(\frac{x_n}{\theta}\right)^{\beta}}\right)\prod_{i=2}^{n}\left[e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\right]\right)^{\frac{1}{n+1}}, \quad (2.19)$$

the natural logarithm of the product spacing function is

$$\ln G=\frac{1}{n+1}\left[\ln\left(1-e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\right)-\alpha\left(\frac{x_n}{\theta}\right)^{\beta}+\sum_{i=2}^{n}\ln\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\right)\right]. \quad (2.20)$$

To obtain the normal equations for the unknown parameters, we differentiate Equation (2.20) partially with respect to the parameters θ, β and α and equate them to zero. The estimators $\hat{\theta}_{MPS}, \beta_{MPS}$ and $\hat{\alpha}_{MPS}$ of θ, β and α respectively, we can obtain the solution of the following equations as

$$\frac{\partial ln\, G}{\partial\theta}=\frac{1}{n+1}\left[\frac{e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\frac{-\alpha\beta}{\theta^{(\beta+1)}}{x_1}^{\beta}}{\left(1-e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\right)}+\frac{\alpha\beta}{\theta^{(\beta+1)}}{x_1}^{\beta}\right.$$
$$\left.+\sum_{i=2}^{n}\frac{\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}\frac{\alpha\beta}{\theta^{(\beta+1)}}{x_{i-1}}^{\beta}\right)-\left(e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\frac{\alpha\beta}{\theta^{(\beta+1)}}{x_i}^{\beta}\right)}{\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\right)}\right], \tag{2.21}$$

$$\frac{\partial \ln G}{\partial\alpha}=\frac{1}{n+1}\left[\frac{e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\left(\frac{x_1}{\theta}\right)^{\beta}}{\left(1-e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\right)}-\left(\frac{x_n}{\theta}\right)^{\beta}\right.$$
$$\left.+\sum_{i=2}^{n}\frac{\left(e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}\right)-\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}\left(\frac{x_{i-1}}{\theta}\right)^{\beta}\right)}{\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\right)}\right], \tag{2.22}$$

$$\frac{\partial \ln G}{\partial\beta}=\frac{1}{n+1}\left[\frac{e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\alpha\left(\frac{x_1}{\theta}\right)^{\beta}\ln\frac{x_1}{\theta}}{\left(1-e^{-\alpha\left(\frac{x_1}{\theta}\right)^{\beta}}\right)}-\alpha\left(\frac{x_n}{\theta}\right)^{\beta}\ln\frac{x_n}{\theta}\right.$$
$$\left.+\sum_{i=2}^{n}\frac{\left(e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\alpha\left(\frac{x_i}{\theta}\right)^{\beta}\ln\frac{x_i}{\theta}\right)-\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}\ln\frac{x_{i-1}}{\theta}\right)}{\left(e^{-\alpha\left(\frac{x_{i-1}}{\theta}\right)^{\beta}}-e^{-\alpha\left(\frac{x_i}{\theta}\right)^{\beta}}\right)}\right]. \tag{2.23}$$

The above nonlinear equations can't be solved analytically so, the $\hat{\theta}_{MPS}$, $\hat{\beta}_{MPS}$ and $\hat{\alpha}_{MPS}$ of θ, β and $\alpha,$ can be use any iterative procedure techniques such as Newton-Raphson type algorithms, to obtained as the numerical solution.

3. Moment Function

The r^{th} non-central moment of the NWPD used in both mechanics and statistics, of the shape of a set of points is given by

$$E(x^r)=\theta^r\alpha^{\frac{-r}{\beta}}\Gamma\left(\frac{\beta+r}{\beta}\right),\qquad r=1,2,3,4. \tag{3.1}$$

Then

$$E(x\,)=\theta\alpha^{\frac{-1}{\beta}}\Gamma\left(\frac{\beta+1}{\beta}\right),$$
$$E(x^2)=\theta^2\alpha^{\frac{-2}{\beta}}\Gamma\left(\frac{\beta+2}{\beta}\right), \tag{3.2}$$

$$Variance(x) = E(x^2) - \left(E(x)\right)^2 = \theta^2 \alpha^{\frac{-2}{\beta}} \left[\Gamma\left(\frac{\beta+2}{\beta}\right) - \left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^2\right],$$

$$E(x^3) = \theta^3 \alpha^{\frac{-3}{\beta}} \Gamma\left(\frac{\beta+3}{\beta}\right),$$

$$E(x^4) = \theta^4 \alpha^{\frac{-4}{\beta}} \Gamma\left(\frac{\beta+4}{\beta}\right).$$

The third moment is used to construct a measure of skewness, which describes whether the probability mass is more to the left or the right of the mean, compared to a normal distribution. The fourth moment is used to construct a measure of kurtosis, which measures the width of the NWPD.

$$\text{skewness}(x) = \frac{E(x^3) - 3E(x)E(x^2) + 2\left(E(x)\right)^3}{\left(Variance(x)\right)^{3/2}}, \tag{3.3}$$

$$\text{kurtosis}(x) = \frac{E(x^4) - 4E(x)E(x^3) + 6\left(E(x)\right)^2 E(x^2) - 3\left(E(x)\right)^4}{\left(Variance(x)\right)^2}. \tag{3.4}$$

4. Confidence Intervals

In this section, we propose different confidence intervals. One is based on the asymptotic distribution of θ, α and $\beta$and bootstrap confidence intervals. To more information in bootstrap see Zheng and Gastwirth (2010) and Yang et al. (2010).

4.1 Asymptotic Confidence Intervals

In this subsection, we propose the asymptotic confidence intervals (ACI) using ML method can be used to construct the confidence interval for the parameters. $I(\hat{\delta})$ is the observed inverse Fishers information matrix and is define as:

$$I(\hat{\delta}) = \begin{bmatrix} -L''_{\theta} & -L''_{\theta\alpha} & -L''_{\theta\beta} \\ -L''_{\alpha\theta} & -L''_{\alpha} & -L''_{\alpha\beta} \\ -L''_{\beta\theta} & -L''_{\beta\alpha} & -L''_{\beta} \end{bmatrix}^{-1} = \begin{bmatrix} I_{\hat{\theta}} & I_{\hat{\theta}\hat{\alpha}} & I_{\hat{\theta}\hat{\beta}} \\ I_{\hat{\alpha}\hat{\alpha}} & I_{\hat{\alpha}} & I_{\hat{\alpha}\hat{\beta}} \\ I_{\hat{\beta}\hat{\theta}} & I_{\hat{\beta}\hat{\alpha}} & I_{\hat{\beta}} \end{bmatrix}, \tag{4.1}$$

where $\hat{\delta} = (\hat{\theta}, \hat{\alpha}, \hat{\beta})$. An approximate 95% two side confidence interval for (θ, α and β) are respectively

$$\hat{\theta} \pm Z_{0.025}\sqrt{I_{\hat{\theta}}}, \qquad \hat{\alpha} \pm Z_{0.025}\sqrt{I_{\hat{\alpha}}}, \qquad \hat{\beta} \pm Z_{0.025}\sqrt{I_{\hat{\beta}}}.$$

