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MAXIMUM PRODUCT SPACING AND BAYESIAN METHOD FOR PARAMETER ESTIMATION FOR GENERALIZED POWER WEIBULL DISTRIBUTION UNDER CENSORING SCHEME

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ABSTRACT

This article discusses the estimation of the Generalized Power Weibull parameters using the maximum product spacing (MPS) method, the maximum likelihood (ML) method and Bayesian estimation method under squares error for loss function. The estimation is done under progressive type-II censored samples and a comparative study among the three methods is made using Monte Carlo Simulation. Markov chain Monte Carlo (MCMC) method has been employed to compute the Bayes estimators of the Generalized Power Weibull distribution. The optimal censoring scheme has been suggested using two different optimality criteria (mean squared of error, Bias and relative efficiency). A real data is used to study the performance of the estimation process under this optimal scheme in practice for illustrative purposes. Finally, we discuss a method of obtaining the optimal censoring scheme.

Keywords: Maximum Likelihood, Maximum Product Spacing, Bayesian Estimation, Generalized Power Weibull and Progressive Type-II Censoring.

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1 Introduction

Many distributions have been used to make inferences about population based on a set of empirical data from these population. Determining an adequate model is a very important problem. The Weibull distribution is commonly used for modeling lifetime data with monotone failure rates. The major weakness of the Weibull distribution is its inability to accommodate non-monotone hazard rates, which has led to new generalizations of this distribution. Recently new classes of distributions based on modifications of the Weibull distribution has been proposed in the literature to provide a good fit to data set which has different hazard failure rates. One of the first extensions allowing for non-monotone hazard rates, including the bathtub shaped hazard rate function, is the exponentiated Weibull distribution (Mudholkar and Srivastava (1993); Mudholkar et al. (1995)). Pham, and Lai, (2007) introduced, the generalized power Weibull (GPW) distribution as a another extension of the Weibull family. Nikulin and Haghighi (2007), introduced a random variable X has the GPW distribution with parameters α, and θ, say if its cumulative distribution function (cdf), probability density function (pdf) and the quantile function are given as follows:

$$F(x;\theta,\alpha) = 1 - e^{1-(1+x^{\alpha})^{\theta}}, \qquad (1)$$

$$f(x;\theta,\alpha) = \theta\alpha x^{\alpha-1}(1+x^{\alpha})^{\theta-1}e^{1-(1+x^{\alpha})^{\theta}}, \qquad (2)$$

and

$$x_u = \left((1-\ln(1-u))^{\frac{1}{\theta}} - 1\right)^{\frac{1}{\alpha}} \quad ; 0 < u < 1, \qquad (3)$$

respectively, where $\alpha, \theta > 0$ and $x > 0$.

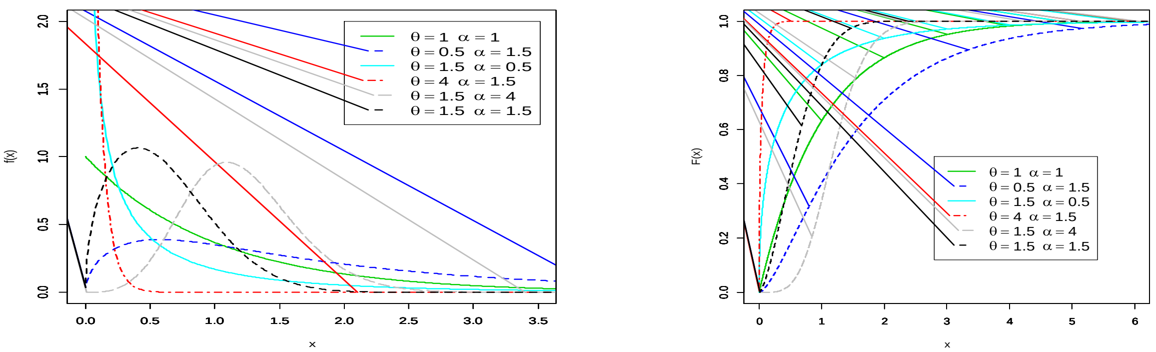

Figure 1: GPW distribution with Various Value of parameters

The Maximum Product of Spacings (MPS) estimation method was introduced by Cheng and Amin (1979, 1983) and independently discussed by Ranneby (1984) as an alternative to maximum likelihood estimation (MLE) method for the estimation of parameters of continuous univariate distributions. it was shown that for some distributions such as a three-parameter Gamma, Lognormal or Weibull distribution where the MLE method breaks down due to unboundedness of the likelihood, the MPS method produces consistent and asymptotically efficient estimators. In situations like mixture of normals where the MLE method is known to produce inconsistent estimators, the MSP estimators are consistent (see Ranneby, 1984). For comprehensive content, one can refer to Ekström (2008).

