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A NEW DISTRIBUTION FOR MODELING EXTREME VALUES

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ABSTRACT

In this work, we introduce a new distribution for modeling the extreme values. Some important mathematical properties of the new model are derived. We assess the performance of the maximum likelihood method in terms of biases and mean squared errors by means of a simulation study. The new model is better than some other important competitive models in modeling the repair times data and the breaking stress data.

Keywords: Fréchet Distribution; Moments; Estimation, Extreme Value Theory

1. Introduction

The extreme value theory (EVT) is very popular in the statistical literature, it is devoted to study of stochastic series of independent and identically distributed random variables (iid RVs). In EVT, we study the behavior of EVs even though these values have a very low chance to be occur, but can turn out to have a very high impact to the observed system. Fields such as finance and insurance are the best fields of research to observe the importance of the EVT. The study of EVT started in the last century as an equivalent theory to the central limit theory (CLT), which is dedicated to the study of the asymptotic distribution of the average of a sequence of RVs. The CLT states that the sum and the mean of the RVs from an arbitrary distribution are normally distributed under the condition that the sample size (n) is sufficiently large. However, in some other studies we are looking for the limiting distribution of maximum $(\max)$ or minimum $(\min)$ values rather than the average. Assume that $Z_1, Z_2, . . ., Z_n$ is a sequence of iid RVs distributed with CDF denote $F(z)$. One of the most interesting statistics in a research is the sample maximum

$$S_n = max\{Z_1, Z_2, \ldots, Z_n\},$$

this theory of extreme values studied the behavior of S_n as the sample size n increases to ∞ where

$$\begin{aligned} p_r\{S_n \le z\} &= p_r\{Z_1 \le z, Z_2 \le z, \ldots, Z_n \le z,\} \\ &= p_r\{Z_1 \le z\} p_r\{Z_2 \le z\} \ldots p_r\{Z_n \le z\} \\ &= F(z)^n \end{aligned}$$

Suppose there are sequences of constants $\{C_n > 0\}$ and $\{D_n\}$ such that

$$p_r\left\{\frac{(S_n - D_n)}{C_n} \leq x\right\} \rightarrow G(z)\ as\, n \rightarrow \infty.$$

Then if $G(z)$ is a non-degenerate distribution function then it will belong to one of the three following fundamental types of classic extreme value family, the Gumbel distribution (Type I); the Fréchet distribution (Type II); the Weibull distribution (Type III). A RV Z is said to have the Fréchet (Fr) distribution if its probability density function (PDF), cumulative distribution function (CDF) are given by

$$g(z; a, b) = ba^b z^{-(b+1)} exp[-(a/z)^b] \quad (1)$$

and

$$G(z; a, b) = exp[-(a/z)^b] \quad (2)$$

The more flexible version of the Fr model is the exponentiated Fréchet (EFr) distribution, with PDF and CDF are given by (for $x \geq 0$)

$$\pi_\theta(x; a, b) = \theta b a^b x^{-(b+1)} exp[-\theta(a/x)^b] \quad (3)$$

and

$$\Pi_\theta(x; a, b) = \exp[-\theta(a/x)^b] \quad (4)$$

respectively, where $a > 0$ is a scale parameter and $\theta, b > 0$ is a shape parameters, respectively.

The rest of this article is outlined as follows: In Section two, we introduce the genesis of the new model, Section 3 introduces a motivation and justification. A useful representation is given in Section 4. Some Mathematical properties are derived in Section 5. Section 6 shows the estimation method. Section 7 display the simulations results. Four applications are provided in Section 8. Finally, Section 9 deals with some concluding remarks.

2. The genesis of the new model

In this paper we will use the Transmuted Topp Leone G (TTL-G) family introduced by Yousof et al. (2017) to generate a new flexible distribution for modeling extreme values data. The TTL-G extends the transmuted class (Shaw and Buckley (2007)) and the TL-G (Rezaei et al. (2017)), these two families are well-known in statistical literature, Yousof et al. (2017) compiled the two families to obtain a more flexible one. We used the TTL-G to establish a new extension of the Fr model. To this end, we can write the PDF of the TTLEFr model (for $x \geq 0$) as

$$F(x) = (1+\lambda)(1 - \{1 - exp[-\theta(a/x)^b]\}^2)^\alpha \\ -\lambda(1 - \{1 - exp[-\theta(a/x)^b]\}^2)^{2\alpha}, x \geq 0, \quad (5)$$

the PDF corresponding to (5) is

$$f(x) = 2\alpha\theta b a^b x^{-(b+1)} exp[-\theta(a/x)^b]\{1 - exp[-\theta(a/x)^b]\}$$
$$\times [1 + \lambda - 2\lambda(1 - \{1 - exp[-\theta(a/x)^b]\}^2)^\alpha]$$
$$\times (1 - \{1 - exp[-\theta(a/x)^b]\}^2)^{\alpha-1}, x > 0 \quad (6)$$

where $, \alpha > 0$ and $|\lambda| \leq 1$. The HRF for the new model can be expressed as

$$\tau(x) = 2\alpha\theta b a^b x^{-(b+1)}$$
$$\times \exp[-\theta(a/x)^b]\{1 - \exp[-\theta(a/x)^b]\}$$
$$\times \frac{1 + \lambda - 2\lambda(1 - \{1 - \exp[-\theta(a/x)^b]\}^2)^\alpha}{(1 - \{1 - \exp[-\theta(a/x)^b]\}^2)^{1-\alpha}}$$
$$\times \begin{bmatrix} 1 - (1+\lambda)(1 - \{1 - \exp[-\theta(a/x)^b]\}^2)^\alpha \\ +\lambda(1 - \{1 - \exp[-\theta(a/x)^b]\}^2)^{2\alpha} \end{bmatrix}^{-1} \quad (7)$$

For simulation of this new model, if $U \sim u(0,1)$ then for $\lambda \neq 0$ we have

$$x_U = a\left[-\frac{1}{\theta}\ln\left(1 - \left\{1 - \left[\frac{1 + \lambda - \sqrt{(1+\lambda)^2 - 4\lambda U}}{2\lambda}\right]^{\frac{1}{\alpha}}\right\}^{0.5}\right)\right]^{-\frac{1}{5}}$$

has CDF (5). We used this Equation for simulating the TTLFr model (see Section 7). Some important extensions of the Fr model can be cited as: the exponentiated Fr by Nadarajah and Kotz (2003), beta Fr by Barreto-Souza et al. (2011), transmuted Fr by Mahmoud and Mandouh (2013), Marshall-Olkin Fr by Krishna et.al. (2013), transmuted exponentiated generalized Fr (2015) by Yousof et el. (2015), beta exponential Fr by Mead et al. (2016), Kumaraswamy Marshall-Olkin Fr by Afify et al. (2016a), Weibull Fr Afify et al. (2016b), Kumaraswamy transmuted Marshall-Olkin Fr by Yousof et al. (2016), Odd Lindley Fr by Korkmaz et al. (2017), odd log-logistic Fr by Yousof et al. (2018a), Transmuted Topp Leone Fr by Yousof et al. (2018b), among others. Many other extensions can be found in Brito et al. (2017), Hamedani et al. (2017), Cordeiro et al. (2018), Chakraborty et al. (2018), Hamedani et al. (2018), Korkmaz et al. (2018), Korkmaz et al. (2019) and Hamedani et al. (2019).