4.2 Bootstrap Confidence Interval

Here, we construct two parametric bootstrap confidence interval for θ, α, β as follow:

I. Normal approximation method (BN)

The confidence interval of the mean of a measurement variable is commonly estimated on the assumption that the statistic follows a normal distribution, and that the variance is therefore independent of the mean. This is known as a normal approximation confidence interval. Here, we construct parametric bootstrap confidence interval for θ, α, β by using percentile bootstrap confidence interval as follow:

1) Compute the MLE of $\delta = (\theta, \alpha, \beta)$.
2) Generated a bootstrap samples using θ, α, β to obtain the bootstrap estimate of θ say $\hat{\theta}^b$, α say $\hat{\alpha}^b$ and β say $\hat{\beta}^b$ using the bootstrap sample.
1) Repeat step (2) B times to have $\left(\theta^{b(1)}, \theta^{b(2)}, \dots, \theta^{b(B)}\right), \left(\alpha^{b(1)}, \alpha^{b(2)}, \dots, \alpha^{b(B)}\right)$ and $\left(\beta^{b(1)}, \beta^{b(2)}, \dots, \beta^{b(B)}\right)$.
3) A two side $100\ (1-\gamma)\%$ normal bootstrap confidence interval for the unknown parameters θ, α, β is given by $\left\{\overline{\hat{\theta}^b} \pm Z_{\gamma/2} \frac{s_{\hat{\theta}^b}}{\sqrt{b}} \sqrt{\left(1-\frac{b}{B}\right)}\right\}, \left\{\overline{\hat{\alpha}^b} \pm Z_{\gamma/2} \frac{s_{\hat{\alpha}^b}}{\sqrt{b}} \sqrt{\left(1-\frac{b}{B}\right)}\right\}$ and $\left\{\overline{\hat{\beta}^b} \pm Z_{\gamma/2} \frac{s_{\hat{\beta}^b}}{\sqrt{b}} \sqrt{\left(1-\frac{b}{B}\right)}\right\}$

II. Percentile Bootstrap Confidence Interval (BP)

Here, we construct parametric bootstrap confidence interval for α, θ by using percentile bootstrap confidence interval as follow:

1) Compute the MLE of $\delta = (\theta, \alpha, \beta)$.
2) Generated a bootstrap samples using θ, α, β to obtain the bootstrap estimate of α say $\hat{\alpha}^b$,θ say $\hat{\theta}^b$ and β say $\hat{\beta}^b$using the bootstrap sample.
3) Repeat step (2) B times to have $\left(\theta^{b(1)}, \theta^{b(2)}, \dots, \theta^{b(B)}\right), \left(\alpha^{b(1)}, \alpha^{b(2)}, \dots, \alpha^{b(B)}\right)$ and $\left(\beta^{b(1)}, \beta^{b(2)}, \dots, \beta^{b(B)}\right)$.
4) Arrange $\left(\theta^{b(1)}, \theta^{b(2)}, \dots, \theta^{b(B)}\right), \left(\alpha^{b(1)}, \alpha^{b(2)}, \dots, \alpha^{b(B)}\right)$ and $\left(\beta^{b(1)}, \beta^{b(2)}, \dots, \beta^{b(B)}\right)$ in ascending order as $\left(\theta^{b[1]}, \theta^{b[2]}, \dots, \theta^{b[B]}\right), \left(\alpha^{b[1]}, \alpha^{b[2]}, \dots, \alpha^{b[B]}\right)$ as$\left(\beta^{b[1]}, \beta^{b[2]}, \dots, \beta^{b[B]}\right)$.
5) A two side $100\ (1-\gamma)\%$ percentile bootstrap confidence interval for the unknown parameters θ, α, β is given by $\left\{\hat{\theta}^{b[B\gamma/2]}, \hat{\theta}^{b[B(1-\gamma/2)]}\right\}, \left\{\hat{\alpha}^{b[B\gamma/2]}, \hat{\alpha}^{b[B(1-\gamma/2)]}\right\}$ and $\left\{\hat{\beta}^{b[B\gamma/2]}, \hat{\beta}^{b[B(1-\gamma/2)]}\right\}$.

5. Reliability Functions

In this section, we propose the estimation of survival and hazard functions using ML, LS, WLS and MPS for specified value of time say $(t = mean)$ where the survival time of a given system because of it analytical structure, it is given by Nasiru and Luguterah (2015)

$$S(x) = e^{-\alpha\left(\frac{t}{\theta}\right)^{\beta}}, \tag{5.1}$$

the hazard function defined by

$$h(x) = \frac{\alpha\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}. \tag{5.2}$$

Reliability functions can be used in many models where they are needed in life testing. There are modern applications such as Almetwaly & Almongy (2018), Almetwally et al. (2018) and Hanagal & Bhalerao (2018).

6. Simulation Study

In this section, we provide a complete algorithm of Monte Carlo simulation (MCS) study. We explain our algorithm through an application in estimation methods framework, especially; we will use Monte Carlo technique to compare between MLE, LS, WLS and MPS estimation methods based on complete data for estimating NWPD in life time by R language.

Simulation Algorithm Scheme: We can build our model by generate all simulation controls. In this stage, we must follow the following steps by order:

Step 1: Suppose any values as true values of the parameters vector of NWPD as (α, β, θ): case 1 $= (0.9, 0.4, 0.6)$, case 2 $= (1.8, 1.3, 0.75)$ and case 3 $= (2.5, 1.4, 3)$.

Step 2: Choose the sample size $n = 10, 20, 30, 50, 70, 100$ and 150.

Step 3: Generate the sample random values of NWPD by using quantile function in equation (1.3).

Step 4: Solve differential equations for each estimation methods, to obtain the estimators of the parameters for NWPD, we calculate $\hat{\delta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta})$.

