# Supplementary Material

Semiparametric Dynamic Copula Models using Rolling-window Portfolio Optimization

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## 1 Supplementary Figures and Tables

This section presents supplementary figures and tables that illustrate and support the proposed semiparametric dynamic copula framework and its empirical implementation across the United States, India, and Hong Kong markets. Figures S1 and S2 provide schematic flowcharts summarizing the six-step modeling pipeline: computation of log returns, estimation of SGT marginals, transformation to pseudo-uniform scores, empirical beta-copula fitting, Monte Carlo simulation of joint returns, and portfolio optimization via constrained quadratic programming under full-investment, long-only, and EWO constraints. Tables S1 to S3 report descriptive statistics for the 20 constituent assets in each economy, including mean return, volatility, and skewness. Figures S3 to S6 correspond to the U.S. market, where Figures S3 and S4 display the evolution of mean returns, volatilities, and Sharpe ratios under daily rebalancing from April 2019 to March 2025, while Figures S5 and S6 present the associated changes in optimal portfolio weights, risk contributions, and SGT marginal diagnostics under weekly rebalancing. Analogous results for the Indian and Hong Kong markets are shown in Figures S11 to S14 and Figures S15 to S18, respectively, capturing time-varying portfolio performance, weight dynamics, and goodness-of-fit assessments across different rebalancing frequencies. Figures S7 to S10 present rolling performance metrics and optimal portfolio weights for the Indian and Hong Kong markets under weekly rebalancing. Together, these materials highlight the dynamic evolution of return–dependence structures across economies and demonstrate the robustness and adaptability of the proposed SGT–EBC framework under changing market conditions. Figures S19 and S20 present the distribution of rolling optimal portfolio weights under the $t$-, Gaussian-, and $\beta$-copulas with SGT marginals for the Indian and Hong Kong markets.

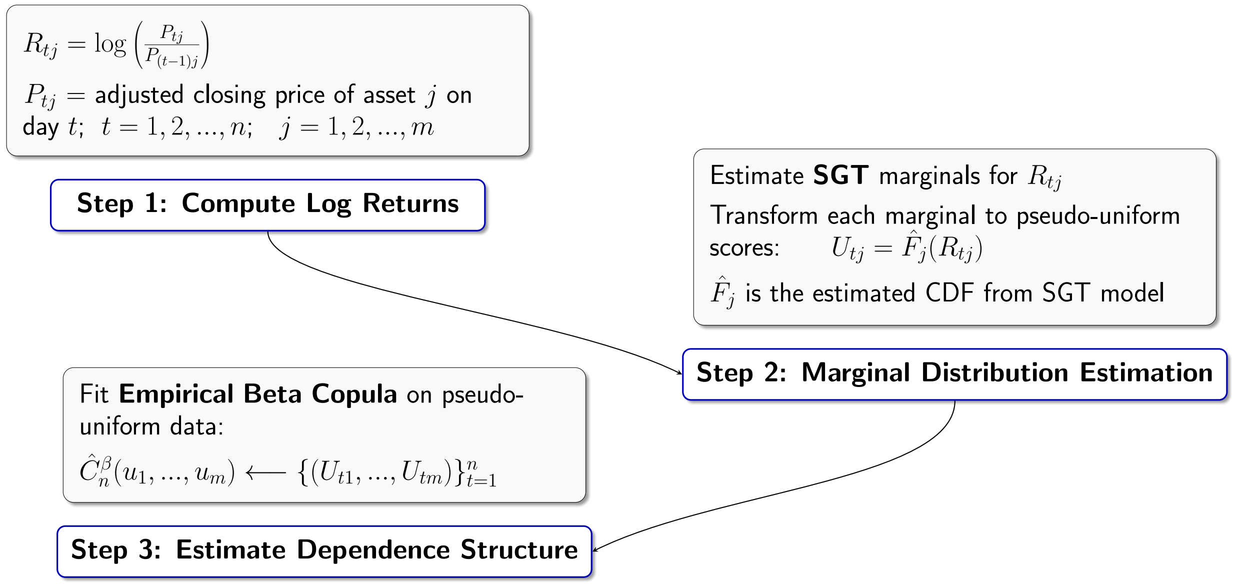


Figure S1: Schematic of Steps 1–3 in the proposed framework (corresponding to Section 2). Log returns are computed from adjusted prices, marginals are estimated via the SGT-distribution and transformed to uniform scores, followed by empirical beta copula fitting to capture joint dependence. Arrows indicate sequential updates across rolling windows.

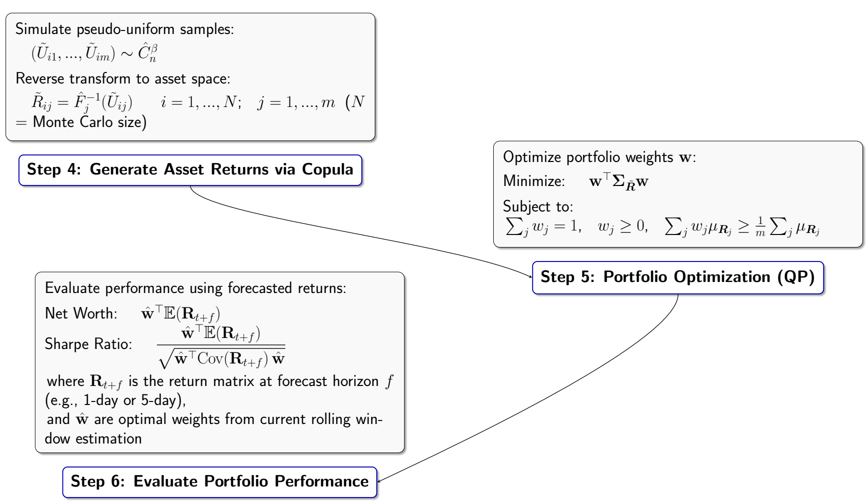


Figure S2: Schematic of Steps 4–6 (corresponding to Section 2): Pseudo-uniform samples are simulated from the fitted copula, transformed to asset returns, optimized via constrained quadratic programming, and evaluated using net worth and Sharpe ratio metrics. Arrows denote iterative estimation and optimization across rolling windows.