3. Motivation and justification

Suppose "$T_1$and T_2" are two independent RVs with CDF (3). Define

$$X = \begin{cases} T_{1:2} & \text{with probability } 0.5(1+\lambda); \\ T_{2:2} & \text{with probability } 0.5(1-\lambda), \end{cases}$$

where

$$T_{1:2} = min\{T_1, T_2\} \text{ and } T_{2:2} = max\{T_1, T_2\}.$$

Then, the CDF of X is given by (5).

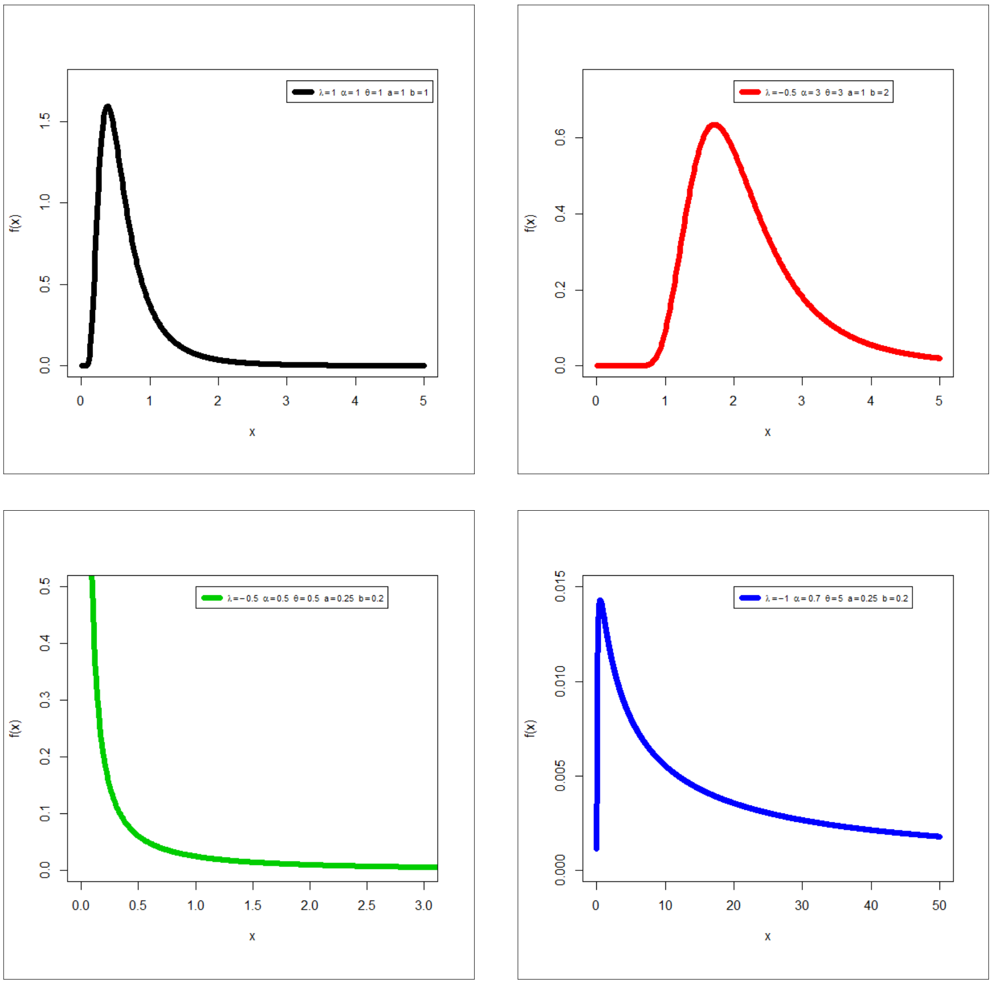

Figure 1: Plots of the TTLEFr PDF.

