Journal of Data Science,17(3). P. 535 -550,2019
DOI:10.6339/JDS.201907_17(3).0005

BIVARIATE GENERALIZED BURR AND RELATED DISTRIBUTIONS: PROPERTIES AND ESTIMATION

Hiba Z. Muhammed[1*]

[1]*Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt.*

Abstract

Compound distributions gained their importance from the fact that natural factors have compound effects, as in the medical, social and logical experiments. Dubey (1968) introduced the compound Weibull by compounding Weibull distribution with gamma distribution. The main aim of this paper is to define a bivariate generalized Burr (compound Weibull) distribution so that the marginals have univariate generalized Burr distributions. Several properties of this distribution such as marginals, conditional distributions and product moments have been discussed. The maximum likelihood estimates for the unknown parameters of this distribution and their approximate variance- covariance matrix have been obtained. Some simulations have been performed to see the performances of the MLEs. One data analysis has been performed for illustrative purpose.

Keywords: Burr distribution; Compound Weibull distribution; Weibull gamma distribution; Generalized Burr distribution; Maximum likelihood estimation.

[*] Corresponding author
Email: hiba_stat@cu.edu.eg.

1. Introduction

Dubey (1968) introduced the generalized Burr distribution for the first time by compounding Weibull distribution with gamma distribution. She derived the compound Weibull distribution by assuming a conditional random variable X follow the Weibull distribution, and its scale parameter follow a gamma distribution. The resulting unconditional pdf is called the compound Weibull (CW) distribution. Because of, the Burr distribution which defined by Burr (1942) is resulted as a special case of CW distribution , she renamed the CW distribution by generalized Burr distribution.

A random variable with the generalized Burr (GB) distribution has a pdf and a cdf, for $x > 0$, in the following form

$$f_{GB}(x;\alpha,\delta,\vartheta)=\frac{\alpha\vartheta}{\delta}x^{\alpha-1}\left(1+\frac{x^{\alpha}}{\delta}\right)^{-\vartheta-1},F_{GB}(x;\alpha,\delta,\vartheta)=1-\left(1+\frac{x^{\alpha}}{\delta}\right)^{-\vartheta}$$

Respectively, where the quantities $\delta>0$ is a scale parameter and $\alpha>0$ and $\vartheta>0$ are shape parameters respectively. From now on it will be denoted by $GB(\alpha,\delta,\vartheta)$.

It is clear that, for $\delta = 1$, the GB distribution reduces to the Burr distribution.

The GB distribution has a considerable attention in one dimension by many authors such as Gottschalk et al (1997), Mahmoud et al (2014) and Qutb and Rajhi (2016).

The aim of this paper is to consider the GB distribution in two dimension by constructing the bivariate GB distribution for the first time . The proposed bivariate generalized Burr (BGB) distribution is constructed from three independent GB distributions using a minimization process according to Marshall and Olkin (1967) . These authors introduced a multivariate exponential distribution whose marginals have exponential distributions and proposed a bivariate Weibull distribution.

The new BGB distribution is a singular distribution, and it can be used quit conveniently if there are ties in the data. The BGB distribution can be interpreted as Competing risk, Shock, Stress and Maintenance Model.

The paper is organized as follows: In Section 2, the BGB distribution is introduced and the representations for the joint survival function and pdf are obtained. The conditional and marginal distributions, joint cdf, joint hazard function and product moments of the BGB model are presented in Section3. The maximum likelihood estimation, approximate variance-covariance matrix and asymptotic confidence intervals for BGB distribution are provided in Section 4. Related distributions to BGB distribution are presented in Section 5. An absolutely continuous BGB distribution is introduced in Section 6. For illustrative purpose an empirical application is presented in Section 7. Finally conclude the paper in Section 8.

2. Bivariate Generalized Burr distribution

Suppose U_1, $U_2$and $U_3$are three independent random variables such that $U_i\sim GB(\alpha,\delta,\vartheta_i)$ for $i = 1,2,3$. Define $X_1 = min(U_1,U_3)$ and $X_2 = min(U_2,U_3)$, then it is said that the bivariate vector (X_1,X_2) has BGB distribution with parameters $(\vartheta_1,\vartheta_2,\vartheta_3,\alpha,\delta)$ denoted by BGB $(\vartheta_1,\vartheta_2,\vartheta_3,\alpha,\delta)$. Then, the joint survivor function of BGB distribution is given as follows

$$S_{BGB}\left(x_1,x_2\right)=S_{GB}\left(x_1;\alpha,\delta,\vartheta_1\right)S_{GB}\left(x_2;\alpha,\delta,\vartheta_2\right)S_{GB}\left(x_3;\alpha,\delta,\vartheta_3\right)$$

$$S_{BGB}(x_1,x_2)=\left(1+\frac{x_1^{\alpha}}{\delta}\right)^{-\vartheta_1}\left(1+\frac{x_2^{\alpha}}{\delta}\right)^{-\vartheta_2}\left(1+\frac{x_3^{\alpha}}{\delta}\right)^{-\vartheta_3} \qquad (2.1)$$

where $x_3 = \max(x_1, x_2)$.

The following Propositions will provide the joint survival function , joint pdf, the marginal distributions and conditional pdf.

Proposition 2.1: If $(X_1, X_2) \sim BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$. Then, the joint survival function of (X_1, X_2) can be written as

$$S_{BGB}(x_1,x_2)=\begin{cases} S_1(x_1,x_2) & if \quad 0<x_1<x_2<\infty \\ S_2(x_1,x_2) & if \quad 0<x_2<x_1<\infty \\ S_3(x) & if \quad 0<x_1=x_2=x<\infty, \end{cases} \qquad (2.2)$$

Where

$$S_1\,(x_1,x_2) = S_{GB}(x_1;\alpha,\delta,\vartheta_1)S_{GB}(x_2;\alpha,\delta,\vartheta_{23})$$

$$S_2\,(x_1,x_2) = S_{GB}\,(x_1;\alpha,\delta,\vartheta_{13})S_{GB}(x_2;\alpha,\delta,\vartheta_2)$$

$$S_3\,(x) = S_{GB}(x\,;\alpha,\delta,\vartheta_{123}),$$

And $\vartheta_{13} = \vartheta_1 + \vartheta_3$, $\vartheta_{23} = \vartheta_2 + \vartheta_3$ and $\vartheta_{123} = \vartheta_1 + \vartheta_2 + \vartheta_3$