Step 5: Repeat this experiment (L-1) times. In each experiment, the same values of the parameters. It is certain that, the values of generating random are varying from experiment to experiment even though n are not changed. In the end, we have L-values of bias and MSE, we restricted the number of repeat this experiment to 10000. Take the averages of these values and call them Monte Carlo estimates: the Bias estimator $Bias = \hat{\delta} - \delta$, where $\hat{\delta}$ is the estimated value of $\delta = (\theta, \alpha, \beta)$, and the mean squared error (MSE) of the estimator. $MSE = Mean(\hat{\delta} - \delta)^2$. After ending the treatment stage, we must check and evaluate the simulation result before put or discuss (display) it in our paper (research).

The simulation methods are compared using MSE, Bias, Length of confidence interval for the parameters, measure skewness and kurtosis and reliability functions using the NWPD, this assess their performance through a Monte Carlo simulation study.

Table 1: Parameters Estimation of the NWPD in Case1

$\boldsymbol{\alpha}$=0.9 ; $\boldsymbol{\beta}$=0.4 $\boldsymbol{\theta}$=0.6									
		MLE		LS		WLS		MPS	
$\boldsymbol{n}$		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	$\hat{\boldsymbol{\alpha}}$	0.00054	0.04343	0.00903	0.04734	0.00413	0.04520	0.01530	0.03607
	$\hat{\boldsymbol{\beta}}$	0.06647	0.02266	0.00210	0.02222	0.00351	0.02049	-0.03004	0.01229
	$\hat{\boldsymbol{\theta}}$	-0.02364	0.03297	0.02109	0.03166	0.01524	0.03272	0.02904	0.02809
20	$\hat{\boldsymbol{\alpha}}$	0.00720	0.02183	0.00701	0.02054	0.00843	0.02030	0.01496	0.01916
	$\hat{\boldsymbol{\beta}}$	0.02592	0.00737	-0.00469	0.00859	-0.00006	0.00803	-0.02898	0.00598
	$\hat{\boldsymbol{\theta}}$	0.00687	0.01660	0.02582	0.01809	0.02379	0.01733	0.03844	0.01639
30	$\hat{\boldsymbol{\alpha}}$	0.01285	0.01370	0.01280	0.01360	0.01395	0.01349	0.01866	0.01247
	$\hat{\boldsymbol{\beta}}$	0.01895	0.00422	-0.00186	0.00535	0.00240	0.00471	-0.02195	0.00363
	$\hat{\boldsymbol{\theta}}$	0.01217	0.01161	0.02531	0.01317	0.02360	0.01203	0.03455	0.01149
50	$\hat{\boldsymbol{\alpha}}$	0.01750	0.00859	0.01808	0.00849	0.01882	0.00848	0.02129	0.00785
	$\hat{\boldsymbol{\beta}}$	0.01122	0.00215	0.00022	0.00310	0.00333	0.00260	-0.01662	0.00203
	$\hat{\boldsymbol{\theta}}$	0.02039	0.00741	0.02673	0.00866	0.02532	0.00784	0.03498	0.00801
70	$\hat{\boldsymbol{\alpha}}$	0.01709	0.00718	0.01700	0.00702	0.01803	0.00703	0.02064	0.00671
	$\hat{\boldsymbol{\beta}}$	0.00722	0.00153	-0.00314	0.00215	-0.00012	0.00178	-0.01421	0.00153
	$\hat{\boldsymbol{\theta}}$	0.02366	0.00607	0.03062	0.00710	0.02946	0.00641	0.03546	0.00672
100	$\hat{\boldsymbol{\alpha}}$	0.01541	0.00490	0.01553	0.00525	0.01612	0.00508	0.01791	0.00479
	$\hat{\boldsymbol{\beta}}$	0.00611	0.00114	0.00102	0.00171	0.00289	0.00141	-0.01044	0.00112
	$\hat{\boldsymbol{\theta}}$	0.02992	0.00512	0.03310	0.00558	0.03250	0.00526	0.03783	0.00549
150	$\hat{\boldsymbol{\alpha}}$	0.01832	0.00319	0.01867	0.00325	0.01899	0.00321	0.02077	0.00302
	$\hat{\boldsymbol{\beta}}$	0.00423	0.00072	-0.00006	0.00099	0.00180	0.00083	-0.00786	0.00072
	$\hat{\boldsymbol{\theta}}$	0.02939	0.00341	0.03260	0.00395	0.03209	0.00363	0.03626	0.00397

Table 2: Parameters Estimation of the NWPD in Case2

$\boldsymbol{\alpha = 1.8}$; $\boldsymbol{\beta = 1.3}$; $\boldsymbol{\theta = 0.75}$									
		MLE		LS		WLS		MPS	
$\boldsymbol{n}$		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	$\hat{\boldsymbol{\alpha}}$	0.01931	0.07201	0.02936	0.10991	0.03747	0.08652	0.06534	0.04485
	$\hat{\boldsymbol{\beta}}$	0.21601	0.23938	0.01048	0.24301	0.01569	0.22585	-0.09762	0.12982
	$\hat{\boldsymbol{\theta}}$	-0.02625	0.02571	0.07708	0.05303	0.07049	0.04692	0.10739	0.04668
20	$\hat{\boldsymbol{\alpha}}$	0.05545	0.02761	0.05933	0.03982	0.06325	0.03381	0.07778	0.02441
	$\hat{\boldsymbol{\beta}}$	0.08424	0.07783	-0.01523	0.09070	-0.00020	0.08482	-0.09419	0.06317
	$\hat{\boldsymbol{\theta}}$	0.00850	0.01206	0.06404	0.02761	0.05277	0.02142	0.08798	0.02304
30	$\hat{\boldsymbol{\alpha}}$	0.06540	0.01856	0.07052	0.02824	0.06703	0.02146	0.08495	0.01927
	$\hat{\boldsymbol{\beta}}$	0.06158	0.04461	-0.00604	0.05652	0.00780	0.04975	-0.07132	0.03838
	$\hat{\boldsymbol{\theta}}$	0.00986	0.00815	0.04633	0.01638	0.03630	0.01270	0.06747	0.01412
50	$\hat{\boldsymbol{\alpha}}$	0.07034	0.01372	0.07333	0.01583	0.07180	0.01564	0.08354	0.01501
	$\hat{\boldsymbol{\beta}}$	0.03645	0.02268	0.00072	0.03273	0.01082	0.02748	-0.05404	0.02147
	$\hat{\boldsymbol{\theta}}$	0.01401	0.00480	0.03330	0.00896	0.02654	0.00714	0.05264	0.00824
70	$\hat{\boldsymbol{\alpha}}$	0.07263	0.01261	0.07506	0.01438	0.07259	0.01314	0.08091	0.01330
	$\hat{\boldsymbol{\beta}}$	0.02346	0.01618	-0.01020	0.02266	-0.00038	0.01876	-0.04619	0.01616
	$\hat{\boldsymbol{\theta}}$	0.01869	0.00387	0.03658	0.00718	0.02969	0.00545	0.04738	0.00620
100	$\hat{\boldsymbol{\alpha}}$	0.06952	0.01049	0.06886	0.01192	0.07311	0.01125	0.07755	0.01154
	$\hat{\boldsymbol{\beta}}$	0.01985	0.01204	0.00333	0.01801	0.00940	0.01490	-0.03394	0.01179
	$\hat{\boldsymbol{\theta}}$	0.02082	0.00289	0.02972	0.00504	0.02720	0.00403	0.04313	0.00462
150	$\hat{\boldsymbol{\alpha}}$	0.07175	0.00883	0.07071	0.00976	0.06843	0.00875	0.07544	0.00925
	$\hat{\boldsymbol{\beta}}$	0.01375	0.00757	-0.00021	0.01047	0.00583	0.00880	-0.02554	0.00757
	$\hat{\boldsymbol{\theta}}$	0.02019	0.00203	0.02670	0.00334	0.02282	0.00253	0.03539	0.00300