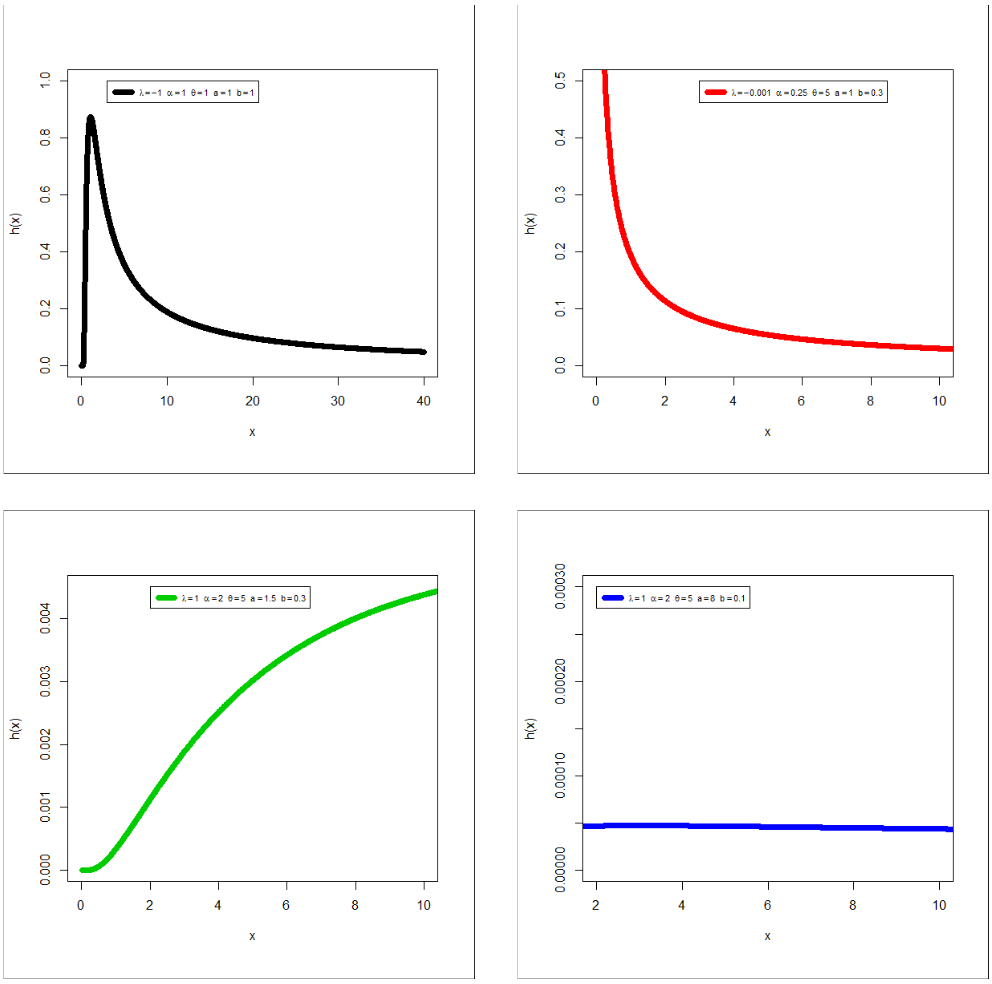

Figure 2: Plots of the TTLEFr HRF.

The justification for the practicality of the TTLEFr model is based on the wider use of the Fr model in modeling extreme values as well as we are motivated to introduce the TTLEFr model because it exhibits the unimodal, decreasing, increasing and constant hazard rate as illustrated in Figure 2. Also, the PDF of the new model are flexible and have many important shapes. The new model is better than the Fr, Kumaraswamy Fr, exponentiated Fr, beta Fr, transmuted Fr, and Marshal-Olkin Fr in modeling four data sets.

4. Useful representations

Equation (5) can be expanded as

$$F(x) = (1+\lambda)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\{1-\exp[-\theta(a/x)^b]\}^{2j}$$

$$-\lambda\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\{1-\exp[-\theta(a/x)^b]\}^{2j}$$

(8)

or

$$F(x) = (1+\lambda)\sum_{j=0}^{\infty}\sum_{k=0}^{2j}(-1)^{j+k}\binom{\alpha}{j}\binom{2j}{k}\{\exp[-\theta(a/x)^b]\}^k$$

$$-\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{2j}(-1)^{j+k}\binom{2\alpha}{j}\binom{2j}{k}\{exp[-\theta(a/x)^b]\}^k$$

and finally

$$F(x) = \sum_{k=0}^{2j} w_{j,k}\exp[-\theta k(a/x)^b] = \sum_{k=0}^{2j}\zeta_k\,\Pi_{\theta k}(x;a,b) \tag{9}$$

where

$$\zeta_k = \sum_{j=0}^{\infty}(-1)^{j+k}\left[\left(1+\lambda\right)\binom{\alpha}{j} - \lambda\binom{2\alpha}{j}\right]\binom{2j}{k}$$

and $\Pi_{\theta k}(x;a,b)$ is the CDF of the Fr distribution with scale parameter $a(\theta k)^{\frac{1}{b}}$ and shape parameter b. The corresponding TTLEFr density function is obtained by differentiating (9)

$$f(x) = \sum_{k=0}^{2j}\zeta_k\,\pi_{\theta(k+1)}(x;a,b) \tag{10}$$

where $\pi_{\theta(k+1)}(x;a,b)$ is the PDF of the Fr model with scale parameter $a[\theta(k+1)]^{\frac{1}{b}}$ and shape parameter b. So, the new density (6) can be expressed as a double linear mixture of the Fr density. Then, several of its structural properties can be obtained from Equation (10) and those properties of the Fr model.

5. Mathematical properties

5.1 Moments and incomplete moments

The rth ordinary moment of X is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx,$$

then we obtain

$$\mu_r' = \sum_{k=0}^{2j}\zeta_k\, a^r[\theta(k+1)]^{r/b}\Gamma(1-r/b), \forall b > r \tag{11}$$

Where

$$\Gamma(1+\omega)|_{(\omega\in\mathbb{R}^+)} = \omega! = \omega \times (\omega-1) \times (\omega-2) \times \ldots \times 1 = \prod_{h=0}^{\omega-1}(\omega-h)$$

Setting $r = 1,2,3$ and 4 in (11), we have

$$\begin{aligned} E(X) &= \mu_1' = \sum_{k=0}^{2j} \zeta_k \, a \, [\theta(k+1)]^{1/b}\Gamma(1-1/b), \forall\, b > 1, \\ E(X^2) &= \mu_2' = \sum_{k=0}^{2j} \zeta_k \, a^2 \, [\theta(k+1)]^{2/b}\Gamma(1-2/b), \forall\, b > 2, \\ E(X^3) &= \mu_3' = \sum_{k=0}^{2j} \zeta_k \, a^3 \, [\theta(k+1)]^{3/b}\Gamma(1-3/b), \forall\, b > 3, \end{aligned}$$

and

$$E(X^4) = \mu_4' = \sum_{k=0}^{2j} \zeta_k \, a^4 \, [\theta(k+1)]^{4/b}\Gamma(1-4/b), \forall\, b > 4.$$

The rth incomplete moment, say $\varphi_r(t)$, of X can be expressed, from (9), as

$$\varphi_r(t) = \int_{-\infty}^{t} x^r \, f(x)dx = \sum_{k=0}^{2j} \zeta_k \int_{-\infty}^{t} x^r \, \pi_{\theta(k+1)}(x; a, b)dx$$

$$= \textstyle\sum_{k=0}^{2j} \zeta_k \, a^r [\theta(k+1)]^{\frac{r}{\delta}}\gamma(1-r/b\,, [\theta(k+1)](a/t)^b), \forall b > r \quad (12)$$

where $\gamma(\omega, q)$ is the incomplete gamma function where

$$\begin{aligned} \gamma(\omega, q)|_{(\omega\neq 0,-1,-2,\ldots)} &= \int_0^q t^{\omega-1}\exp(-t)dt \\ &= \frac{q^\omega}{\omega}\{{}_1\mathbf{F}_1[\omega; \omega+1; -q]\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(\omega+k)} q^{\omega+k}, \end{aligned}$$

and ${}_1\mathbf{F}_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. The first incomplete moment given by (12) with $r = 1$ as

$$\varphi_1(t) = \sum_{k=0}^{2j} w_k a\,[\theta(k+1)]^{\frac{1}{b}} \Gamma\left(1-\frac{1}{b}, [\theta(k+1)]\left(\frac{a}{t}\right)^b\right), \forall\, b > 1.$$