Proposition 2.2: If $(X_1, X_2) \sim BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$. Then, the joint pdf of (X_1, X_2) is given as

$$f_{BGB}\left(x_1,x_2\right)=\begin{cases} f_1\left(x_1,x_2\right) & \text{if} \quad 0<x_1<x_2<\infty \\ f_2\left(x_1,x_2\right) & \text{if} \quad 0<x_2<x_1<\infty \\ f_3(x) & \text{if} \quad 0<x_1=x_2=x<\infty \end{cases} \qquad (2.3)$$

Where

$$f_1\,(x_1,x_2) = f_{GB}\,(x_1;\alpha,\delta,\vartheta_1)\,f_{GB}(x_2;\alpha,\delta,\vartheta_{23})$$

$$f_2\,(x_1,x_2) = f_{GB}(x_1;\alpha,\delta,\vartheta_{13})\,f_{GB}(x_2;\alpha,\delta,\vartheta_2)$$

and

$$f_3\,(x) = \frac{\vartheta_3}{\vartheta_{123}}\;f_{GB}(x\;;\alpha,\delta,\vartheta_{123}).$$

Proof. The expressions for $f_1(\cdot,\cdot)$ and $f_1(\cdot,\cdot)$ can be obtained simply by taking $\frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1,X_2}(x_1,x_2)$ for $x_1<x_2$ and $x_2<x_1$ respectively. But $f_3(.)$ cannot be obtained in the same way. Using the fact that

$$\int_0^{\infty}\int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 + \int_0^{\infty}\int_0^{x_1} f_2(x_1,x_2)dx_2dx_1 + \int_0^{\infty} f_3(x)dx = 1,$$

$$\int_0^{\infty}\int_0^{x_2} f_1(x_1,x_2)dx_1dx_2=\frac{\vartheta_1}{\vartheta_{123}} \text{ and } \int_0^{\infty}\int_0^{x_1} f_2(x_1,x_2)dx_2dx_1=\frac{\vartheta_2}{\vartheta_{123}}$$

Hence

$$\int_0^{\infty} f_3(x)dx=\frac{\vartheta_3}{\vartheta_{123}}$$

Note that

$$\int_0^{\infty} f_3(x)dx=\frac{\alpha}{\delta}\vartheta_3\int_0^{\infty} x^{\alpha-1}(1+\frac{x^{\alpha}}{\delta})^{\vartheta_{123}-1}dx=\frac{\vartheta_3}{\vartheta_{123}}.$$

Therefore, the results follow.

It should be mentioned that the BGB distribution has both an absolute continuous part and a singular part, similar to Marshall and Olkin's bivariate exponential model. The joint survival function of (X_1, X_2) can be expressed explicitly as a mixture of an absolutely continuous part and a singular part in the following form

$$S_{X_1,X_2}\left(x_1,x_2\right)=\frac{\vartheta_{12}}{\vartheta_{123}}S_a\left(x_1,x_2\right)+\frac{\vartheta_3}{\vartheta_{123}}S_s\left(x_3\right) \tag{2.4}$$

where $x_3=\max(x_1,x_2)$,$S_3(x_3)=S_{GB}(x_3,\alpha,\delta,\vartheta_{123})$ and

$$S_{\mathrm{a}}(x_1,x_2)=\frac{\vartheta_{123}}{\vartheta_{12}}S_{GB}(x_1;\alpha,\delta,\vartheta_1)S_{GB}(x_2;\alpha,\delta,\vartheta_2)S_{GB}(x_3;\alpha,\delta,\vartheta_3)$$
$$-\frac{\vartheta_3}{\vartheta_{12}}S_{GB}(x_3;\alpha,\delta,\vartheta_{123})$$

Here $S_s(.,.)$ and $S_a(\cdot,\cdot)$ are the singular and the absolutely continuous part respectively.

As a result, the joint pdf of (X_1, X_2)can be also expressed as a mixture of an absolutely continuous part and a singular part in the following form

$$f_{X_1,X_2}\left(x_1,x_2\right)=\frac{\vartheta_{12}}{\vartheta_{123}}f_a\left(x_1,x_2\right)+\frac{\vartheta_3}{\vartheta_{123}}f_s\left(x_3\right) \tag{2.5}$$

where

$$f_a(x_1,x_2)=\frac{\vartheta_{123}}{\vartheta_{12}}\times\begin{cases} f_{GB}(x_1;\vartheta_1)\cdot f_{GB}(x_2;\vartheta_2+\vartheta_3) & if \quad x_1<x_2 \\ f_{GB}(x_1;\vartheta_1+\vartheta_3)\cdot f_{GB}(x_2;\vartheta_2) & if \quad x_1>x_2 \end{cases}$$

and

$$f_s(x_3)=f_{GB}(x_3;\vartheta_{123}).$$

Clearly, here $f_a(x_1,x_2)$ and $f_s(x_3)$ are the absolutely continuous and singular part respectively.