Table S1: Summary Statistics of Selected US Stocks (April 2018 – March 2025)

| Ticker | Company Name | Sector | Mean Return (%) | Mean SD (%) | Skewness |
|---|---|---|---|---|---|
| MSFT | Microsoft Corporation | Information Technology | 0.09 | 1.81 | -0.29 |
| GOOGL | Alphabet Inc. (Class A) | Communication Services | 0.06 | 1.95 | -0.24 |
| NVDA | NVIDIA Corporation | Information Technology | 0.17 | 3.29 | -0.24 |
| AMZN | Amazon.com, Inc. | Consumer Discretionary | 0.06 | 2.16 | -0.14 |
| META | Meta Platforms, Inc. | Communication Services | 0.07 | 2.67 | -1.34 |
| TSLA | Tesla, Inc. | Consumer Discretionary | 0.16 | 4.06 | -0.07 |
| AAPL | Apple Inc. | Information Technology | 0.10 | 1.93 | -0.23 |
| HD | The Home Depot, Inc. | Consumer Discretionary | 0.05 | 1.70 | -1.43 |
| ADBE | Adobe Inc. | Information Technology | 0.03 | 2.33 | -0.83 |
| NFLX | Netflix, Inc. | Communication Services | 0.07 | 2.83 | -2.18 |
| BRK-B | Berkshire Hathaway Inc. | Financials | 0.06 | 1.30 | -0.21 |
| V | Visa Inc. | Financials | 0.06 | 1.64 | -0.07 |
| MA | Mastercard Incorporated | Financials | 0.07 | 1.84 | 0.03 |
| JNJ | Johnson & Johnson | Health Care | 0.03 | 1.22 | -0.33 |
| UNH | UnitedHealth Group | Health Care | 0.06 | 1.80 | -0.53 |
| JPM | JPMorgan Chase & Co. | Financials | 0.06 | 1.86 | -0.02 |
| PG | Procter & Gamble Company | Consumer Staples | 0.05 | 1.27 | -0.03 |
| PEP | PepsiCo, Inc. | Consumer Staples | 0.03 | 1.31 | -0.57 |
| XOM | Exxon Mobil Corporation | Energy | 0.05 | 1.95 | -0.16 |
| COST | Costco Wholesale Corporation | Consumer Staples | 0.10 | 1.45 | -0.53 |

Table S2: Summary Statistics of Selected Indian Stocks (April 2018 – March 2025)

| Ticker | Company Name | Sector | Mean Return (%) | Mean SD (%) | Skewness |
|---|---|---|---|---|---|
| RELIANCE.NS | Reliance Industries | Oil, Gas & Consumable Fuels | 0.07 | 1.82 | 0.05 |
| TCS.NS | Tata Consultancy Services | Information Technology | 0.06 | 1.53 | -0.02 |
| INFY.NS | Infosys Ltd | Information Technology | 0.07 | 1.74 | -0.64 |
| HDFCBANK.NS | HDFC Bank Ltd | Financial Services | 0.04 | 1.58 | -0.35 |
| ICICIBANK.NS | ICICI Bank Ltd | Financial Services | 0.10 | 1.95 | -0.49 |
| HINDUNILVR.NS | Hindustan Unilever Ltd | Fast Moving Consumer Goods | 0.04 | 1.44 | 0.73 |
| ITC.NS | ITC Limited (Ltd) | Fast Moving Consumer Goods | 0.04 | 1.56 | -0.65 |
| KOTAKBANK.NS | Kotak Mahindra Bank | Financial Services | 0.04 | 1.77 | -0.28 |
| LT.NS | Larsen & Toubro Ltd | Construction | 0.06 | 1.74 | -0.62 |
| SBIN.NS | State Bank of India | Financial Services | 0.07 | 2.07 | -0.39 |
| BHARTIARTL.NS | Bharti Airtel Ltd | Telecommunication | 0.09 | 1.89 | 0.31 |
| ASIANPAINT.NS | Asian Paints Ltd | Consumer Durables | 0.04 | 1.60 | -0.41 |
| BAJFINANCE.NS | Bajaj Finance Ltd | Financial Services | 0.09 | 2.35 | -0.95 |
| MARUTI.NS | Maruti Suzuki India Ltd | Automobile & Auto Parts | 0.02 | 1.86 | -0.21 |
| HCLTECH.NS | HCL Technologies Ltd | Information Technology | 0.08 | 1.72 | -0.10 |
| AXISBANK.NS | Axis Bank Limited | Financial Services | 0.05 | 2.19 | -1.65 |
| SUNPHARMA.NS | Sun Pharma Industries | Healthcare | 0.07 | 1.76 | -0.02 |
| ULTRACEMCO.NS | UltraTech Cement Ltd | Construction Materials | 0.06 | 1.75 | -0.16 |
| TITAN.NS | Titan Company Ltd | Consumer Durables | 0.07 | 1.82 | -0.31 |
| NTPC.NS | NTPC Limited | Power | 0.07 | 1.77 | -0.43 |

Table S3: Summary Statistics of Selected Hong Kong Stocks (April 2018 – March 2025)