5.2 Moment generating function

Here, we will introduce two methods for getting the moment generating function (MGF) of the new model. The 1^{st} one: The MGF $M_X(t) = E(e^{t\,X})$ of X can be derived from equation (9) as

$$M_X(t) = \sum_{k=0}^{2j} \zeta_k\; M_{\theta(k+1)}(t),$$

where $M_{\theta(k+1)}(t)$ is the MGF of the Fr model with scale parameter$a[\theta(k+1)]^{\frac{1}{b}}$ and shape parameter b, then

$$M_X(t) = \sum_{k=0}^{2j}\sum_{r=0}^{\infty} (t^r \zeta_k / r!)\; a^r\, [\theta(k+1)]^{r/b} \Gamma(1 - r/b), \forall\, b > r.$$

The 2^{nd} method: First, we determine the generating function of (1). Setting $y = x^{-1}$, we can write this MGF as

$$M(t; a, b) = b a^b \int_0^{\infty} \exp(t/y)\, y^{(b-1)} \exp\{-(ay)^b\}.$$

By expanding the first exponential and calculating the integral, we have

$$M(t; a, b) = b a^b \int_0^{\infty} \sum_{m=0}^{\infty} (t^m / m\,!)\, exp(t/y)\, y^{b-m-1}\, exp\{-(ay)^b\}$$

$$= \sum_{m=0}^{\infty} (a^m\; t^m / m\,!)\; \Gamma(1 - m/b),$$

where the gamma function is well-defined for any non-integer b. Consider the Wright generalized hypergeometric function defined by

$${}_p\Psi_q \left[\begin{matrix} a_1, A_1, \dots, a_p, A_p \\ b_1, B_1, \dots, b_q, B_q \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + A_j\, n\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + B_j\, n\right)} \frac{x^n}{n!}$$

Then, we can write $M(t;a,\beta)$ as

$$M(t;a,b) =_1 \Psi_0[(1,-1/b) \quad ;at] \tag{13}$$

Combining expressions (10) and (13), we obtain the MGF of X, say $M(t)$, as

$$M(t) = \sum_{k=0}^{2j} \zeta_k \left\{ {}_1\Psi_0 \left[\left(1,-\frac{1}{b}\right); a\,[\theta(k+1)]^{\frac{1}{b}}\, t \right] \right\}.$$

5.3 Moments of order statistics

Let $X_1,\dots,X_n$ be a random sample from the TTLEFr distribution and let $X_{1:n},\dots,X_{n:n}$ be their corresponding order statistics. The PDFof i^{th} order statistic, $X_{i:n}$, can be written as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} B^{-1}(i,n-i+1) f(x)(-1)^j \binom{n-i}{j} F^{i+i-1}(x) \tag{14}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in equation (13) and using a power series expansion, we have

$$f(x)\,F(x)^r = \sum_{k=0}^{2m+1} c_k\ \pi_{\theta(k+1)}(x;a,b),$$

where

$$c_k = \sum_{k=0}^{j+1-1} \sum_{n=0}^{\infty} 2\,\alpha\lambda^h (k+1)^{-1}(-1)^{h+m+k}(1+\lambda)^{j+i-h-1} \binom{j+i-1}{h} \binom{2m+1}{k}$$

$$\times \left\{ (1+\lambda) \binom{\alpha(j+i+h)-1}{m} - 2\lambda \binom{\alpha(j+i+h+1)-1}{m} \right\}$$

The PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k=0}^{2m+1} \mathrm{B}^{-1}(i,n-i+1)(-1)^j \binom{n-i}{j} c_k\, \pi_{\theta(k+1)}(x;a,b).$$

Based on the last equation, we note that the properties of $X_{i:n}$ follow from those of Fr density. So, the moments of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^{\tau}) = \sum_{j=0}^{n-i}\sum_{k=0}^{2m+1} \frac{(-1)^j \binom{n-i}{j}}{B(i,n-i+1)} c_k a^{\tau} [\theta(k+1)]^{\tau/b} \Gamma(1-\tau/b), \forall b > \tau$$

5.4 Probability weighted moments (PWMs)

The (s,r)th PWMs of X following the TTLEFr distribution, say $p_{s,r}$, is formally defined by

$$p_{s,r} = E\{X^s\, F(X)^r\} = \int_{-\infty}^{\infty} x^s\; F(x)^r\, f(x)\, dx.$$

Using equations (5) and (6), we can write

$$f(x)\, F(x)^r = \sum_{k=0}^{2j+1} d_k\, \pi_{\theta(k+1)}(x; a, b),$$

where

$$d_k = \sum_{i=0}^{r}\sum_{j=0}^{\infty} 2\, \alpha\lambda^i (k+1)^{-1} (-1)^{i+j+k} (1+\lambda)^{r-i} \binom{r}{i}\binom{2j+1}{k} \times \left\{ (1+\lambda)\binom{\alpha(r+i+1)-1}{j} - 2\lambda \binom{\alpha(r+i+2)-1}{j} \right\}.$$

Then, the (s,r)th PWMs of X can be expressed as

$$p_{s,r} = \sum_{k=0}^{2j+1} d_k\, a^s\, [\theta(k+1)]^{\frac{s}{b}} \Gamma\left(1-\frac{s}{b}\right), \forall\, b > s.$$