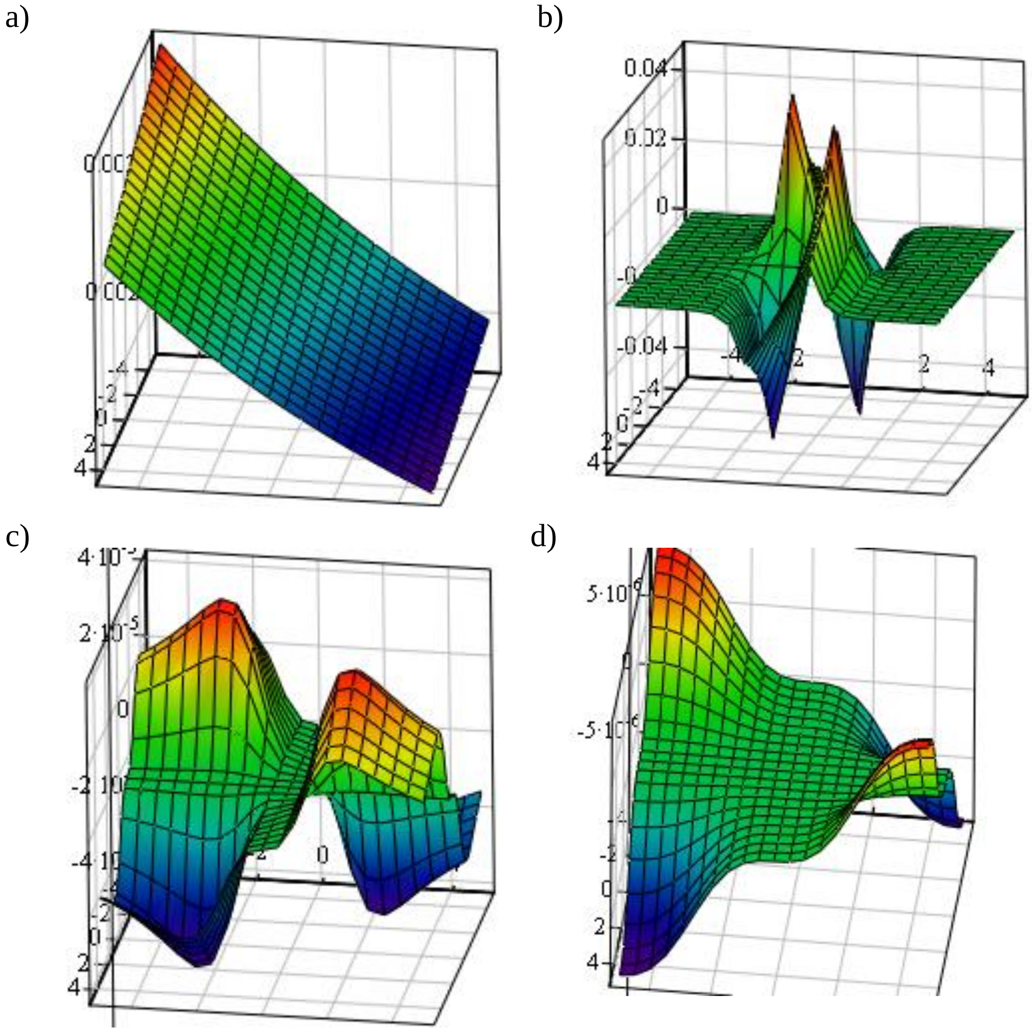

Figure1: Surface plots of the absolutely continuous part of the joint pdf of the BGB distribution for different values of ($v_1, v_2, v_3, \alpha, \delta$): (a) (2, 3, 4, 1, 10), (b) (0.2, 0.3, 0.4, 4,0.002), (c) (0.2, 0.3, 0.4, 4, 1) and (d) (0.2, 0.3, 0.4, 4, 2).

The absolutely continuous part of the BGB density may be unimodal depending on the values of α, δ ϑ_1, ϑ_2 and ϑ_3 that is $f_a(x_1, x_2)$ is unimodal and the respective modes are

$$\left\{[\frac{\delta\,(\alpha-1)}{\alpha\,\vartheta_1+1}]^{-\alpha}, [\frac{\delta\,(\alpha-1)}{\alpha(\vartheta_{23})+1}]^{-\alpha}\right\} \quad \text{and} \quad \left\{[\frac{\delta\,(\alpha-1)}{\alpha(\vartheta_{13})+1}]^{-\alpha}, [\frac{\delta\,(\alpha-1)}{\alpha\,\vartheta_2+1}]^{-\alpha}\right\}$$

The median for the BGB distribution is obtained as

$$\left[\delta\left(2^{1/\vartheta_{123}}-1\right)\right]^{\frac{1}{\alpha}}.$$

3. Different Properties

This section is devoted to introduce some basic properties of the BGB model. First, the marginal and conditional distributions of BGB model will provide.

Proposition 3.1: If $(X_1, X_2) \sim BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$.Then,

1. $X_1 \sim GB(\alpha, \delta, \vartheta_{13})$ and $X_2 \sim GB(\alpha, \delta, \vartheta_{23})$

2. $min(X_1, X_2) \sim GB(\alpha, \delta, \vartheta_{123})$
3. Conditional density is given by

$$f_{i/j}(x_1, x_2) = \begin{cases} f_{i/j}^{(1)}(x_i / x_j) & if \quad x_i < x_j \\ f_{i/j}^{(2)}(x_i / x_j) & if \quad x_j < x_i \\ f_{i/j}^{(3)}(x_i / x_j) & if \quad x_i = x_j, \end{cases}$$

where

$$f_{i/j}^{(1)}(x_i / x_j) = \frac{\alpha\, \vartheta_1}{\delta} x_i^{\alpha-1} (1 + \frac{x_i^{\alpha}}{\delta})^{-\vartheta_1 - 1},$$

$$f_{i/j}^{(2)}(x_i / x_j) = \frac{\alpha\, (\vartheta_{13})\vartheta_2}{\delta(\vartheta_{23})} x_i^{\alpha-1} (1 + \frac{x_i^{\alpha}}{\delta})^{-(\vartheta_{13})-1} (1 + \frac{x_j^{\alpha}}{\delta})^{\vartheta_3},$$

$$f_{i/j}^{(3)}(x_i / x_j) = \frac{\vartheta_3 x_i^{\alpha-1}}{(\vartheta_{23}) x_j^{\alpha-1}} (1 + \frac{x_i^{\alpha}}{\delta})^{-(\vartheta_{123})-1} (1 + \frac{x_j^{\alpha}}{\delta})^{\vartheta_{123}+1} .$$

Proof: They can be obtained by routine calculation.
The joint cdf of the BGB distribution is given a

$$F_{BGB}(x_1, x_2) = \begin{cases} F_{GB}(x_1; \vartheta_{13}) - F_{GB}(x_1; \vartheta_1)[1 - F_{GB}(x_2; \vartheta_{23})], & x_1 < x_2 \\ F_{GB}(x_2; \vartheta_{23}) - F_{GB}(x_2; \vartheta_2)[1 - F_{GB}(x_1; \vartheta_{13})], & x_2 < x_1 \\ 1 - F_{GB}(x; \vartheta_{123}), & x_1 = x_2 = x. \end{cases}$$

The joint hazard function of the BGB distribution is given as

$$h_{BGB}(x_1, x_2) = \begin{cases} \left(\frac{\alpha}{\delta}\right)^2 (\vartheta_{23})\vartheta_1 \; x_1^{\alpha-1} x_2^{\alpha-1} \left(1 + \frac{x_1^{\alpha}}{\delta}\right)^{-1} \left(1 + \frac{x_2^{\alpha}}{\delta}\right)^{-1}, x_1 < x_2 \\ \left(\frac{\alpha}{\delta}\right)^2 (\vartheta_{13})\vartheta_2 \; x_1^{\alpha-1} x_2^{\alpha-1} \left(1 + \frac{x_1^{\alpha}}{\delta}\right)^{-1} \left(1 + \frac{x_2^{\alpha}}{\delta}\right)^{-1}, x_1 > x_2 \\ \frac{\alpha}{\delta} \vartheta_3 \, x^{\alpha-1} \left(1 + \frac{x^{\alpha}}{\delta}\right)^{-1}, & x_1 = x_2 = x. \end{cases}$$

Algorithm to generate from BGB distribution

Step 1. Generate U_1, U_2 and U_3 from $U(0,1)$,

Step 2. Compute

$$Z_1 = \left[\delta\left(U_1^{\frac{-1}{\vartheta_1}} - 1\right)\right]^{\frac{1}{\alpha}}, Z_2 = \left[\delta\left(U_2^{\frac{-1}{\vartheta_2}} - 1\right)\right]^{\frac{1}{\alpha}} and\ Z_3 = \left[\delta\left(U_3^{\frac{-1}{\vartheta_3}} - 1\right)\right]^{\frac{1}{\alpha}}$$

Step3. Obtain $X_1 = \min(Z_1, Z_3)$ and $X_2 = \min(Z_2, Z_3)$.