| **Ticker** | **Company Name** | **Sector** | **Mean Return (%)** | **Mean SD (%)** | **Skewness** |
|---|---|---|---|---|---|
| 0005.HK | HSBC Holdings plc | Financial Services | 0.03 | 1.50 | -0.22 |
| 0700.HK | Tencent Holdings Limited | Communication Services | 0.02 | 2.40 | 0.41 |
| 0001.HK | CK Hutchison Holdings Limited | Diversified Holdings | -0.02 | 1.54 | 1.32 |
| 2318.HK | Ping An Insurance Co. of China | Financial Services | -0.01 | 2.20 | 0.33 |
| 1299.HK | AIA Group Limited | Financial Services | -0.0002 | 1.96 | 0.001 |
| 0388.HK | HK Exchanges and Clearing Ltd. | Financial Services | 0.03 | 2.05 | 0.47 |
| 1398.HK | Industrial & Commercial Bank of China | Financial Services | 0.02 | 1.35 | 0.38 |
| 2388.HK | BOC Hong Kong (Holdings) Ltd. | Financial Services | 0.01 | 1.43 | 0.19 |
| 0002.HK | CLP Holdings Limited | Utilities | 0.005 | 1.10 | -0.23 |
| 2888.HK | Standard Chartered PLC | Financial Services | 0.03 | 1.93 | -0.18 |
| 0941.HK | China Mobile Limited | Communication Services | 0.03 | 1.39 | 0.70 |
| 0003.HK | HK & China Gas Co. Ltd. | Utilities | -0.02 | 1.26 | -1.41 |
| 0823.HK | Link REIT | Real Estate Investment | -0.01 | 1.50 | -0.37 |
| 0016.HK | Sun Hung Kai Properties Ltd. | Real Estate | -0.01 | 1.46 | -0.07 |
| 0267.HK | CITIC Limited | Diversified Holdings | 0.02 | 1.85 | -0.03 |
| 0011.HK | Hang Seng Bank Limited | Financial Services | -0.01 | 1.45 | -0.10 |
| 0836.HK | China Resources Power Holdings | Utilities | 0.04 | 2.46 | 0.14 |
| 0027.HK | Galaxy Entertainment Group Ltd. | Consumer Discretionary | -0.04 | 2.53 | -0.19 |
| 0012.HK | Henderson Land Development Co. Ltd. | Real Estate | -0.01 | 1.60 | -0.03 |
| 1038.HK | CK Infrastructure Holdings Ltd. | Infrastructure | 0.004 | 1.39 | -0.56 |

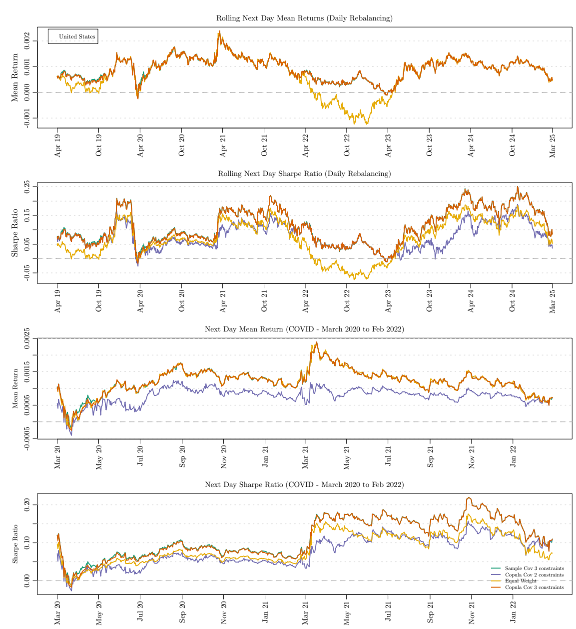


Figure S3: Rolling performance metrics for the U.S. market under daily rebalancing (April 2019–March 2025) and during the COVID-19 period (March 2020–February 2022). The top two panels show rolling next-day mean returns and Sharpe ratios for the U.S. market using daily rebalancing across portfolio strategies: equal weight, sample covariance, and copula-based covariance with two and three constraints. Temporal patterns mirror those under weekly rebalancing, with copula-based strategies achieving more stable and elevated Sharpe ratios during high-volatility periods. The bottom two panels zoom into the COVID-19 period.

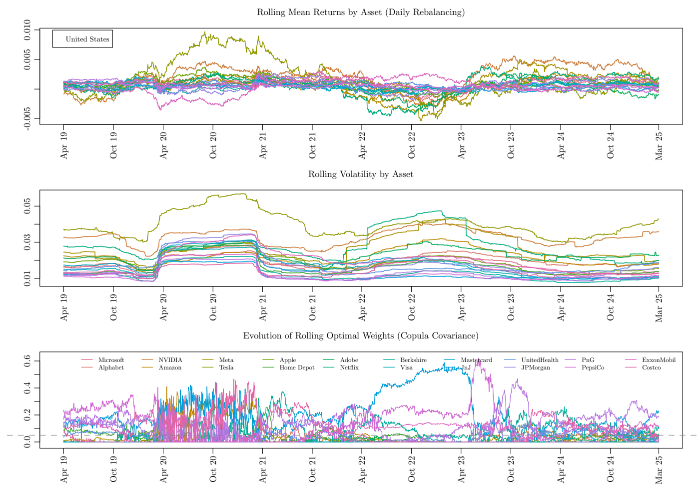


Figure S4: Rolling asset-level statistics and optimal weights for the U.S. market under daily rebalancing (April 2019–March 2025). The top panels show rolling mean returns and volatilities by asset under daily rebalancing, capturing finer temporal fluctuations in asset behavior. The bottom panel illustrates the evolution of optimal portfolio weights derived from copula-based covariance estimation with three constraints. The dynamic reallocation patterns highlight the model's responsiveness to short-term changes in asset-level risk and dependence.

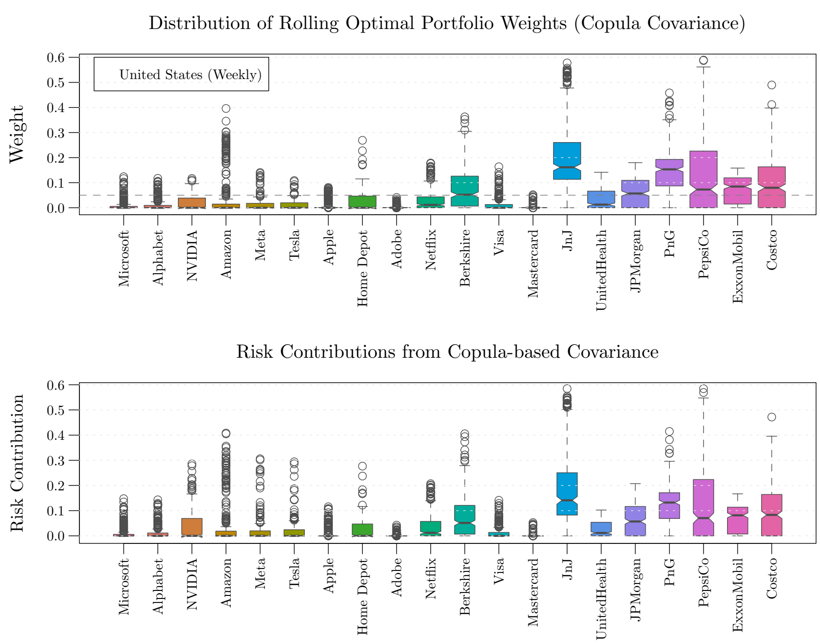


Figure S5: Time-averaged distribution of rolling optimal portfolio weights and corresponding risk contributions under the`copula_cov_3constraint` strategy for the U.S. market.