5.5 Residual life and reversed residual life functions

The nth moment of the residual life

$$m_n(t) = E\left[(X-t)^n \;\; \Big|_{(X>t,\, n=1,2,\dots)}\right]$$

the nth moment of the residual life of X is given by

$$m_n(t) = \frac{\int_t^{\infty} (x-t)^n dF(x)}{1-F(t)}.$$

Therefore,

$$m_n(t) = \frac{1}{1-F(t)} \sum_{k=0}^{2\,j} \zeta_k^{(m)} \int_t^{\infty} x^r \, \pi_{\theta(k+1)}(x; a, b) dx$$

$$= \frac{a^n}{1-F(t)} \sum_{k=0}^{2\,j} \zeta_k^{(m)} \; [\theta(k+1)]^{n/b} \Gamma(1 - n/b \, , [\theta(k+1)](a/t)^b), \forall \; b > n,$$

where

$$\zeta_k^{(m)} = \zeta_k \sum_{r=0}^{n} \binom{n}{r} (-t),$$

$$\Gamma(\omega, q)|_{(x>0)} = \int_q^{\infty} t^{\omega-1} \exp(-t) dt,$$

and

$$\Gamma(\omega, q) + \gamma(\omega, q) = \Gamma(\omega).$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by

$$m_1(t) = E\big[(X-t)|_{(X>t,\, n=1)}\big],$$

which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in the last equation. The nth moment of the reversed residual life, say

$$M_n(t) = E\left[(t-X)^n|_{(X \le t, t>0 \; and \; n=1,2,\dots)}\right]$$

uniquely determines $F(x)$. We obtain

$$M_n(t) = \frac{\int_0^t (\,t-x)^n dF(x)}{F(t)}.$$

Then, the nth moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{k=0}^{2\,j} \zeta_k^{(M)} \int_0^{t} x^r \, \pi_{\theta(k+1)}(x; a, b) dx$$

$$= \frac{a^n}{F(t)} \sum_{k=0}^{2\,j} \zeta_k^{(M)} \; [\theta(k+1)]^{n/b} \gamma(1 - n/b \, , [\theta(k+1)](a/t)^b), \forall \; b > n,$$

Where

$$\zeta_k^{(M)} = \zeta_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.$$

5.6 Stress-strength model

Let X_1 and X_2 be two independent RVs with TTLEFr $(\lambda_1, \alpha_1, \theta, a, b)$ and TTLEFr$(\lambda_1, \alpha_1, \theta, a, b)$ distributions. Then, the reliability, $\mathbf{R}_{X_1,X_2}$, is defined by

$$\mathbf{R}_{X_1,X_2} = \int_0^\infty f_1\,(x; \lambda_1, \alpha_1, \psi) F_2(x; \lambda_1, \alpha_1, \psi) dx$$

We can write

$$\mathbf{R}_{X_1,X_2} = \sum_{k=0}^{2j} \sum_{m=0}^{2w} r_{k,m} \int_0^\infty \pi_{\theta(k+m)}\,(x) dx$$

where

$$r_{k,m} = \sum_{j,w=0}^{\infty} (k+1)\,(-1)^{j+k+w+m} (k+m+1)^{-1} \binom{2j}{k} \binom{2w}{m}$$

$$\times \left[(1+\lambda_1)\binom{\alpha_1}{j} - \lambda_1 \binom{2\alpha_1}{j}\right] \left[(1+\lambda_2)\binom{\alpha_2}{w} - \lambda_2 \binom{2\alpha_2}{w}\right]$$

Thus, $\mathbf{R}_{X_1,X_2}$ can be expressed as

$$\mathbf{R}_{X_1,X_2} = \sum_{k=0}^{2\,j} \sum_{m=0}^{2\,w} r_{k,m}.$$

6. Estimation

Let $x_1, \dots, x_n$ be a random sample from the TTLEFr distribution with parameters $\lambda, \alpha, \theta, a$ and b. Let $\Theta = (\lambda, \alpha, \theta, a, b)^\intercal$ be the 5×1 parameter vector. For determining the MLE of Θ, we have the log-likelihood function

$$\ell = \ell(\Theta) = n\log(2\alpha\theta b a^b)$$

$$+\textstyle\sum_{i=1}^{n} \log\{1 - \exp[-\theta(a/x_i)^b]\}$$

$$-(b+1)\textstyle\sum_{i=1}^{n} \log x_i - \theta\, \sum_{i=1}^{n} (a/x_i)^b$$

$$+(\alpha-1)\ \sum_{i=1}^{n}(1-\{1-\exp[-\theta(a/x_i)^b]\}^2)$$

$$+\sum_{i=1}^{n}\log\left[1+\lambda-2\lambda(1-\{1-\exp[-\theta(a/x_i)^b]\}^2)^{\alpha}\right]$$

The components of the score vector, $\mathbf{U}(\Theta)=\frac{\partial\ell}{\partial\Theta}=\left(\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b}\right)^{\top}$,are available if needed. Setting $U_\lambda=U_\alpha=U_\theta=U_a=$ and $U_b=\mathbf{0}$ and solving them simultaneously yields the MLE $\hat{\Theta}=(\hat{\lambda},\hat{\alpha},\hat{\theta},\hat{a},\hat{b})^{\top}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 5×5 observed information matrix

$$J(\Theta)=\{\partial^2\ell/\partial r\,\partial s\}\quad(\forall r,s=\lambda,\alpha,\theta,a,b),$$

whose elements can be computed numerically.

7. Simulation studies

Using the inversion method, we simulate the TTLEFr model by taking n=20, 50, 200 and 500. For each sample size, we evaluate the MLEs of the parameters using the optim function of the R software. Then, we repeat this process 1000 times and compute the biases (Bias) and mean squared errors (MSEs). Table 1 gives all simulation results. The values in Table 1 indicate that the MSEs and the Bias of $\hat{\lambda}$, $\hat{\alpha}$, $\hat{\theta}$, $\hat{a}$ and $\hat{b}$ decay toward zero when the n increases for all settings of a and b, as expected under first-under asymptotic theory. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. Table 1 gives biases and MSEs based on 1000 simulations of the TTLEFr distribution for some values of a and b when $\lambda=0.5$, $\alpha=1.5$ and $\theta=2.5$ by taking $n=20$, 50, 200 and 500.

8 Real data modeling

In this section, we provide four applications to real data sets to illustrate the importance of the TTLEFr distribution. To evaluate performance of considered model, the MLEs of the parameters for the considered models are calculated and three goodness-of-fit statistics are used to compare the new distribution.

The following measures of goodness of fit are computed to compare the fitted models:

1-Anderson-Darling (A^*);

2-Cramér-von Mises (W^*);

3-Akaike Information Criterion (AIC);

4-Bayesian information criterion (BIC);

5-Kolmogrov-Smirnov (K-S) statistics (and its corresponding p-value).

In general, the smaller are the values of these statistics (A^*, W^*, AIC, BIC and K-S), the better the fit to the data. The required computations are carried out in the R-language for the all application. The numerical values of the model selection statistics A^*, W^*, AIC, BIC, K-S and its corresponding p-value are listed in Tables 2 and 4. Tables 3 and 5 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The total time test (TTT) plots for the two data sets indicates that the HRFs are upside down for the 1^{st} data and increasing for the 2^{nd} data.