Proposition 3.2. If $(X_1, X_2) \sim BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$.Then, the r^{th} and s^{th} moment of X_1 and X_2 is given as:

$$E(X_1^r X_2^s) = \frac{\vartheta_1\vartheta_{23}\delta^{\frac{r+s}{\alpha}}}{\frac{r}{\alpha}+1}\, B\left[\left(\frac{r+s}{\alpha}+2\right),\left(\vartheta_{23}-\frac{r+s}{\alpha}-1\right)\right]$$

$$ {}_3F_2\left[\left(\frac{r+s}{\alpha}+2\right),\left(\frac{r}{\alpha}+1\right),\left(1-\vartheta_1+\frac{r}{\alpha}\right);\left(\frac{r}{\alpha}+2\right)(\vartheta_{23}+1);1\right]$$

$$+\frac{\vartheta_2\vartheta_{13}\delta^{\frac{r+s}{\alpha}}}{\frac{s}{\alpha}+1}\, B\left[\left(\frac{r+s}{\alpha}+2\right),\left(\vartheta_{13}-\frac{r+s}{\alpha}-1\right)\right]$$

$$.\quad {}_3F_2\left[\left(\frac{r+s}{\alpha}+2\right),\left(\frac{s}{\alpha}+1\right),\left(1-\vartheta_2+\frac{s}{\alpha}\right);\left(\frac{s}{\alpha}+2\right)(\vartheta_{13}+1);1\right]$$

$$+\vartheta_3\delta^{\frac{r+s}{\alpha}}\, B\left[\left(\frac{r+s}{\alpha}+1\right),\left(\vartheta_{123}-\frac{r+s}{\alpha}\right)\right].$$

Where

$B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is the beta function,

$pFq(b_1,\ldots,b_P;c_1,\ldots,c_q;u)=\sum_{i=0}^{\infty}\frac{(b_1)_i\ldots(b_P)_i}{(c_1)_i\ldots(c_q)_i}\frac{u^i}{i!}$ is a hypergeometric function,

$(b)_i=b(b+1)\ldots(b+i-1)=\frac{\Gamma(b+i)}{\Gamma(b)}(b\neq 0, i=1,2,\ldots).$ and p,q are nonnegative integers.

4. Maximum Likelihood Estimation

In this section, the maximum likelihood estimators (MLEs) of the unknown parameters of the BGB distribution will be considered. Suppose $\{(x_{11},x_{21}),\ldots,(x_{1n},x_{2n})\}$ is a random sample from $BGB(\vartheta_1,\vartheta_2,\vartheta_3,\alpha,\delta)$ distribution. Consider the following notation

$$I_1=\{i;x_{1i}<x_{2i}\},\quad I_2=\{i;x_{1i}>x_{2i}\},\quad I_3=\{x_{1i}=x_{2i}=x_i\},\quad I=I_1\cup I_2\cup I_3,$$

$$|I_1|=n_1,\quad |I_2|=n_2,\quad |I_3|=n_3,\qquad \text{and}\ n_1+n_2+n_3=n.$$

The log-likelihood function of the sample of size n is given by

$$\ln L(\underline{\theta})=\sum_{i\in I_1}\ln f_1(x_{1i},x_{2i})+\sum_{i\in I_2}\ln f_2(x_{1i},x_{2i})+\sum_{i\in I_3}\ln f_3(x_i)$$

$$\ln L(\underline{\theta}) = (2n_1 + 2n_2 + n_3)\ln(\alpha) - (2n_1 + 2n_2 + n_3)\ln(\delta) + n_1 \ln(\vartheta_1) + n_2 \ln(\vartheta_2)$$
$$+ n_3 \ln(\vartheta_3) + n_1 \ln(\vartheta_{23}) + n_2 \ln(\vartheta_{13}) + (\alpha - 1)[\sum_{I_1 \cup I_2} \ln x_{1i} + \sum_{I_1 \cup I_2} \ln x_{2i}$$
$$+ \sum_{I_3} \ln x_i] - (\vartheta_1 + 1)\sum_{I_1} c(x_{1i};\alpha,\delta) - (\vartheta_{23} + 1)\sum_{I_1} c(x_{2i};\alpha,\delta)$$
$$-(\vartheta_{13} + 1)\sum_{I_2} c(x_{1i};\alpha,\delta) - (\vartheta_2 + 1)\sum_{I_2} c(x_{2i};\alpha,\delta) - (\vartheta_{123} + 1)\sum_{I_3} c(x_i;\alpha,\delta).$$

where $\underline{\theta} = (\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$ and $c(x_i;\alpha,\delta) = \ln(1 + \frac{x_i^\alpha}{\delta})$.

The likelihood equations are as follows

$$\frac{n_1}{\hat{\vartheta}_1} + \frac{n_2}{\hat{\vartheta}_{13}} - \sum_{I_1} c(x_{1i};\hat{\alpha},\hat{\delta}) - \sum_{I_2} c(x_{1i};\hat{\alpha},\hat{\delta}) - \sum_{I_3} c(x_i;\hat{\alpha},\hat{\delta}) = 0,$$

$$\frac{n_2}{\hat{\vartheta}_2} + \frac{n_1}{\hat{\vartheta}_{23}} - \sum_{I_1} c(x_{2i};\hat{\alpha},\hat{\delta}) - \sum_{I_2} c(x_{2i};\hat{\alpha},\hat{\delta}) - \sum_{I_3} c(x_i;\hat{\alpha},\hat{\delta}) = 0,$$