Table 3: Parameters Estimation of the NWPD in Case3

$\alpha = 2.5$; $\beta = 1.4$; $\theta = 3$									
		MLE		LS		WLS		MPS	
n		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	$\hat{\alpha}$	0.10541	0.16682	0.00463	0.26089	-0.00070	0.19990	-0.05549	0.15356
	$\hat{\beta}$	0.23262	0.27760	0.01137	0.28222	0.01745	0.26756	-0.10513	0.15056
	$\hat{\theta}$	-0.14786	0.26329	0.17064	0.37294	0.14710	0.33647	0.23859	0.28018
20	$\hat{\alpha}$	0.09104	0.07603	0.02849	0.09680	0.04973	0.09084	-0.01867	0.07256
	$\hat{\beta}$	0.09072	0.09026	-0.01640	0.10519	-0.00022	0.09837	-0.10143	0.07326
	$\hat{\theta}$	-0.00731	0.11749	0.15791	0.20862	0.13475	0.17576	0.23161	0.16749
30	$\hat{\alpha}$	0.10702	0.05868	0.06003	0.06560	0.07502	0.05752	0.02195	0.04676
	$\hat{\beta}$	0.06631	0.05174	-0.00650	0.06555	0.00840	0.05769	-0.07681	0.04451
	$\hat{\theta}$	0.00976	0.07456	0.12117	0.13240	0.10016	0.11419	0.18589	0.10650
50	$\hat{\alpha}$	0.10686	0.03954	0.08656	0.04851	0.09115	0.04237	0.04970	0.03057
	$\hat{\beta}$	0.03926	0.02630	0.00077	0.03796	0.01166	0.03187	-0.05819	0.02490
	$\hat{\theta}$	0.03585	0.04650	0.09915	0.08483	0.07750	0.06748	0.15741	0.06933
70	$\hat{\alpha}$	0.10089	0.03284	0.08277	0.04059	0.08603	0.03184	0.05805	0.02535
	$\hat{\beta}$	0.02526	0.01877	-0.01099	0.02628	-0.00041	0.02176	-0.04975	0.01874
	$\hat{\theta}$	0.05312	0.03554	0.11412	0.06877	0.09251	0.05187	0.14796	0.05454
100	$\hat{\alpha}$	0.09511	0.02651	0.08775	0.03503	0.08667	0.02690	0.06230	0.01956
	$\hat{\beta}$	0.02138	0.01396	0.00359	0.02089	0.01012	0.01728	-0.03655	0.01368
	$\hat{\theta}$	0.06288	0.02779	0.09513	0.04872	0.07905	0.03738	0.13553	0.04345
150	$\hat{\alpha}$	0.09385	0.01966	0.08946	0.02308	0.09334	0.02102	0.07110	0.01528
	$\hat{\beta}$	0.01480	0.00878	-0.00022	0.01214	0.00628	0.01021	-0.02751	0.00878
	$\hat{\theta}$	0.06098	0.01931	0.08748	0.03176	0.07747	0.02508	0.11425	0.02941

Table 4: The Average 95%C Lengths Interval of Case 1 for Different Methods

n		ACI	BN	BP	n		ACI	BN	BP
10	$\hat{\alpha}$	0.8173	0.0264	0.0256	50	$\hat{\alpha}$	0.3779	0.0116	0.0120
	$\hat{\beta}$	0.5297	0.0171	0.0169		$\hat{\beta}$	0.1823	0.0058	0.0058
	$\hat{\theta}$	0.7061	0.0219	0.0216		$\hat{\theta}$	0.3394	0.0107	0.0113
20	$\hat{\alpha}$	0.5792	0.0190	0.0178	70	$\hat{\alpha}$	0.3160	0.0100	0.0100
	$\hat{\beta}$	0.3193	0.0101	0.0103		$\hat{\beta}$	0.1517	0.0049	0.0048
	$\hat{\theta}$	0.5032	0.0163	0.0160		$\hat{\theta}$	0.2957	0.0094	0.0102
30	$\hat{\alpha}$	0.4771	0.0149	0.0157	100	$\hat{\alpha}$	0.2598	0.0083	0.0085
	$\hat{\beta}$	0.2425	0.0078	0.0082		$\hat{\beta}$	0.1257	0.0038	0.0039
	$\hat{\theta}$	0.4165	0.0135	0.0134		$\hat{\theta}$	0.2449	0.0077	0.0081
150	$\hat{\alpha}$	0.2160	0.0069	0.0069					
	$\hat{\beta}$	0.1011	0.0031	0.0033					
	$\hat{\theta}$	0.2041	0.0064	0.0066					

Table 5: The Average 95%C Lengths Interval of Case 2 for Different Methods

n		ACI	BN	BP	n		ACI	BN	BP
10	$\hat{\alpha}$	1.0497	0.0343	0.0327	50	$\hat{\alpha}$	0.3789	0.0116	0.0115
	$\hat{\beta}$	1.7217	0.0556	0.0550		$\hat{\beta}$	0.5924	0.0187	0.0189
	$\hat{\theta}$	0.6204	0.0191	0.0193		$\hat{\theta}$	0.2752	0.0086	0.0087
20	$\hat{\alpha}$	0.6050	0.0199	0.0181	70	$\hat{\alpha}$	0.3397	0.0106	0.0109
	$\hat{\beta}$	1.0378	0.0327	0.0336		$\hat{\beta}$	0.4931	0.0158	0.0155
	$\hat{\theta}$	0.4489	0.0148	0.0139		$\hat{\theta}$	0.2256	0.0071	0.0072
30	$\hat{\alpha}$	0.4771	0.0151	0.0158	100	$\hat{\alpha}$	0.2826	0.0092	0.0091
	$\hat{\beta}$	0.7881	0.0252	0.0267		$\hat{\beta}$	0.4085	0.0123	0.0128
	$\hat{\theta}$	0.3523	0.0115	0.0111		$\hat{\theta}$	0.1882	0.0059	0.0061
150	$\hat{\alpha}$	0.2373	0.0075	0.0074					
	$\hat{\beta}$	0.3285	0.0101	0.0108					
	$\hat{\theta}$	0.1577	0.0050	0.0051					