Table 1: The biases and MSEs based on 1000 simulations.

	a = 0:5 and b = 2			a = 2 and b = 0:5		
n	$\boldsymbol{\Theta}$	Bias	MSE	$\boldsymbol{\Theta}$	Bias	MSE
20	a	0:53014	0:93380	a	0:42651	0:59439
	b	0:17543	0:88234	b	0:11634	0:18498
	$\boldsymbol{\lambda}$	0:53447	0:54144	$\boldsymbol{\lambda}$	0:38225	0:34516
	$\boldsymbol{\alpha}$	0:63432	0:41383	$\boldsymbol{\alpha}$	0:35410	0:59115
	$\boldsymbol{\theta}$	0:19230	0:42330	$\boldsymbol{\theta}$	0:22650	0:44762
50	a	0:13148	0:62591	a	0:12849	0:48654
	b	0:14120	0:48210	b	0:06632	0:09892
	$\boldsymbol{\lambda}$	0:33576	0:39034	$\boldsymbol{\lambda}$	0:22435	0:22410
	$\boldsymbol{\alpha}$	0:32490	0:35515	$\boldsymbol{\alpha}$	0:25717	0:28321
	$\boldsymbol{\theta}$	0:16819	0:31923	$\boldsymbol{\theta}$	0:13933	0:24230
200	a	0:01675	0:30729	a	0:02883	0:20473
	b	0:11082	0:27327	b	0:00914	0:03759
	$\boldsymbol{\lambda}$	0:21642	0:15430	$\boldsymbol{\lambda}$	0:13149	0:10073
	$\boldsymbol{\alpha}$	0:12669	0:20421	$\boldsymbol{\alpha}$	0:15670	0:08871
	$\boldsymbol{\theta}$	0:03278	0:02689	$\boldsymbol{\theta}$	0:10316	0:12169
500	a	0:01325	0:12266	a	0:00570	0:05110
	b	0:09016	0:04649	b	0:00281	0:00591
	$\boldsymbol{\lambda}$	0:03113	0:06324	$\boldsymbol{\lambda}$	0:01192	0:01410
	$\boldsymbol{\alpha}$	0:07687	0:00960	$\boldsymbol{\alpha}$	0:06100	0:01020
	$\boldsymbol{\theta}$	0:00213	0:00425	$\boldsymbol{\theta}$	0:00201	0:00151

8.1 Application 1: Repair times data

The 1^{st} data set (repair times data) represents an active repair times (hours) for an air borne communication transceiver originally given by Chhikara and Folks (1989). This data set is reproduced as follows: 0.20, 0.3, 0.50, 0.5, 0.50, 0.5, 0.60, 0.6, 0.70, 0.7, 0.70, 0.8, 0.8, 1.00, 1.0, 1.00, 1.0, 1.10, 1.30, 1.50, 1.5, 1.50,1.50, 2.0, 2.0, 2.20, 2.50, 2.7, 3.00, 3.0, 3.30, 3.3, 4.00, 4.0, 4.50, 4.7, 5.00, 5.4, 5.4, 7.00, 7.5, 8.80, 9.0, 10.30, 22.0 and 24.50. The statistics of the fitted models for the 1^{st} data set are presented in Table 2 and the MLEs and corresponding standard errors are given in Table 3. We note from the values in Table 2 that the TTLEFr model has the lowest values of the A^*, W^*, AIC, BIC, and K-S statistics (for the 1^{st} data set). The histogram and other related important plots of the first data of the TTLEFr model Figure 3. We compare the fits of the TTLEFr distribution with other models such as Fréchet (Fr), Kumaraswamy Fréchet (KFr), exponentiated Fréchet (EFr), beta Fréchet (BFr), transmuted Fré chet (TFr), and Marshal-Olkin Fréchet (MOFr) distributions given by:

EFr :

$$f_{EFr}(x;\alpha,a,b)=\alpha b a^{b}x^{-(b+1)}\exp[-(a/x)^{b}]\{1-\exp[-(a/x)^{b}]\}^{\alpha-1}$$

BFr :

$$f_{BFr}(x;\alpha,\theta,a,b)=ba^{b}B^{-1}(\alpha,\theta)x^{-(b+1)}\exp[\,-\alpha(a/x)^{b}]\{1-\exp[\,-(a/x)^{b}]\}^{\theta-1};$$

KFr :

$$f_{KFr}(x;\alpha,\theta,a,b)=\alpha\theta b a^{b}x^{-(b+1)}\exp[\,-\alpha(a/x)^{b}]\{1-\exp[\,-\alpha(a/x)^{b}]\}^{\theta-1};$$

TFr :

$$f_{TFr}(x;\lambda,a,b)=ba^{b}x^{-(b+1)}\exp[\,-(a/x)^{b}]\{1+\lambda-2\lambda\exp[\,-(a/x)^{b}]\};$$

MOFr :

$$f_{MOFr}(x;\alpha,a,b)=\alpha b a^{b}x^{-(b+1)}\exp[-(a/x)^{b}]\{\alpha+(1-\alpha)\exp[-(a/x)^{b}]\}^{-2}$$

The parameters of the above densities are all positive real numbers except for the TFr distribution for which $|\lambda|\leq 1$.

Table 2: The statistics A^*, W^*, AIC, BIC, K-S and K-S p-value for the repair times data set.

Model	Goodness of fit criteria					
	W^*	A^*	AIC	BIC	K-S	p-value
TTLEFr	0.0445	0.2771	206.4	214.6	0.07432	0.9613
Fr	0.0576	0.3806	207.4	215.0	0.0807	0.9252
KFr	0.0467	0.2847	207.4	214.6	0.0826	0.9118
EFr	0.0576	0.3805	207.4	214.9	0.0806	0.9259
BFr	0.0463	0.2831	207.4	214.7	0.0836	0.9040
TFr	0.0568	0.3586	207.8	215.3	0.0818	0.9174
MOFr	0.0478	0.2972	207.9	214.7	0.0785	0.9392

Table 3: MLEs and their standard errors (in parentheses) for the first data set.