$$\frac{n_3}{\hat{\vartheta}_3} + \frac{n_1}{\hat{\vartheta}_{23}} + \frac{n_2}{\hat{\vartheta}_{13}} - \sum_{I_1} c(x_{2i};\hat{\alpha},\hat{\delta}) - \sum_{I_2} c(x_{1i};\hat{\alpha},\hat{\delta}) - \sum_{I_3} c(x_i;\hat{\alpha},\hat{\delta}) = 0,$$

$$\frac{(2n_1 + 2n_2 + n_3)}{\hat{\alpha}} - (\hat{\vartheta}_1 + 1)\sum_{I_1} d(x_{1i};\hat{\alpha},\hat{\delta}) - (\hat{\vartheta}_2 + \hat{\vartheta}_3 + 1)\sum_{I_1} d(x_{2i};\hat{\alpha},\hat{\delta})$$
$$-(\hat{\vartheta}_2 + 1)\sum_{I_2} d(x_{2i};\hat{\alpha},\hat{\delta}) - (\hat{\vartheta}_{13} + 1)\sum_{I_2} d(x_{1i};\hat{\alpha},\hat{\delta}) - (\hat{\vartheta}_{123} + 1)\sum_{I_3} d(x_i;\hat{\alpha},\hat{\delta})$$
$$+ \sum_{I_1 \cup I_2} \ln x_{1i} + \sum_{I_1 \cup I_2} \ln x_{2i} + \sum_{I_3} \ln x_i,$$

and

$$-\frac{(2n_1 + 2n_2 + n_3)}{\hat{\delta}} + (\hat{\vartheta}_1 + 1)\sum_{I_1} g(x_{1i};\hat{\alpha},\hat{\delta}) + (\hat{\vartheta}_{23} + 1)\sum_{I_1} g(x_{2i};\hat{\alpha},\hat{\delta})$$
$$+(\hat{\vartheta}_2 + 1)\sum_{I_2} g(x_{2i};\hat{\alpha},\hat{\delta}) + (\hat{\vartheta}_{13} + 1)\sum_{I_2} g(x_{1i};\hat{\alpha},\hat{\delta}) + (\hat{\vartheta}_{123} + 1)\sum_{I_3} g(x_i;\hat{\alpha},\hat{\delta}).$$

where

$$d(x;\hat{\alpha},\hat{\delta}) = \frac{x^\alpha}{\delta}(1 + \frac{x^\alpha}{\delta})^{-1} \ln x$$

and

$$g(x;\hat{\alpha},\hat{\delta}) = \frac{x^\alpha}{\delta}(1 + \frac{x^\alpha}{\delta})^{-1}.$$

The numerical solutions for these equations will be considered to obtain $\hat{\vartheta}_1, \hat{\vartheta}_2, \hat{\vartheta}_3, \hat{\alpha}$ and $\hat{\delta}$.The evaluation of the MLEs was performed based on the following quantities for each sample size: the Average Estimates (AE) , the Mean Squared Error, (MSE) and Relative Absolute Bias (RAB) are estimated from R replications for $\hat{\vartheta}_1, \hat{\vartheta}_2, \hat{\vartheta}_3, \hat{\alpha}$ and $\hat{\delta}$ the sample size has been considered at $n = 20{,}40$ and 70, and some values for the parameters $\vartheta_1, \vartheta_2, \vartheta_3, \alpha$ and δ have been considered. It can be noted that from Table 1 that the estimates are work well and E , RAB decreases as the sample size increases.

The approximate variance-covariance matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}^{-1}$$

Where

$$a_{11} = -\frac{\partial^2 \ln L}{\partial \vartheta_1^2}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = \frac{n_1}{\hat{\vartheta}_1^{\,2}} + \frac{n_2}{(\hat{\vartheta}_{13})^2},$$

$$a_{12} = -\frac{\partial^2 \ln L}{\partial \vartheta_1 \partial \vartheta_2}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = 0,$$

$$a_{13} = -\frac{\partial^2 \ln L}{\partial \vartheta_1 \partial \vartheta_3}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = \frac{n_2}{(\hat{\vartheta}_{13})^2},$$

$$a_{14} = -\frac{\partial^2 \ln L}{\partial \vartheta_1 \partial \alpha}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = \sum_{I_1} d(x_{1i};\hat{\alpha},\hat{\delta}) + \sum_{I_2} d(x_{1i};\hat{\alpha},\hat{\delta}) + \sum_{I_3} d(x_i;\hat{\alpha},\hat{\delta})$$

$$a_{15} = -\frac{\partial^2 \ln L}{\partial \vartheta_1 \partial \delta}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = \sum_{I_1} g(x_{1i};\hat{\alpha},\hat{\delta}) + \sum_{I_2} g(x_{2i};\hat{\alpha},\hat{\delta}) + \sum_{I_3} g(x_i;\hat{\alpha},\hat{\delta})$$

$$a_{22} = -\frac{\partial^2 \ln L}{\partial \vartheta_2^2}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = \frac{n_2}{\hat{\vartheta}_1^{\,2}} + \frac{n_1}{(\hat{\vartheta}_{23})^2},$$

$$a_{23} = -\frac{\partial^2 \ln L}{\partial \vartheta_2 \partial \vartheta_3}\bigg|_{\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\hat{\alpha},\hat{\delta}} = \frac{n_2}{(\hat{\vartheta}_{23})^2},$$

$$a_{24}=-\left.\frac{\partial^2 \ln L}{\partial\vartheta_2\partial\alpha}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=\sum_{I_1} d(x_{1i};\hat\alpha,\hat\delta)+\sum_{I_2} d(x_{2i};\hat\alpha,\hat\delta)+\sum_{I_3} d(x_i;\hat\alpha,\hat\delta)$$

$$a_{25}=-\left.\frac{\partial^2 \ln L}{\partial\vartheta_2\partial\delta}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=\sum_{I_1} g(x_{2i};\hat\alpha,\hat\delta)+\sum_{I_2} g(x_{2i};\hat\alpha,\hat\delta)+\sum_{I_3} g(x_i;\hat\alpha,\hat\delta)$$

$$a_{33}=-\left.\frac{\partial^2 \ln L}{\partial\vartheta_3^2}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=\frac{n_3}{\hat\vartheta_3^{\,2}}+\frac{n_1}{(\hat\vartheta_{23})^2}+\frac{n_2}{(\hat\vartheta_{13})^2},$$

$$a_{34}=-\left.\frac{\partial^2 \ln L}{\partial\vartheta_3\partial\alpha}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=\sum_{I_1} d(x_{2i};\hat\alpha,\hat\delta)+\sum_{I_2} d(x_{1i};\hat\alpha,\hat\delta)+\sum_{I_3} d(x_i;\hat\alpha,\hat\delta),$$