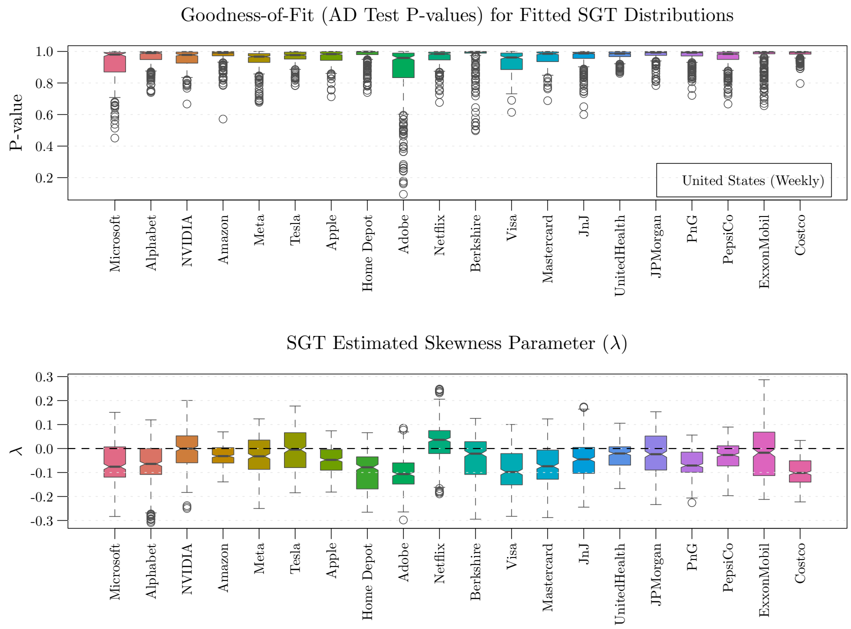


Figure S6: Goodness-of-fit and skewness estimates for SGT models - United States. $p$-values from the AD test and estimated skewness parameters ($\lambda$) for weekly return distributions fitted using the SGT-distribution across U.S. equities. The results demonstrate strong model fit and highlight moderate to negative skewness in several technology and consumer stocks.

Table S4: Differences ($\Delta$) in Average Return and Sharpe Ratio between empirical $\beta$ Copula and Gaussian (G) or $t$ Copula Models during the COVID period (March 2020–March 2022).

| | **U.S.** | | **India** | | **Hong Kong** | |
|---|---|---|---|---|---|---|
| **Metric (%)** | $\Delta_{\beta-G}$ | $\Delta_{\beta-t}$ | $\Delta_{\beta-G}$ | $\Delta_{\beta-t}$ | $\Delta_{\beta-G}$ | $\Delta_{\beta-t}$ |
| Average Return | −0.0006 | −0.0011 | +0.0007 | −0.0008 | −0.0016 | −0.0025 |
| Average Sharpe Ratio | +0.3131 | +0.2575 | +0.2894 | +0.1663 | +0.1175 | +0.0547 |

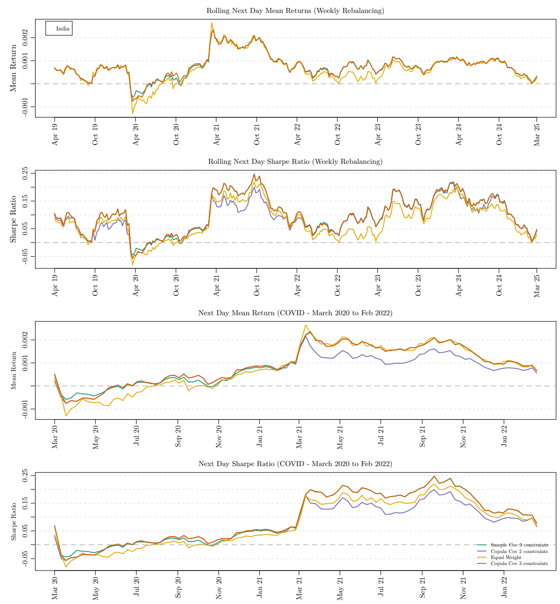


Figure S7: Rolling performance metrics for the Indian market under weekly rebalancing (April 2019–March 2025) and during the COVID-19 period (March 2020–February 2022). The top two panels report rolling next-day mean returns and Sharpe ratios based on portfolios constructed using equal weighting, sample covariance, and copula-based covariance matrices with two and three constraints. The bottom two panels focus on the COVID-19 period, revealing that copula-based strategies outperform others in terms of risk-adjusted performance, particularly during recovery phases following market stress.

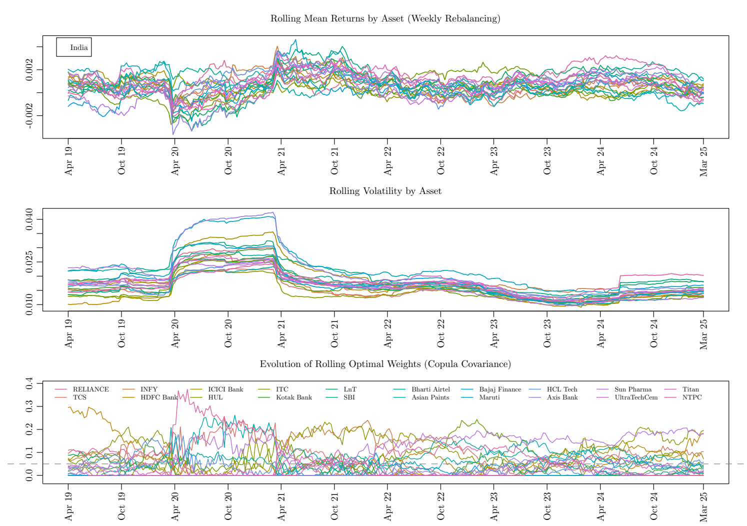


Figure S8: Rolling asset-level statistics and optimal weights for the Indian market under weekly rebalancing (April 2019–March 2025). The top panels show rolling mean returns and volatilities by asset, capturing heterogeneous dynamics across time and securities. The bottom panel depicts the evolution of optimal portfolio weights based on `copula_cov_3constraint`. The time-varying allocations highlight the model's responsiveness to shifts in risk and dependence.