Model	Estimates				
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$
TTLEFr	-0.1490	0.3913	2.7172	1.3586	0.8119
	(0.677)	(0.534)	(2.664)	(1.661)	(0.170)
Fr				1.1297	1.0128
				(0.1740)	(0.1129)
KFr		1.1619	3.8034	4.0226	0.5401
		(7.452)	(4.604)	(47.459)	(0.2753)
EFr		0.9881		1.1433	1.0125
		(23.679)		(27.057)	(0.1129)
BFr		2.3521	5.8362	3.4905	0.4147
		(8.581)	(14.877)	(13.461)	(0.5619)
TFr	-0.6364			0.7747	1.0853
	(0.1173)			(0.3633)	(0.1226)
MOFr		4.9168		0.5066	1.3384
		(6.1834)		(0.3068)	(0.2574)

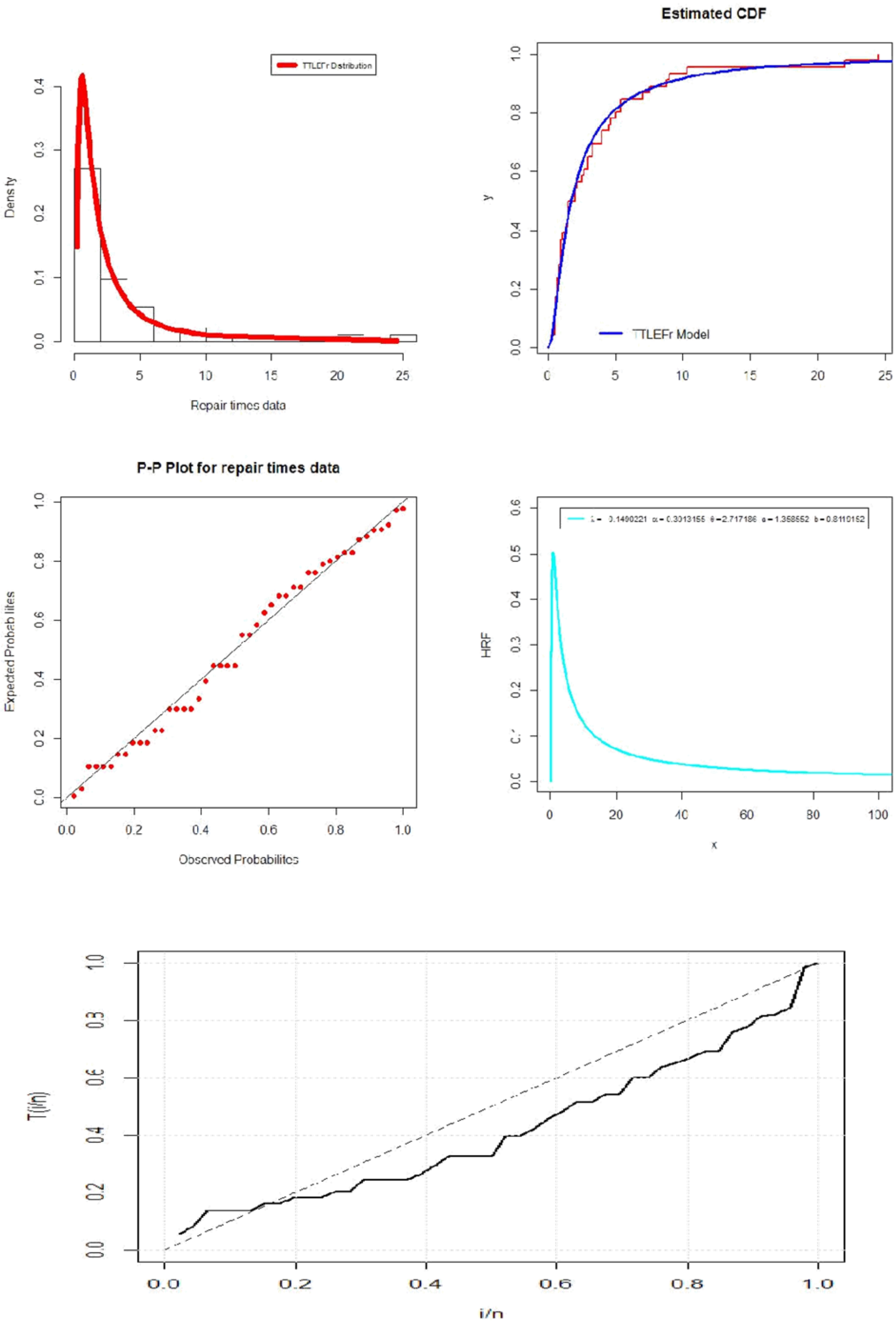

Figure 3: Estimated PDF, estimated CDF, P-P plot, estimated HRF and TTT plot of the repair times data.

8.2 Application 2: Breaking stress data

The 2^{nd} set is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006) and these data are used by Mahmoud and Mandouh (2013) to fit the transmuted Fr distribution. The data are: 0.920, 0.9280, 0.997, 0.99710, 1.061, 1.1170, 1.162, 1.1830, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.21990, 1.22, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.2590, 1.261, 1.2630, 1.276, 1.310, 1.321, 1.3290, 1.331, 1.3370, 1.351, 1.3590, 1.388, 1.4080, 1.449, 1.44970, 1.45, 1.4590, 1.471, 1.4750, 1.477, 1.480, 1.489, 1.5010, 1.507, 1.5150, 1.53, 1.53040, 1.533, 1.5440, 1.5443, 1.5520, 1.556, 1.562, 1.5660, 1.585, 1.5860, 1.5990, 1.602, 1.6140, 1.6160, 1.6170, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.759, 1.7770, 1.794, 1.7990, 1.806, 1.8140, 1.8160, 1.828, 1.830, 1.884, 1.8920, 1.944, 1.9720, 1.9840, 1.9870, 2.02, 2.03040, 2.0290, 2.03500, 2.037, 2.0430, 2.0460, 2.059, 2.1110, 2.165, 2.6860, 2.778, 2.9720, 3.504, 3.8630 and 5.3060. The statistics of the fitted models are presented in Table 4 and the MLEs and corresponding standard errors are given in Table 5. We note from Table 4 that the TTLEFr gives the lowest values the A^*, W^*, AIC, BIC, and K-S statistics (for the 2^{nd} data set) as compared to further models, and therefore the new one can be chosen as the best one. The histogram and other related important plots of the 2^{nd} data are displayed in Figure 4. We compare the fits of the TTLEFr distribution with other related Fr models such as Fr, EFr, KFr, BFr, MOFr, TFr, and McDonald Fréchet (McFr):

$$f_{McFr}(x;\alpha,\theta,\lambda,a,b) = \lambda b a^b x^{-(b+1)} B^{-1}(\alpha,\theta)\exp[-(a/x)^b](\exp[-(a/x)^b])^{\alpha\lambda-1}$$

$$\times\left(1-(\exp[-(a/x)^b])^{\lambda}\right)^{\theta-1}$$

Table 4: The statistics A^*, W^*, AIC, BIC, K-S and K-S p-value for the carbon fibers data set.