$$a_{35}=-\left.\frac{\partial^2 \ln L}{\partial\vartheta_3\partial\delta}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=\sum_{I_1} g(x_{2i};\hat\alpha,\hat\delta)+\sum_{I_2} g(x_{1i};\hat\alpha,\hat\delta)+\sum_{I_3} g(x_i;\hat\alpha,\hat\delta)$$

$$a_{44}=-\left.\frac{\partial^2 \ln L}{\partial\alpha^2}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=\frac{2n_1+2n_2+n_3}{\hat\alpha^2}+(\hat\vartheta_1+1)\sum_{I_1} h(x_{1i};\hat\alpha,\hat\delta)+(\hat\vartheta_2+1)\sum_{I_2} h(x_{2i};\hat\alpha,\hat\delta)$$
$$+(\hat\vartheta_{23}+1)\sum_{I_1} h(x_{2i};\hat\alpha,\hat\delta)+(\hat\vartheta_{13}+1)\sum_{I_2} h(x_{1i};\hat\alpha,\hat\delta)+(\hat\vartheta_{123}+1)\sum_{I_3} h(x_i;\hat\alpha,\hat\delta)$$

$$a_{45}=-\left.\frac{\partial^2 \ln L}{\partial\alpha\partial\delta}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=-(\hat\vartheta_1+1)\sum_{I_1} p(x_{1i};\hat\alpha,\hat\delta)-(\hat\vartheta_2+1)\sum_{I_2} p(x_{2i};\hat\alpha,\hat\delta)$$
$$-(\hat\vartheta_{23}+1)\sum_{I_1} p(x_{2i};\hat\alpha,\hat\delta)-(\hat\vartheta_1+\hat\vartheta_3+1)\sum_{I_2} p(x_{1i};\hat\alpha,\hat\delta)-(\hat\vartheta_{123}+1)\sum_{I_3} p(x_i;\hat\alpha,\hat\delta),$$

$$a_{55}=-\left.\frac{\partial^2 \ln L}{\partial\delta^2}\right|_{\hat\vartheta_1,\hat\vartheta_2,\hat\vartheta_3,\hat\alpha,\hat\delta}=-\frac{2n_1+2n_2+n_3}{\hat\delta^2}+(\hat\vartheta_1+1)\sum_{I_1} q(x_{1i};\hat\alpha,\hat\delta)+(\hat\vartheta_2+1)\sum_{I_2} q(x_{2i};\hat\alpha,\hat\delta)$$
$$+(\hat\vartheta_{23}+1)\sum_{I_1} q(x_{2i};\hat\alpha,\hat\delta)+(\hat\vartheta_{13}+1)\sum_{I_2} q(x_{1i};\hat\alpha,\hat\delta)+(\hat\vartheta_{123}+1)\sum_{I_3} q(x_i;\hat\alpha,\hat\delta),$$

$$h(x;\hat\alpha,\hat\delta)=(\ln x)^2\frac{x^\alpha}{\delta}\left(1+\frac{x^\alpha}{\delta}\right)^{-2},\quad p(x;\hat\alpha,\hat\delta)=\delta^2x^\alpha(\ln x)(\delta^2+\delta x^\alpha)^{-2},$$

And

$$q(x;\hat\alpha,\hat\delta)=(2\delta x^\alpha+x^{2\alpha})(\delta^2+\delta x^\alpha)^{-2}.$$

Now, The asymptotic normality results will be considered to obtain the asymptotic confidence intervals of $\vartheta_1,\vartheta_2,\vartheta_3,\alpha$ and δ, It can be stated as follows

$$\sqrt{n}\left[(\hat{\alpha}-\alpha),(\hat{\delta}-\delta),\left(\hat{\vartheta}_1-\vartheta_1\right),\left(\hat{\vartheta}_2-\vartheta_2\right),\left(\hat{\vartheta}_3-\vartheta_3\right)\right]\to N_5\left(0,I(\underline{\theta})^{-1}\right) \text{ as } n\to\infty \tag{4.1}$$

Where $I^{-1}(\underline{\theta})$ is the variance-covariance matrix, $\underline{\hat{\theta}}=(\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3,\alpha,\hat{\delta})$ and $\underline{\theta}=(\vartheta_1,\vartheta_2,\vartheta_3,\alpha,\delta)$,Since $\underline{\theta}$ is unknown in (6), $I^{-1}(\underline{\theta})$ is estimated by $I^{-1}(\underline{\hat{\theta}})$; the asymptotic variance-covariance matrix that defined above and this can be used to obtain the asymptotic confidence intervals of $\vartheta_1,\vartheta_2,\vartheta_3,\alpha$ and δ.

5. Bivariate Distributions Related to BGB Distribution

The interrelations between particular cases of the BGB distribution and other distributions will be considered in this section such as bivariate Burr Type XII distribution, bivariate Pareto II distribution, bivariate Lomax distribution, bivariate Degum distribution and bivariate Burr Type III.

The BGB distribution has the following survival function

$$S_{X_1,X_2}(x_1,x_2)=\left(1+\frac{x_1^{\alpha}}{\delta}\right)^{-\vartheta_1}\left(1+\frac{x_2^{\alpha}}{\delta}\right)^{-\vartheta_2}\left(1+\frac{x_3^{\alpha}}{\delta}\right)^{-\vartheta_3},\quad x_3=\max(x_1,x_2).$$

By some changes to α and δ the following cases will be considered

i) Bivariate Burr Type XII (Signh-Maddala) Distribution

Set $\delta=1$,

$$S_{X_1,X_2}(x_1,x_2)=\left(1+x_1^{\alpha}\right)^{-\vartheta_1}\left(1+x_2^{\alpha}\right)^{-\vartheta_2}\left(1+x_3^{\alpha}\right)^{-\vartheta_3},\quad x_3=\max(x_1,x_2).$$

ii) Bivariate Pareto Type II

Set $\delta=1$ and $\alpha=1$

$$S_{X_1,X_2}(x_1,x_2)=\left(1+x_1\right)^{-\vartheta_1}\left(1+x_2\right)^{-\vartheta_2}\left(1+x_3\right)^{-\vartheta_3},\quad x_3=\max(x_1,x_2).$$

iii) Bivariate Lomax Distribution

Set $\alpha=1$

$$S_{X_1,X_2}(x_1,x_2)=\left(1+\frac{x_1}{\delta}\right)^{-\vartheta_1}\left(1+\frac{x_2}{\delta}\right)^{-\vartheta_2}\left(1+\frac{x_3}{\delta}\right)^{-\vartheta_3},\quad x_3=\max(x_1,x_2).$$