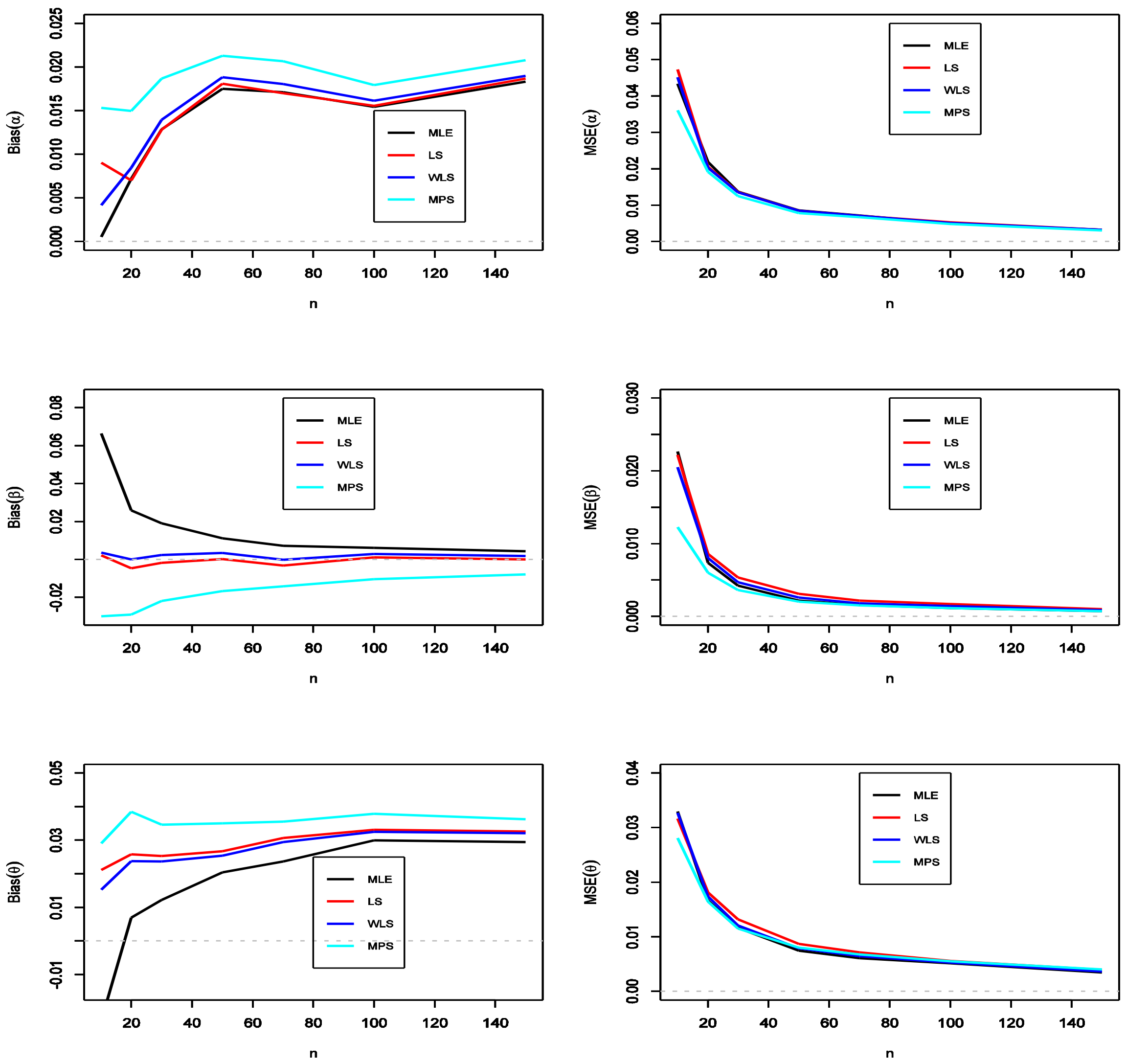

Figure 1: Bias and MSE of the Estimates with Variation of Sample Size in case 1

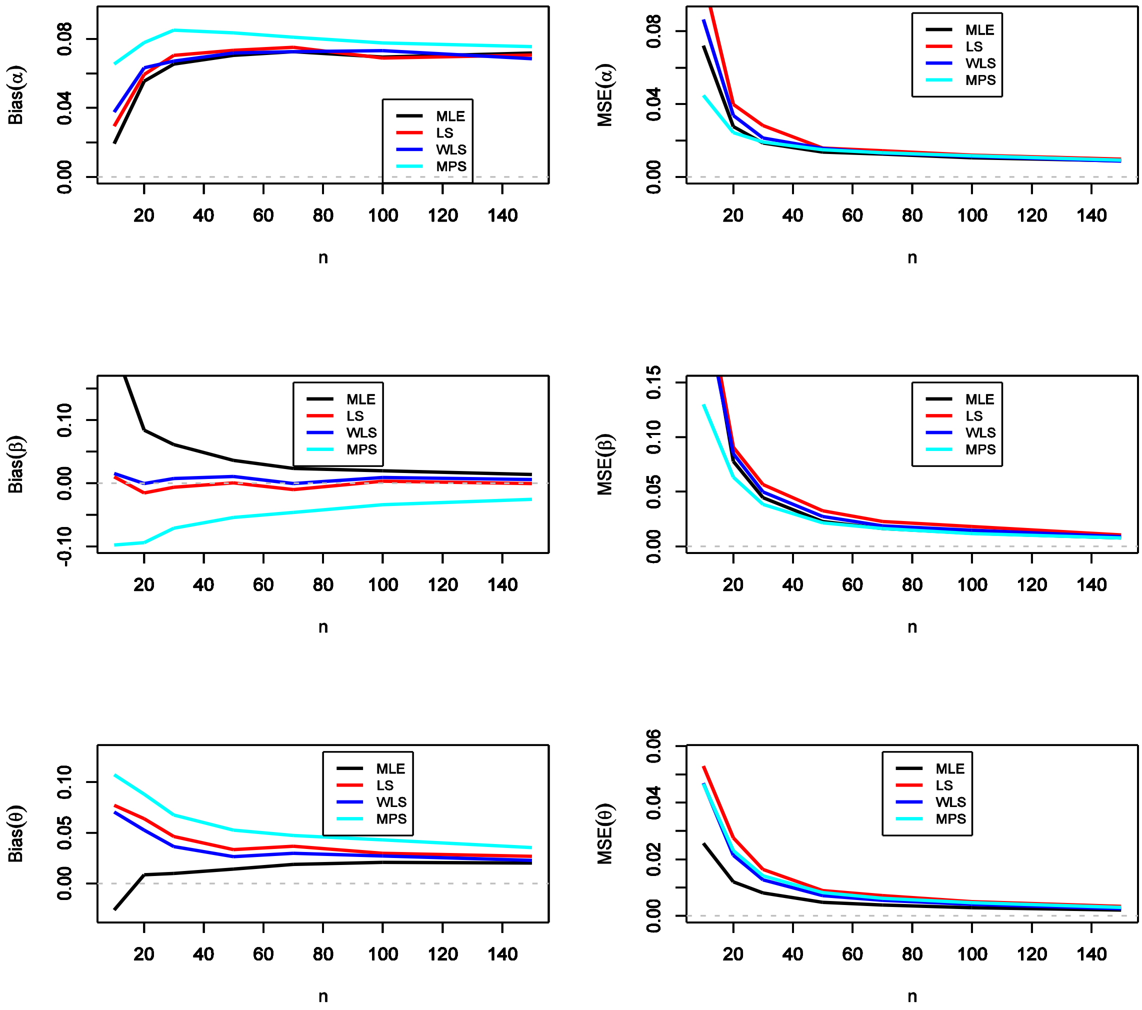

Figure 2: Bias and MSE of the Estimates with Variation of Sample Size in case 2

Table 6: The Average 95%C Lengths Interval of Case 3 for Different Methods

n		ACI	BN	BP	n		ACI	BN	BP
10	$\hat{\alpha}$	1.5476	0.0487	0.0488	50	$\hat{\alpha}$	0.6861	0.0211	0.0221
	$\hat{\beta}$	1.8541	0.0599	0.0592		$\hat{\beta}$	0.6379	0.0202	0.0204
	$\hat{\theta}$	1.9271	0.0589	0.0616		$\hat{\theta}$	0.8378	0.0264	0.0272
20	$\hat{\alpha}$	1.0455	0.0331	0.0351	70	$\hat{\alpha}$	0.5795	0.0181	0.0193
	$\hat{\beta}$	1.1176	0.0352	0.0362		$\hat{\beta}$	0.5310	0.0170	0.0167
	$\hat{\theta}$	1.3707	0.0450	0.0436		$\hat{\theta}$	0.7084	0.0225	0.0227
30	$\hat{\alpha}$	0.8553	0.0268	0.0285	100	$\hat{\alpha}$	0.4833	0.0157	0.0156
	$\hat{\beta}$	0.8487	0.0272	0.0288		$\hat{\beta}$	0.4400	0.0133	0.0137
	$\hat{\theta}$	1.0776	0.0351	0.0349		$\hat{\theta}$	0.5883	0.0187	0.0190
150	$\hat{\alpha}$	0.4137	0.0133	0.0133					
	$\hat{\beta}$	0.3538	0.0109	0.0116					
	$\hat{\theta}$	0.4840	0.0154	0.0157					