Right censoring is one of the censoring techniques used in life-testing experiments. A sample is said to be censored if while it is drawn from a complete population, the item values of some of its members are unknown. Kundu and Pradhan (2009) discussed the two most common censoring schemes termed as type-I and type-II censoring schemes. Ng et al (2012) introduced estimation of parameters for a three-parameter Weibull distribution based on progressively Type-II right censored samples using the ML method, corrected ML method, weighted ML method, MPS method and least squares estimation method. Singh et al (2016) proposed estimation of generalized inverted exponential distribution (GIED) based on progressive type-II censored samples. Basu et al (2018) discussed the maximum product of spacing estimator for a Progressive hybrid Type-I censoring scheme for inverse Lindley distribution. Almetwaly and Almongy (2018) discussed the complete censoring as a special case of the progressive type-II censoring scheme when estimating the GPW distribution parameters. Progressive Type-II censoring scheme can be described as follows: Suppose n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of failures to be observed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n-R_1-1$ units are randomly removed from the experiment. Finally, at the time of the $m-th$ failure, all the remaining surviving units $R_m = n - m - R_1 - \cdots R_m - 1$ are removed from the experiment. Therefore, a progressive Type-II censoring scheme consists of m, and $R_1, \dots, R_m$ such that $R_1 + \cdots + R_m = n - m$. And to more example see Dey at al (2016).

The aim of this paper is to estimate the parameters of the GPW model under Progressive Type-II Censoring Schemes. The maximum likelihood estimators (MLE) and the maximum product of spacing estimation (MPS) method are used as alternative methods. On the other hand, Bayesian estimators for the GPW parameters are considered under the assumptions of independent gamma priors are considered under squared errors of loss function. To evaluate the performance of the estimators, a simulation study is carried out. The optimal censoring scheme has been suggested using three different optimality criteria (mean squared error (MSE), Bias and relative efficiency (RE)) and a Markov chain Monte Carlo (MCMC) method is utilized for computing the Bayes estimates. The final motivation of the paper is to develop a guideline for introducing the best estimation method for in general distribution, which we think would be of deep interest to statisticians and also, a real data set is introduced and analyzed to investigate the model.

The paper is organized as follows: section 2 is devoted for the estimation of the GPW parameters using the MLE method and the MPS method while in section 3 the Bayesian estimation based on MCMC is considered. In section 4, we present Monte Carlo simulation study to compare the performance of the estimators of the GPW distribution parameters for all estimation methods, which are used. Finally, we show the results and the conclusion of the current study.

2 Estimation the parameter of the GPW distribution

The estimation problem of the unknown parameters of the GPW distribution has been discussed.

2.1 MLE method

Based on the observed sample $x_1 < \cdots < x_m$ from a progressive Type-II censoring scheme, $R_1, \dots, R_m$ the likelihood function can be written as

$$L_{ML} = A\prod_{i=1}^{m} f(t_i;\ \theta,\alpha)\ \ (1 - F(t_i;\ \theta,\alpha)^{R_i} \quad ; \quad \theta,\alpha > 0, \tag{4}$$

$$A = n(n - R_1 - 1)\dots\left(n - \sum_{i=1}^{m-1} R_i - (m-1)\right),$$

where

$$L_{ML} = A\theta^m\alpha^m e^{\sum_{i=1}^{m}\left(1-(1+x_i{}^{\alpha})^{\theta}\right)^{R_i+1}} \prod_{i=1}^{m}\left[x_i{}^{\alpha-1}(1 + x_i{}^{\alpha})^{\theta-1}\right], \tag{5}$$

the natural logarithm of the likelihood function is

$$\ln L_{ML} = \ln A + m\ln\theta + m\ln\alpha + (\alpha-1)\sum_{i=1}^{m}\ln x_i + (\theta-1)\sum_{i=1}^{m}\ln(1+{x_i}^{\alpha}) + \sum_{i=1}^{m}(R_i+1)\left(1-(1+{x_i}^{\alpha})^{\theta}\right), \tag{6}$$

to obtain the normal equations for the unknown parameters, we differentiate (6) partially with respect to the parameters θ and α and equate them to zero. The estimators for θ and α can be obtained as the solution of the following equations.