Which is the joint survival function of bivariate Lomax distribution that defined by Attia et al (2014)

iv) Bivariate Inverted Generalized Burr (Bivariate Dagum) Distribution

Set $Z=\frac{1}{X}$ and $\lambda=\frac{1}{\delta}$

$$F_{Z_1,Z_2}\left(z_1,z_2\right)=\left(1+\lambda z_1^{-\alpha}\right)^{-\vartheta_1}\left(1+\lambda z_2^{-\alpha}\right)^{-\vartheta_2}\left(1+\lambda z_3^{-\alpha}\right)^{-v_3},\quad z_3=\min\left(z_1,z_2\right) \tag{5.1}$$

Which is the joint cdf of the bivariate Dagum distribution that introduced by Muhammed (2017).

v) If $\lambda=1$ Bivariate Dagum distribution defined by eq.(7) reduces to bivariate Burr Type III (bivariate inverted Burr type XII) distribution with the following joint cdf

$$F_{Z_1,Z_2}(z_1,z_2)=\left(1+z_1^{-\alpha}\right)^{-\vartheta_1}\left(1+z_2^{-\alpha}\right)^{-\vartheta_2}\left(1+z_3^{-\alpha}\right)^{-\vartheta_3}, \quad z_3=\min(z_1,z_2)$$

6. Bivariate Absolutely Continuous BGB Distribution

Based on the idea of Block and Basu (1974), an absolutely continuous bivariate Generalized Burr (BGB_{ac}) distribution will be introduced by removing the singular part and remaining only the absolutely continuous part.

A random vector (Y_1, Y_2) follows a BGB_{ac} distribution if its pdf is given by

$$f_{Y_1,Y_2}(y_1,y_2)=\begin{cases} c\ f_1(y_1,y_2) & if \quad y_1<y_2 \\ c\ f_2(y_1,y_2) & if \quad y_1>y_2 \end{cases}$$

$$=c.\begin{cases} f_{GB}(y_1;\vartheta_1)\cdot f_{GB}(y_2;\vartheta_{23}) & if \quad y_1<y_2 \\ f_{GB}(y_1;\vartheta_{13})\cdot f_{GB}(y_2;\vartheta_2) & if \quad y_1>y_2 \end{cases},$$

Where c is the normalizing constant and $c=\frac{\vartheta_{123}}{\vartheta_{12}}$.

It will be denoted by $(Y_1, Y_2) \sim BGB_{ac}(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$.

Proposition 6.1. Let $(Y_1, Y_2) \sim BGB_{ac}(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$. The associated survival function is given by

$$S_{Y_1,Y_2}(y_1,y_2)=\frac{\vartheta_{123}}{\vartheta_{12}}S_{GB}(y_1;\alpha,\delta,\vartheta_1)\ S_{GB}(y_2;\alpha,\delta,\vartheta_2)S_{GB}(y;\alpha,\delta,\vartheta_3)$$
$$-\frac{\vartheta_3}{\vartheta_{12}}S_{GB}(y;\alpha,\delta,\vartheta_{123});$$

Where $y=\max(y_1,y_2)$. moreover, the marginal survival functions are given by

$$S_{Y_1}(y_1)=\frac{\vartheta_{123}}{\vartheta_{12}}S_{GB}(y_1;\alpha,\delta,\vartheta_{13})\ -\frac{\vartheta_3}{\vartheta_{12}}S_{GB}(y_1;\alpha,\delta,\vartheta_{123})$$

$$S_{Y_2}(y_2)=\frac{\vartheta_{123}}{\vartheta_{12}}S_{GB}(y_2;\alpha,\delta,\vartheta_{23})\ -\frac{\vartheta_3}{\vartheta_{12}}S_{GB}(y_2;\alpha,\delta,\vartheta_{123})$$

The marginal pdfs associated with the survival function given in Proposition 6.1 are as follows

$$f_{Y_1}(y_1)=cf_{GB}(y_1;\alpha,\delta,\vartheta_{13})\ -c\frac{\vartheta_3}{\vartheta_{123}}f_{GB}(y_1;\alpha,\delta,\vartheta_{123}),$$

And

$$f_{Y_2}(y_2)=cf_{GB}(y_2;\alpha,\delta,\vartheta_{23})\ -c\frac{\vartheta_3}{\vartheta_{123}}f_{GB}(y_2;\alpha,\delta,\vartheta_{123}).$$

Unlike those of the BGB distribution, the marginals of the $\mathrm{BGB}_{\mathrm{ac}}$ distribution are not GB distributions. If $\vartheta_3 \to 0^+$, then Y_1 and Y_2 follow GB distributions and in this case, Y_1 and Y_2 become independent .

Proposition 6.2. The product moments of $(Y_1, Y_2) \sim BGB_{ac}(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$. are given by

$$E(Y_1^r Y_2^s) = c\frac{\vartheta_1\vartheta_{23}\delta^{\frac{r+s}{\alpha}}}{\frac{r}{\alpha}+1} B\left[\left(\frac{r+s}{\alpha}+2\right),\left(\vartheta_{23}-\frac{r+s}{\alpha}-1\right)\right]$$

$$_3F_2\left[\left(\frac{r+s}{\alpha}+2\right),\left(\frac{r}{\alpha}+1\right),\left(1-\vartheta_1+\frac{r}{\alpha}\right);\left(\frac{r}{\alpha}+2\right)(\vartheta_{23}+1);1\right]$$

$$+c\frac{\vartheta_2\vartheta_{13}\delta^{\frac{r+s}{\alpha}}}{\frac{s}{\alpha}+1} B\left[\left(\frac{r+s}{\alpha}+2\right),\left(\vartheta_{13}-\frac{r+s}{\alpha}-1\right)\right]$$

$$. \quad {}_3F_2\left[\left(\frac{r+s}{\alpha}+2\right),\left(\frac{s}{\alpha}+1\right),\left(1-\vartheta_2+\frac{s}{\alpha}\right);\left(\frac{s}{\alpha}+2\right)(\vartheta_{13}+1);1\right].$$

Proposition 6.3. Let $(Y_1, Y_2) \sim BGB_{ac}(\vartheta_1, \vartheta_2, \vartheta_3, \lambda, \delta)$. Then

i. Stress- Strength parameter has the following form;

$$R = P(Y_1 < Y_2) = \frac{\vartheta_1}{\vartheta_1 + \vartheta_2},$$

ii. $min(Y_1, Y_2) \sim GB(\vartheta_{231})$.