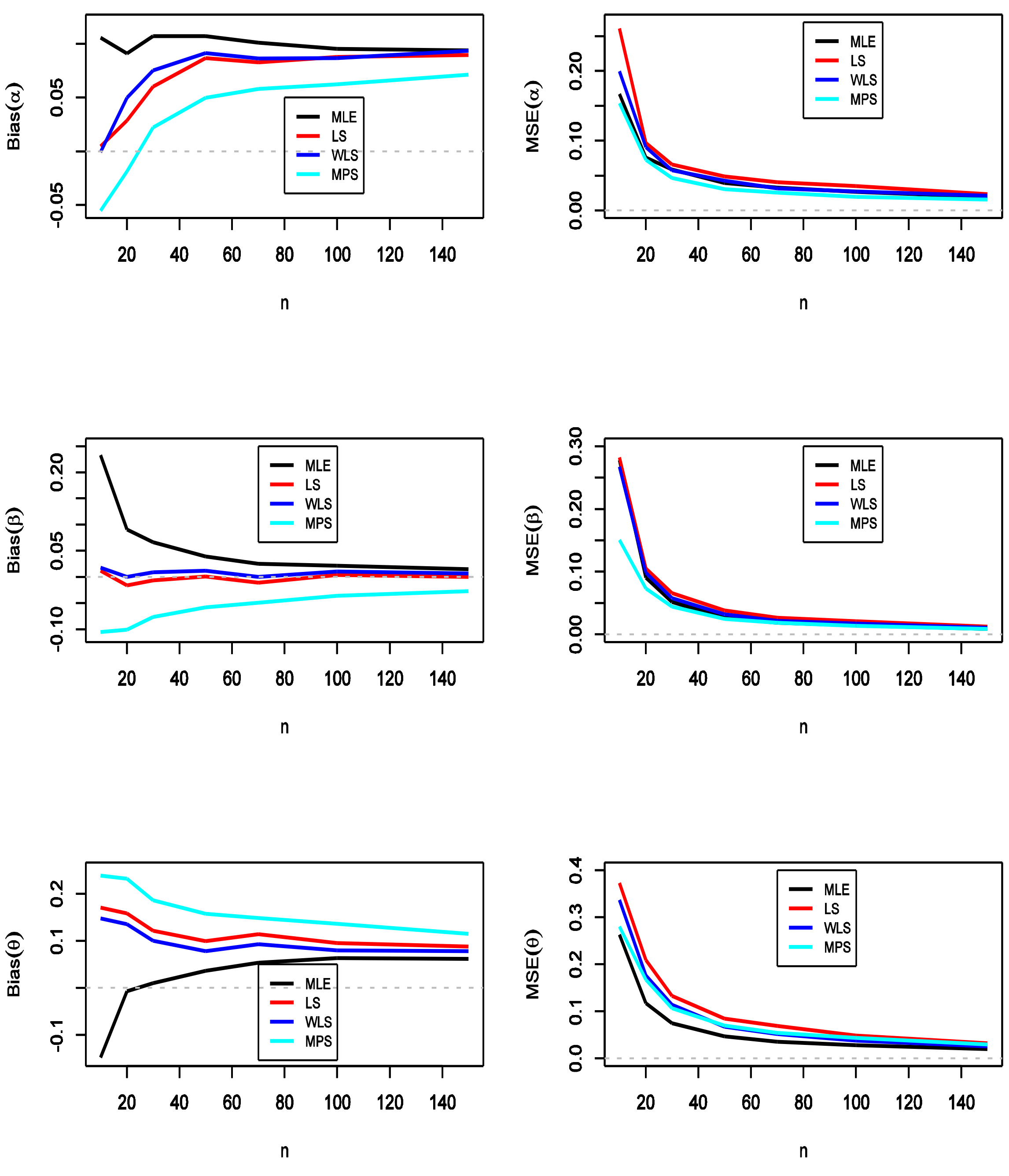

Figure 3: Bias and MSE of the Estimates with Variation of Sample Size in Case 3

Table 7: Moment Generation Function and Reliability Function of NWPD case1

n		Mean	Variance	Skewness	Kurtosis	Survival	hazard
10	MLE	5.3727	98.7379	4.9559	47.3680	0.0780	0.2215
	LS	5.6774	91.4838	4.3108	35.5632	0.1094	0.1567
	WLS	5.7602	96.3839	4.3858	36.8302	0.1076	0.1562
	MPS	6.3089	109.5282	4.2129	33.9497	0.1167	0.1260
20	MLE	5.2917	83.9008	4.4881	38.6032	0.1001	0.1884
	LS	5.4830	82.9983	4.2231	34.1160	0.1172	0.1567
	WLS	5.4313	81.6339	4.2306	34.2369	0.1148	0.1613
	MPS	5.8572	91.1746	4.1056	32.2341	0.1225	0.1351
30	MLE	5.5591	91.4141	4.4454	37.8571	0.0999	0.1739
	LS	5.6403	88.8767	4.2605	34.7281	0.1111	0.1558
	WLS	5.6044	88.2109	4.2771	35.0028	0.1090	0.1596
	MPS	5.9635	95.8611	4.1489	32.9188	0.1171	0.1362
50	MLE	5.4104	82.0562	4.2712	34.9049	0.1095	0.1681
	LS	5.4416	80.0266	4.1570	33.0484	0.1165	0.1579
	WLS	5.4096	79.5010	4.1730	33.3057	0.1148	0.1613
	MPS	5.6841	85.1329	4.0797	31.8281	0.1208	0.1425
70	MLE	5.4565	82.6445	4.2402	34.3944	0.1099	0.1649
	LS	5.5149	82.7444	4.1774	33.3763	0.1138	0.1574
	WLS	5.4791	81.8147	4.1828	33.4627	0.1128	0.1600
	MPS	5.6825	85.6364	4.0992	32.1331	0.1184	0.1449
100	MLE	5.4504	81.7202	4.2119	33.9335	0.1110	0.1636
	LS	5.4547	79.9201	4.1381	32.7478	0.1154	0.1579
	WLS	5.4317	79.5550	4.1500	32.9367	0.1140	0.1604
	MPS	5.6368	84.6415	4.1128	32.3474	0.1171	0.1480
150	MLE	5.5643	85.2922	4.2164	34.0061	0.1091	0.1602
	LS	5.6261	86.7768	4.2013	33.7614	0.1102	0.1562
	WLS	5.5809	85.3189	4.1988	33.7201	0.1099	0.1584
	MPS	5.7097	87.8573	4.1482	32.9085	0.1135	0.1487

Table 8: Moment Generation Function and Reliability Function of NWPD case2

n		Mean	Variance	Skewness	Kurtosis	Survival	hazard
10	MLE	1.2568	3.1310	3.3089	21.1752	0.0150	5.0663
	LS	1.4607	3.1565	2.6712	14.3588	0.0212	3.4584
	WLS	1.4642	3.2259	2.7049	14.6751	0.0195	3.5375
	MPS	1.6281	3.6336	2.5261	13.0468	0.0177	2.9784
20	MLE	1.3922	3.5007	3.0911	18.6501	0.0126	4.4254
	LS	1.5035	3.4822	2.7517	15.1235	0.0159	3.5884
	WLS	1.4934	3.5467	2.8164	15.7585	0.0147	3.7205
	MPS	1.6200	3.8319	2.6457	14.1218	0.0144	3.2068
30	MLE	1.4623	3.8015	3.0555	18.2567	0.0105	4.2526
	LS	1.5288	3.7839	2.8536	16.1313	0.0125	3.7261
	WLS	1.5184	3.8271	2.9062	16.6686	0.0118	3.8388
	MPS	1.6312	4.0545	2.7298	14.9125	0.0121	3.3293
50	MLE	1.4922	3.8600	2.9997	17.6521	0.0108	4.0513
	LS	1.5331	3.8380	2.8716	16.3132	0.0121	3.7429
	WLS	1.5215	3.8651	2.9184	16.7946	0.0116	3.8379
	MPS	1.6107	4.0348	2.7709	15.3106	0.0118	3.4320
70	MLE	1.5135	3.9524	2.9894	17.5418	0.0103	4.0017
	LS	1.5460	3.9477	2.8955	16.5577	0.0109	3.7930
	WLS	1.5368	3.9648	2.9302	16.9168	0.0106	3.8610
	MPS	1.6051	4.0949	2.8153	15.7469	0.0111	3.5170

100	MLE	1.5221	3.9913	2.9861	17.5068	0.0102	3.9714
	LS	1.5460	3.9847	2.9154	16.7628	0.0109	3.7927
	WLS	1.5382	3.9969	2.9435	17.0564	0.0105	3.8630
	MPS	1.5920	4.1077	2.8557	16.1523	0.0107	3.6015
150	MLE	1.5412	4.0977	2.9892	17.5392	0.0097	3.9357
	LS	1.5555	4.1102	2.9556	17.1829	0.0100	3.8357
	WLS	1.5481	4.1096	2.9759	17.3977	0.0098	3.8854
	MPS	1.5928	4.1852	2.8931	16.5327	0.0101	3.6634