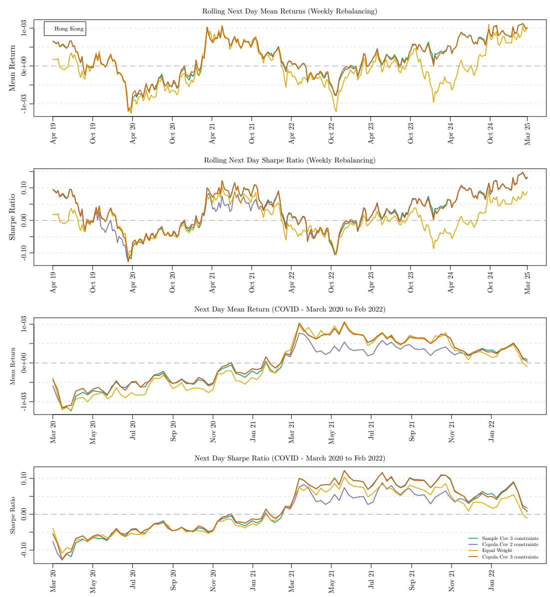


Figure S9: Rolling performance metrics for the Hong Kong market under weekly rebalancing (April 2019–March 2025) and during the COVID-19 period (March 2020–February 2022). The top panels show rolling next-day mean returns and Sharpe ratios across portfolio strategies: equal weight, sample covariance, and copula-based covariance with two and three constraints. Copula-based portfolios with three constraints outperform consistently during volatile phases

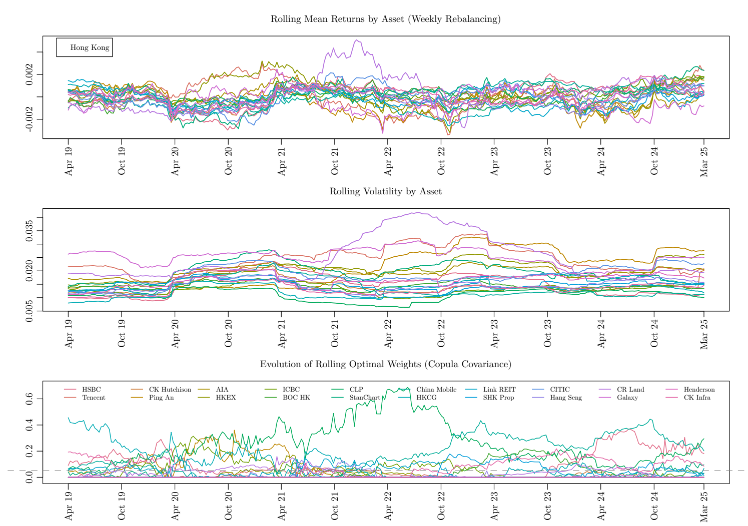


Figure S10: Rolling asset-level statistics and optimal weights for the Hong Kong market under weekly rebalancing (April 2019–March 2025). The top panels display rolling mean returns and volatilities by asset, with sharp volatility spikes during early 2020 and late 2021. The bottom panel illustrates time-varying optimal weights based on `copula_cov_3constraint`.

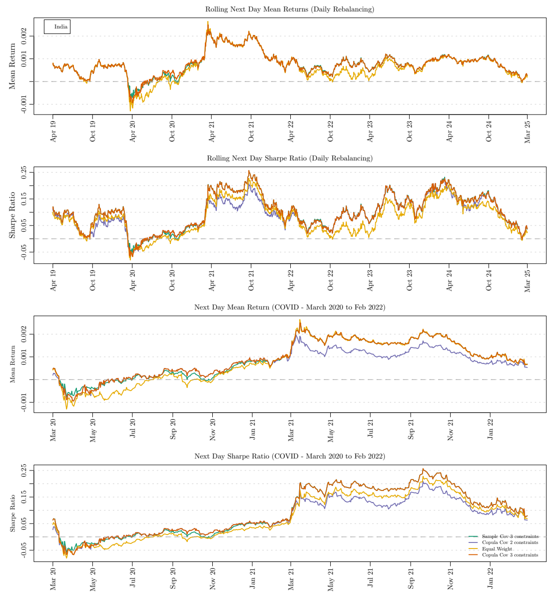


Figure S11: Rolling performance metrics for the Indian market under daily rebalancing (April 2019–March 2025) and during the COVID-19 period (March 2020–February 2022). The top two panels report rolling next-day mean returns and Sharpe ratios based on portfolios constructed using equal weighting, sample covariance, and copula-based covariance matrices with two and three constraints. The bottom two panels focus on the COVID-19 period, revealing that copula-based strategies outperform others in terms of risk-adjusted performance, particularly during recovery phases following market stress.

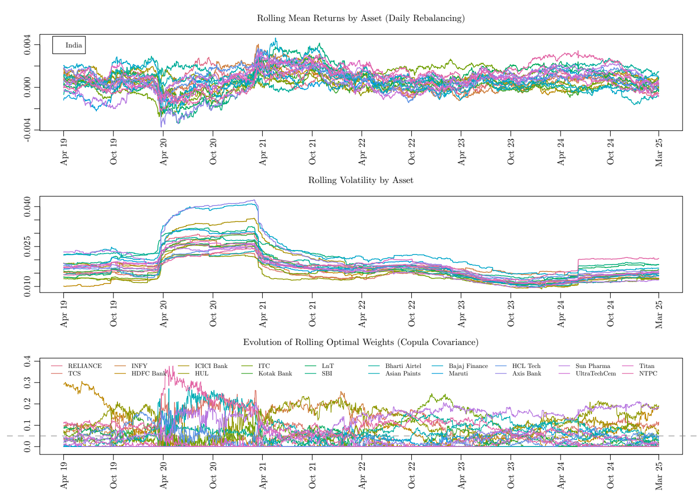


Figure S12: Rolling asset-level statistics and optimal weights for the Indian market under daily rebalancing (April 2019–March 2025). The top panels show rolling mean returns and volatilities by asset, capturing heterogeneous dynamics across time and securities. The bottom panel depicts the evolution of optimal portfolio weights based on copula-implied covariance with three constraints.