Model	Goodness of fit criteria					
	W^*	A^*	AIC	BIC	K-S	p-value
TTLEFr	0.0655	0.5010	114.0	114.8	0.0662	0.7728
Fr	0.1090	0.7657	114.4	124.6	0.0874	0.4282
KFr	0.0812	0.6217	114.0	124.4	0.0759	0.6118
EFr	0.1091	0.7658	114.0	124.3	0.0874	0.4287
BFr	0.0809	0.6207	114.0	124.4	0.0757	0.6147
TFr	0.0871	0.6209	114.4	124.3	0.0782	0.5734
MOFr	0.0886	0.6142	114.0	124.8	0.0763	0.5168
McFr	0.1333	1.0608	123.97	137.0	0.0807	0.5332

Table 5: MLEs and their standard errors (in parentheses) for the second data set.

Model	Estimates				
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$
TTLEFr	0.7410 (0.236)	3.03880 (8.092)	0.8569 (0.000)	1.4858 (0.000)	2.2965 (0.708)
Fr				1.3968 (0.0336)	4.3724 (0.3278)
KFr		0.8489 (16.083)	1.6239 (0.6979)	1.6341 (9.049)	3.4208 (0.7635)
EFr		0.9395 (3.543)		1.4169 (2.568)	0.9395 (0.3278)
BFr		0.7346 (1.5290)	1.5830 (0.7132)	1.6684 (0.7662)	3.5112 (0.9683)
TFr	-0.7166 (0.2616)			1.2656 (0.0579)	4.7121 (0.3657)
MOFr		0.0033 (0.0009)		6.2296 (1.0134)	1.2419 (0.1181)
McFr	0.8503 (0.1353)	44.423 (25.100)	19.859 (6.706)	0.0203 (0.0060)	46.974 (21.871)

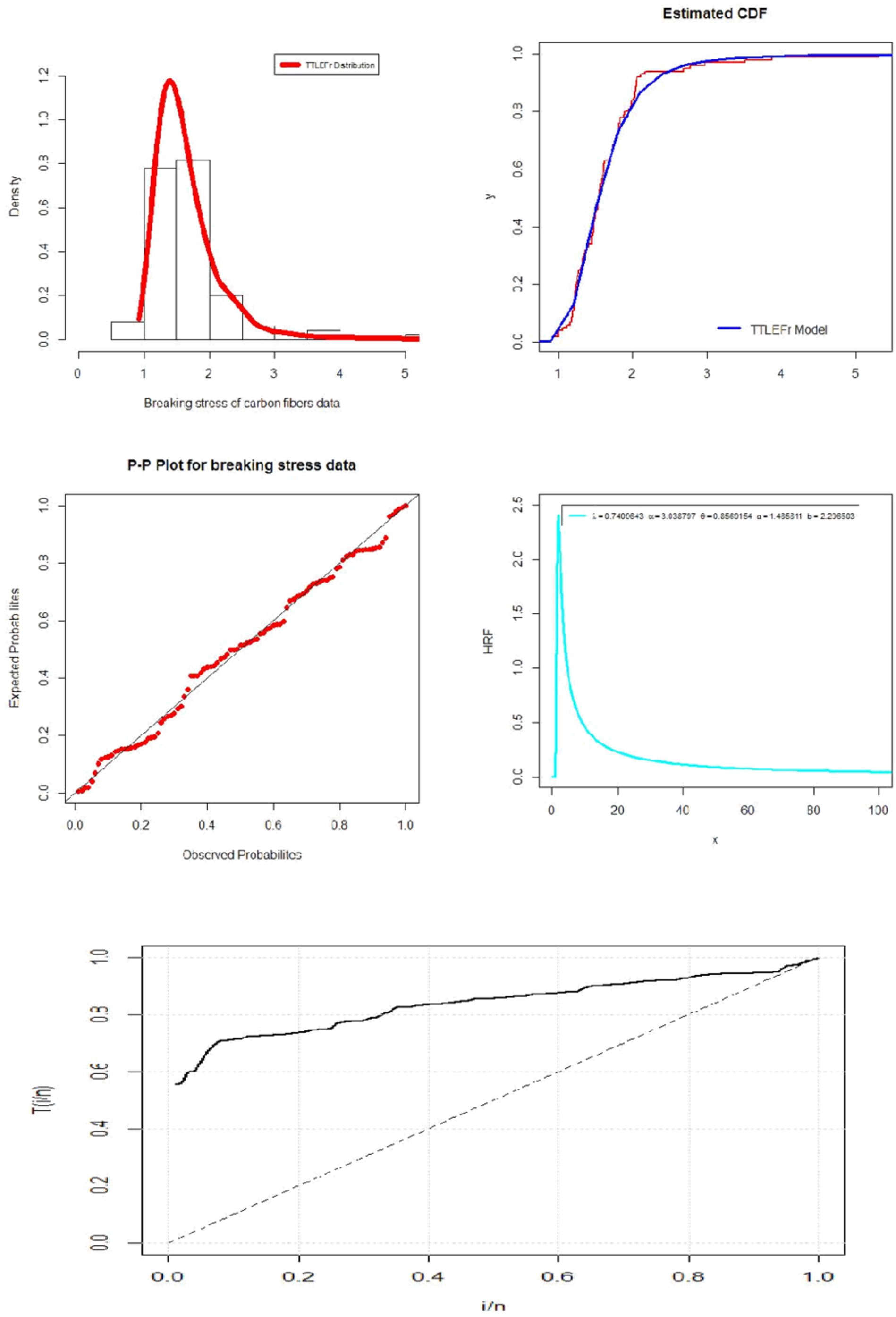

Figure 4: Estimated PDF, estimated CDF, P-P plot, estimated HRF and TTT plot of the carbon fibers data.

9. Concluding remarks

In this work, we introduce a new distribution for modeling the extreme values. Some important mathematical properties of the new model are derived. We assess the performance of the maximum likelihood method in terms of biases and mean squared errors by means of a simulation study. The new model is better than some other important competitive models in modeling the repair times data and the breaking stress data. We hope that the new model will attract a wider application in areas such as survival and lifetime data, engineering, meteorology, hydrology, economics and others.

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