Table 9: Moment Generation Function and Reliability Function of NWPD case3

n		Mean	Variance	Skewness	Kurtosis	Survival	hazard
10	MLE	1.9550	0.5528	0.2184	2.7515	0.2450	1.1744
	LS	2.0114	0.4839	0.1150	2.7153	0.2678	0.9246
	WLS	2.0020	0.4859	0.1221	2.7166	0.2681	0.9320
	MPS	2.0227	0.4711	0.0952	2.7124	0.2648	0.8507
20	MLE	2.0127	0.5439	0.1770	2.7328	0.2379	1.0813
	LS	2.0364	0.5029	0.1222	2.7167	0.2525	0.9479
	WLS	2.0362	0.5106	0.1304	2.7184	0.2489	0.9676
	MPS	2.0435	0.4892	0.1041	2.7136	0.2536	0.8858
30	MLE	2.0457	0.5512	0.1665	2.7289	0.2276	1.0626
	LS	2.0610	0.5241	0.1314	2.7187	0.2372	0.9774
	WLS	2.0637	0.5321	0.1381	2.7203	0.2332	0.9966
	MPS	2.0681	0.5080	0.1114	2.7147	0.2405	0.9129
50	MLE	2.0548	0.5425	0.1531	2.7245	0.2299	1.0298
	LS	2.0726	0.5292	0.1306	2.7185	0.2330	0.9832
	WLS	2.0606	0.5309	0.1385	2.7204	0.2341	0.9939
	MPS	2.0713	0.5128	0.1147	2.7152	0.2386	0.9281
70	MLE	2.0637	0.5440	0.1499	2.7236	0.2274	1.0236
	LS	2.0756	0.5345	0.1344	2.7194	0.2297	0.9907
	WLS	2.0713	0.5380	0.1400	2.7208	0.2288	1.0015
	MPS	2.0748	0.5205	0.1207	2.7164	0.2346	0.9438
100	MLE	2.0656	0.5438	0.1487	2.7232	0.2270	1.0193
	LS	2.0713	0.5343	0.1364	2.7199	0.2302	0.9901
	WLS	2.0695	0.5383	0.1413	2.7211	0.2287	1.0015
	MPS	2.0765	0.5265	0.1259	2.7174	0.2318	0.9588
150	MLE	2.0761	0.5476	0.1471	2.7227	0.2231	1.0174
	LS	2.0824	0.5434	0.1397	2.7207	0.2240	1.0019
	WLS	2.0779	0.5454	0.1439	2.7218	0.2238	1.0095
	MPS	2.0811	0.5341	0.1311	2.7186	0.2275	0.9720

In the first case, we note the efficiency of MPS method compared to MLE, LS and WLS, based on values of MSE and Bias. We also note the difference in the Bias values with the efficiency of the estimated MSE, this is due to the low variance of the parameters in this model, as shown in Table 1 and Figure 1. Table7 confirm the efficiency of the MPS method in estimating the quality of the model compared to other methods. A number of Standards have been used, where the most effective and useful in the practical life are the Survival and Hazard. As MPS has high survival and low Hazard, therefore it is the best method. Standards of normality distribution (Skewness, Kurtosis) by using parameter estimation of MPS are better than other methods. With regard to the interval estimation, the Bootstrap is more efficient than the traditional method ACI, as shown in Table 4. In case 2 and 3 also MPS is more efficient than other methods, but in parameter estimation when α parameter increase, the scale parameter (θ) is better in MLE. Finally, MPS is the best method followed by LS, WLS and MLE according to survival and hazard life testing measurement.

7. Application of Real Data

Mahmoud et al. (2016) discussed the real data set of sample size 63 observed failure times, but we use 62 observation failure times where we remove the observation (1.137) because The MPS method is difference between the observations of the cumulative function leads to observations of density function, Almetwally and Almongy (2018) used this data to fit of the Marshall–Olkin Extended Weibull distribution under adaptive type-II progressive censoring scheme. The data is represented the strength data measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows.

We have estimated parameters of NWPD, computed the Kolmogorov-Smirnov (KS) distance, survival, hazard, measure of skewness and measure of kurtosis for this model under different estimation methods.

Table 10: Parameter Estimation of NWPD

	MLE	LS	WLS	MPS
$\hat{\boldsymbol{\alpha}}$ (std)	1.0697 (0.1591)	0.9636 (0.1084)	0.9533 (0.0055)	0.9519 (0.0196)
$\hat{\boldsymbol{\beta}}$ (std)	2.1423 (0.2105)	2.0301 (1.0218)	2.0917 (0.0461)	2.0035 (0.0952)
$\hat{\boldsymbol{\theta}}$ (std)	1.4704 (0.0528)	1.4043 (0.1509)	1.3947 (0.0079)	1.3973 (0.0285)
D	0.0744	0.0625	0.0690	0.0613
p-value	0.8573	0.9562	0.9095	0.9626
Mean	0.50586	0.53018	0.51217	0.52782
variance	0.1347	0.14640	0.13837	0.14645
Skewness	1.2035	1.19264	1.20559	1.2020
Kurtosis	4.85847	4.8174	4.8663	4.8528
Survival	0.86589	0.87512	0.8893	0.8734
hazard	0.53494	0.51074	0.4789	0.5138

From Table 10 we can see that all estimates provides a good fit to this data set. We note that MPS and WLS estimators give the lowest standard deviation (std) of parameters. We also present the results of goodness-of-fit tests, the Kolmogorov-Smirnov (KS) test, in order to show that the NWPD can be used to model these data sets, where the MPS has the highest p-value and lowest distances (D) followed by LS, WLS and MLE. Finally, survival and hazard life testing measurement are used.

8. Conclusion

In this paper, we estimated the parameters of the NWPD under MLE, LS, WLS and MPS methods. The simulation study have performed an extensive to compare famous methods of estimation for parameters of NWPD model. Besides, we have obtained confidence interval for parameters of model the ACI and two parametric bootstrap. Furthermore, we have obtained estimates for the parameters α, β and θ along with 95% confidence intervals. Furthermore, we have obtained survival and hazard life testing measurement to compare these methods. The standards of moment function (Mean, Variance, Skewness, and Kurtosis) were obtained for NWPD by using MLE, LS, WLS and MPS. We note that, when the sample size is small, the MPS estimators behave quite better than the ML, LS and WLS estimators, where the bias and MSE are lower than the other methods, and in the large sample size, we find the behavior of the MPS and other methods have approximately similar result. We have also noted that increasing the value of the shape parameter α will lead to increase in the bias and MSE for scale parameter of the different estimators of the NWPD. In general, MPS is the best followed by LS, WLS and MLE according to survival and hazard life testing measurement. Finally, we provided a real data example where the parameter estimation of NWPD has been obtained. From both simulation study and real data examples, we observe that the performance of MPS estimators is good alternative to the usual MLE method in this model.

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