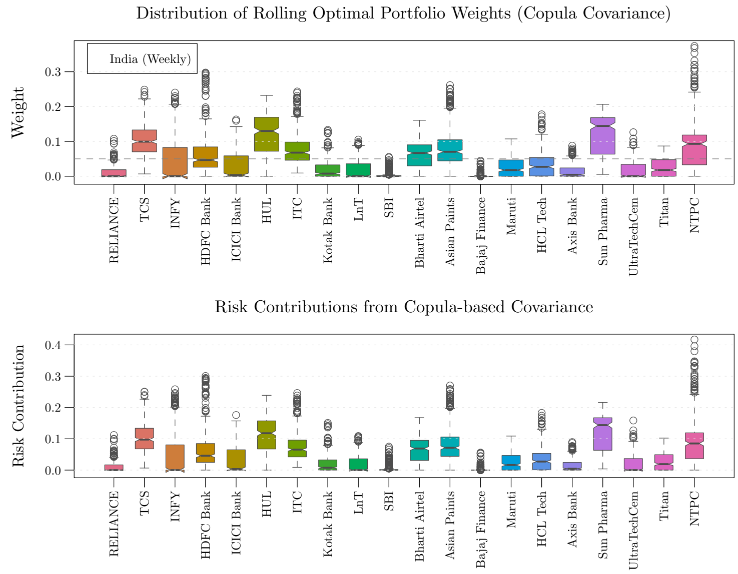


Figure S13: Time-averaged distribution of rolling optimal portfolio weights and corresponding risk contributions under the `copula_cov_3constraint` strategy for the Indian market.

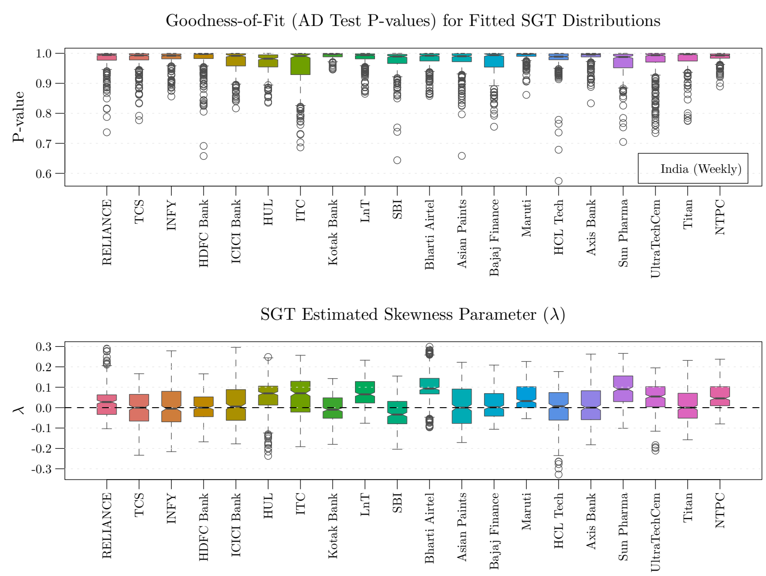


Figure S14: Goodness-of-fit and skewness estimates for SGT models — India. AD test $p$-values and skewness estimates for SGT-fitted weekly return distributions across major Indian stocks. Most assets exhibit good distributional fit and mildly positive skewness, suggesting upside asymmetry among defensives and large-cap financials.

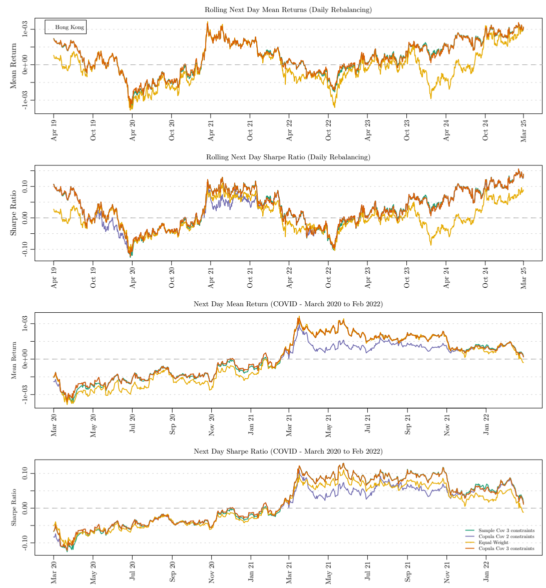


Figure S15: Rolling performance metrics for the Hong Kong market under daily rebalancing (April 2019–March 2025) and during the COVID-19 period (March 2020–February 2022). The top two panels display rolling next-day mean returns and Sharpe ratios across portfolio strategies: equal weight, sample covariance, and copula-based covariance estimators with two and three constraints. Copula-based portfolios with three constraints show superior stability and performance, particularly during the COVID-19 shock and recovery periods. The bottom panels zoom in on the COVID-19 phase, highlighting performance differentials under extreme market conditions.