7. Data Analysis

For illustrative purposes one data set has been analyzed to see how the proposed model works in practice. The data set has been obtained from Meintanis (2007). The data represent the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. In this case all possibilities are open, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

These data were analysed by Meintanis (2007), who considered the Marshall–Olkin bivariate exponential distribution, and by many authors such as Kundu and Dey (2009), Kundu and Gupta (2009), Muhammed (2016) and Muhammed (2017) they considered the Marshall–Olkin bivariate Weibull, bivariate generalized exponential, bivariate inverse Weibull and bivariate Dagum distributions, respectively. Here, these data will be analyzed using the BGB distribution.

The Kolmogorov-Smirnov distances between the fitted distribution and the empirical distribution function for X_1 , X_2 and $\min(X_1, X_2)$ with GB(1.163,3.73,0.013), GB(1.01,3.73,0.013) and GB(1.697,3.73,0.013) are(0.278) , (0.283) and (0.235) respectively. It indicates that the GB distribution can be used for analyzing X_1, $X_2$and

$\min(X_1, X_2)$. Although it does not guarantee that (X_1, X_2)will have BGB distribution, but at least it gives an indication that the BGB model may be used to analyze this bivariate data set.

A 95% confidence intervals of $\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta$ also computed and they are as follows; (0.558, 0.816), (0.501,0.567), (0.445, 0.508), (-0.031, 0.131), (0.011, 0.015)

The Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion

(HQIC) are also calculated for BGB distribution, based on the above estimates the log-likelihood value is -43.509, The corresponding AIC, BIC, CAIC and HQIC values are (97.018, 105.073, 98.954, and 99.858) respectively.

Now from the confidence intervals, from the log-likelihood values and also from the Kolmogorov-Smirnov distances, it is clear that BGB is preferable in this case.

8. Conclusions

In this paper the BGB distribution has been introduced for the first time, whose marginals are univariate GB distributions. The BGB distribution is a singular distribution and it has an absolute continuous part and a singular part. Since the joint distribution survival function and the joint density function are in closed forms, therefore this distribution can be used in practice for non-negative and positively correlated random variables. The interrelations between particular cases of the BGB distribution and other distributions have been considered such as bivariate Burr Type XII distribution, bivariate Pareto II distribution, bivariate Lomax distribution, bivariate Degum distribution and bivariate Burr Type III.

The maximum likelihood estimates for the five unknown parameters of this distribution and their approximate variance- covariance matrix are obtained. Some simulations are performed to see the performances of the MLEs. One data analysis has been performed for illustrative purpose. An absolute continuous version of the BGB distribution also obtained. Work is in progress in this direction and it will be reported elsewhere.

References

[1] Attia A. F., Shaban S. A. and Amer Y.M. (2014). Parameter Estimation for the Bivariate Lomax Distribution Based on Censored Samples. Applied Mathematical Sciences, 8(35), 1711 – 1721.

[2] Alizadeh, M., Cordeiro, G. M., Nascimento, A. D., Lima, M. D. C. S., & Ortega, E. M. (2017). Odd-Burr generalized family of distributions with some applications. Journal of statistical computation and simulation, 87(2), 367-389.

[3] Burr, I. W. (1942). Cumulative Frequency Functions. Ann. Math. Stat., 13, 215-232.

[4] Dubey, S.D.(1968). A compound Weibull distribution, Naval Research Logistics Quarterly, 15, 179-188.

[5] Gottschalk, L., Tallaksen, L.M., Perzyna, G.,(1997). Derivation of low flow distribution functions using recession curves, Journal of hydrology, 194, 239-262.

[6] Kundu D. and Dey, A.K. (2009). Estimating the parameters of the Marshall–Olkin bivariate Weibull distribution by EM algorithm, Comput. Statist. Data Anal. 53, 956–965.

[7] Kundu D. and Gupta R.D. (2009). Bivariate generalized exponential distribution,J.Multivariate Anal. 100, 581–593.

[8] Kumar D. (2017). THE BURR TYPE XII DISTRIBUTION WITH SOME STATISTICAL PROPERTIES. Journal of data science.16, 509-534.

[9] Mahmoud, M.A., Moshref, M., Yhiea, N.M., Mohamed, N.M., (2014) Progressively Censored Data from the Weibull Gamma Distribution Moments and Estimation, Journal of Statistics Applications & Probability, 3, 45-60.

[10] Marshall, A.W. and Olkin, I. (1967). A multivariate exponential distribution. Journal of the American Statistical Association, 62, 30- 44.

[11] Muhammed H. Z. (2016). "Bivariate inverse Weibull distribution", Journal of Statistical Computation and Simulation, 86(12), 1-11.

[12] Muhammed H. Z. (2017). Bivariate Dagum Distribution. International Journal of Reliability and Applications, 18(2) , 65-82,

[13] Muhammed H. Z. (2017). The Distribution of Sum, Product and Ratio for the Absolutely Continuous Bivariate Generalized Exponential Random Variables . The Egyptian Statistical Journal. Under publication.

[14] Silva, O. , da Silva, L. and Cordeiro G. (2015) The Extended Dagum Distribution: Properties and Application. Journal of Data Science 13, 53-72

[15] Qutb, N.S and Rajhi,E (2016). Estimation of the parameters of compound Weibull distribution. IOSR Journal of Mathematics, 12(4), 11-18.

Table 1: The AE, MSE and RAB of $\beta_1, \beta_2, \beta_3, \lambda$ and δ

for BGB Model

n	parameters	AE	MSE	RAB
20	ϑ_1 (0.3)	0.177	0.015	0.409
	ϑ_2 (1.1)	1.328	0.052	0.208
	ϑ_3 (1.5)	1.405	0.009	0.064
	α (2.5)	3.512	0.425	0.261
	δ (0.05)	0.015	0.000 1	0.694
40	ϑ_1 (0.3)	0.244	0.000 45	0.92
	ϑ_2 (1.1)	1.35	0.000 54	0.91
	ϑ_3 (1.5)	2.387	0.017 9	0.97
	α (1)	1.384	0.002	0.95
	δ (0.05)	0.014	0.03	0.89
70	ϑ_1 (0.7)	0.913	0.045	0.304
	ϑ_2 (1.41)	1.417	0.000 045	0.00477
	ϑ_3 (1.5)	2.533	0.477	0.689
	α (2)	2.954	0.91	0.477
	δ (0.8)	1.111	0.097	0.388