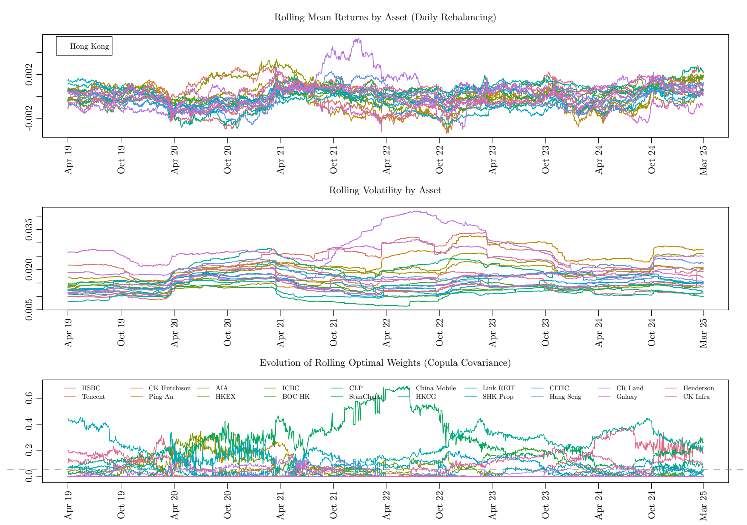


Figure S16: Rolling asset-level statistics and optimal portfolio weights for the Hong Kong market under daily rebalancing (April 2019–March 2025). The top panels display rolling mean returns and volatilities by asset, with elevated volatility observed during periods of market stress, notably in early 2020 and late 2021. The bottom panel shows the evolution of optimal portfolio weights based on copula-implied covariance with three constraints.

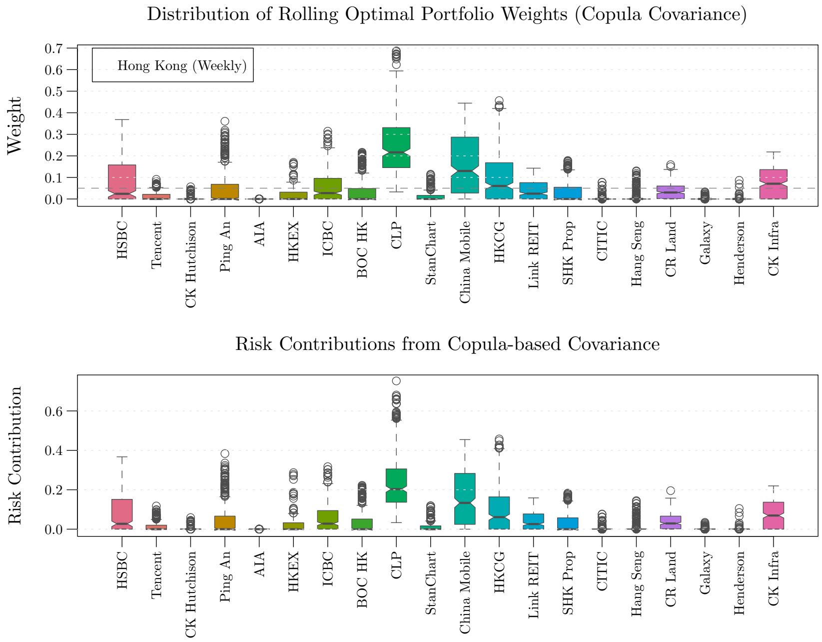


Figure S17: Time-averaged distribution of rolling optimal portfolio weights and corresponding risk contributions under the `copula_cov_3constraint` strategy for the Hong Kong market.

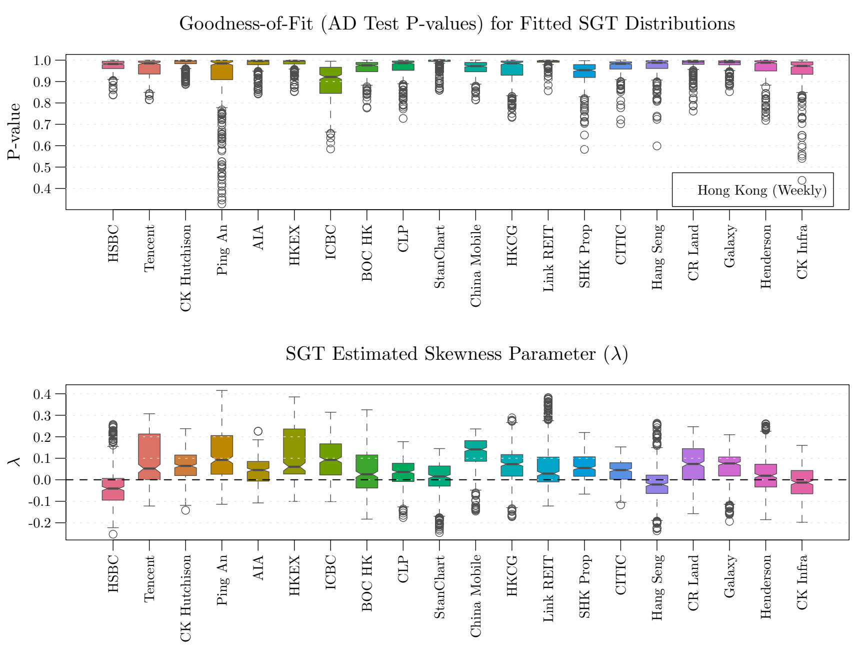


Figure S18: Goodness-of-fit and skewness estimates for SGT models for Hong Kong. SGT model evaluation for Hong Kong equities using AD test $p$-values and estimated $\lambda$ parameters. The fitted models show consistent goodness-of-fit, with moderate positive skewness evident in utilities and telecom sectors, indicating asymmetric tail behavior typical of the region's market structure.

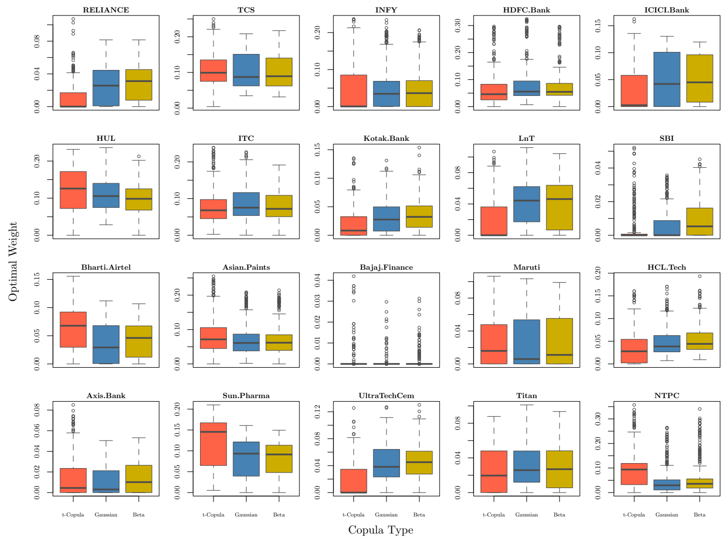


Figure S19: Distribution of rolling optimal portfolio weights under the $t$-, Gaussian-, and empirical $\beta$-copulas with SGT marginals at the weekly rebalancing frequency for the Indian market. The $t$-copula generates the widest dispersion of weights, indicating stronger tail dependence, whereas the Gaussian copula yields more concentrated allocations. The empirical $\beta$-copula displays an intermediate pattern, maintaining moderate and consistently positive weights across a broader set of assets, thereby enhancing diversification.

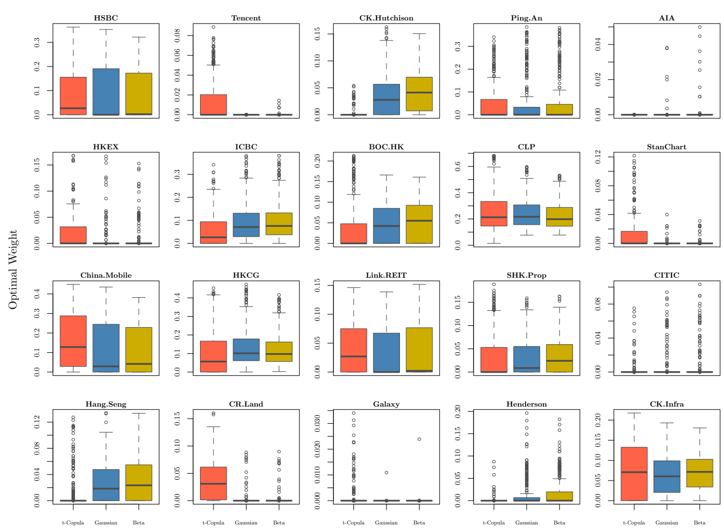


Figure S20: Distribution of rolling optimal portfolio weights under the $t$-, Gaussian-, and empirical $\beta$-copulas with SGT marginals at the weekly rebalancing frequency for the Hong Kong market. The $t$-copula produces the most dispersed weights, reflecting higher tail dependence, whereas the Gaussian copula results in tighter, more concentrated allocations. The empirical $\beta$-copula again exhibits an intermediate structure, preserving balanced and positive exposures across multiple assets, consistent with the diversification pattern observed in the U.S. and Indian markets.

## 2 Software Implementation Details

All computations were performed in R (version 4.0.2) using established packages for data acquisition, distribution fitting, copula modeling, and constrained portfolio optimization. The complete R code and replication materials are available at `GitHub`.

**Data acquisition and preprocessing.** Historical daily adjusted closing prices were retrieved from Yahoo Finance using the `quantmod` package (Ryan and Ulrich (2023)). Log returns were computed for the top 20 stocks from each market (S&P 500, NIFTY 50, and HKEX) over a seven-year period ending in March 2025. The data were aligned and merged to construct a multivariate log-return matrix serving as the input for subsequent modeling and optimization. Multivariate normality was evaluated using the Shapiro–Wilk and energy tests implemented in the `assetsTest()` function of the `fPortfolio` package (Wuertz et al. (2023)). Each test was applied to rolling windows of 250 trading days with weekly rebalancing, and the $(0.1, 0.3, 0.5, 0.7, 0.9)$ quantiles of the resulting $p$-values were reported for one representative market. For the U.S. (S&P 500) market, the energy test yielded quantiles below $2.2{\times}10^{-16}$, while the Shapiro–Wilk quantiles were ($6.1{\times}10^{-25}$, $2.7{\times}10^{-22}$, $4.8{\times}10^{-21}$, $2.6{\times}10^{-20}$, $1.2{\times}10^{-19}$), indicating rejection of multivariate normality in at least 90% of the rolling windows.

**Marginal modeling.** Each asset's return distribution was estimated using the SGT-model implemented in the `sgt.mle()` function from the `sgt` package (Davis (2015)). The Anderson–Darling goodness-of-fit test, via the `ad.test()` function from the `ADGofTest` package (Bellosta (2011)), was applied to assess marginal adequacy. Standardized residuals were then transformed into pseudo-uniform scores for dependence modeling.

**Dependence modeling.** Cross-asset dependence was implemented using the `copula` package (Hofert et al., 2024), where Gaussian and $t$-copulas were fitted via maximum likelihood (`fitCopula()`) and the empirical $\beta$-copula was estimated using `empCopula()` function. This estimator provides smooth, finite-sample-valid copulas with uniform margins and supports both evaluation and simulation. Monte Carlo samples ($m = 10^5$) were generated using `rCopula()` to simulate joint return realizations consistent with the estimated dependence structure.

**Portfolio optimization.** Constrained mean–variance optimization was carried out using the `solve.QP()` function from the `quadprog` package (Turlach and Weingessel (2019)). The optimization was subject to full-investment ($\mathbf{1}^\top \boldsymbol{\omega} = 1$), long-only ($\omega_i \geq 0$), and EWO constraints.

**Rolling-window implementation and parallelization.** The entire framework was implemented within a rolling-window structure with daily and weekly (five-day) rebalancing. For each window, the SGT parameters, empirical beta copula, and portfolio weights were re-estimated and optimized, allowing the model to adapt dynamically to evolving market conditions. To ensure computational efficiency, each rolling window was executed as an independent array job on the **Auburn University High Performance Computing** cluster (AUHPC (2024)). This job-level parallelization enabled simultaneous processing of multiple windows, markedly reducing overall runtime and enhancing the scalability of the proposed framework for longer time horizons and higher-dimensional asset portfolios.

This reproducible computational pipeline integrates distributional modeling, dependence estimation, and constrained optimization into a unified, data-driven framework consistent with the methodological development in Section 2.

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