$$\frac{\partial \ln L_{ML}}{\partial\theta} = \frac{m}{\theta} + \sum_{i=1}^{m}\ln(1+x_i^{\alpha}) - \sum_{i=1}^{m}(1+{x_i}^{\alpha})^{\theta}\ \ln(1+{x_i}^{\alpha}) - \sum_{i=1}^{m}(R_i+1)\ (1+{x_i}^{\alpha})^{\theta}\ \ln(1+{x_i}^{\alpha}). \tag{7}$$

Differentiating the log-likelihood function in (16) with respect to α is given as

$$\frac{\partial \ln L_{ML}}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m}\frac{(\theta-1)\ {x_i}^{\alpha}}{(1+{x_i}^{\alpha})}\ln x_i + \sum_{i=1}^{m}\theta {x_i}^{\alpha}\ln x_i\,(1+{x_i}^{\alpha})^{\theta-1} - \sum_{i=1}^{m}\theta {x_i}^{\alpha}\ (R_i+1)\ln x_i\,(1+{x_i}^{\alpha})^{\theta-1}. \tag{8}$$

The MLE $\hat{\alpha}$,$\hat{\theta}$ can be obtained by solving simultaneously the likelihood equations

$$\left.\frac{\partial L_{ML}}{\partial\theta}\right|_{\theta=\hat{\theta}} = 0\ , \qquad \left.\frac{\partial L_{ML}}{\partial\alpha}\right|_{\alpha=\hat{\alpha}} = 0.$$

But the equation (7) and (8) have to be performed numerically using a nonlinear optimization algorithm.

2.2 MPS method

Ng et al (2012) introduced MPS method Based on Progressive Type-II Censored Sample method, MPS method chooses the parameter values that make the observed data as uniform as possible, according to a specific quantitative measure of uniformity.

$$S = \prod_{i=1}^{m+1} \left(F(t_i;\theta,\alpha) - F(t_{i-1};\theta,\alpha)\right) \prod_{i=1}^{m} \left(1 - F(t_i;\theta,\alpha)\right)^{R_i} \quad . \tag{9}$$

Cheng and Amin (1983) as follows the geometric mean of the spacing defined as:

$$G = \left(\prod_{i=1}^{m+1} D_i\right)^{\frac{1}{m+1}} , \tag{10}$$

where

$$D_i = \begin{cases} D_1 = F(x_1) \\ D_i = F(x_i) - F(x_{i-1}) = F\left(x_{(2:m)}\right); \qquad i = 2 \dots m, \\ D_{m+1} = 1 - F(x_m) \end{cases} \tag{11}$$

such that $\sum D_i = 1$, depending on MPS method that was introduced by Cheng and Amin (1983) and Progressive Type-II Censored scheme that was discussed by Balakrishnan et al (2000) and Ng et al (2004), to more example of complete sample see Singh et al (2014).

$$L_{MPS} = A \left(\left(1 - e^{1-(1+x_1{}^{\alpha})^{\theta}}\right) \left(e^{1-(1+x_m{}^{\alpha})^{\theta}}\right) \prod_{i=2}^{m} \left[e^{1-(1+x_{i-1}{}^{\alpha})^{\theta}} - e^{1-(1+x_i{}^{\alpha})^{\theta}} \right] \right) \prod_{i=1}^{m} \left(e^{1-(1+x_i{}^{\alpha})^{\theta}}\right)^{R_i} . \tag{12}$$

The natural logarithm of the likelihood function is

$$\ln L_{MPS} = \ln A + \left(\ln\left(1 - e^{1-(1+x_1{}^{\alpha})^{\theta}}\right) + \left(1 - (1 + x_m{}^{\alpha})^{\theta}\right) + \sum_{i=2}^{m} \ln\left[e^{1-(1+x_{i-1}{}^{\alpha})^{\theta}} - e^{1-(1+x_i{}^{\alpha})^{\theta}}\right]\right) + \sum_{i=1}^{m} R_i \ \left(1 - (1 + x_i{}^{\alpha})^{\theta}\right), \tag{13}$$

to obtain the normal equations for the unknown parameters, we differentiate (13) partially with respect to the parameters θ and α and equate them to zero. The estimators for θ and α can be obtained as the solution of the following equations.

$$\frac{\partial \ln L_{MPS}}{\partial \theta} = \left(\frac{(1 + x_1{}^{\alpha})^{\theta} \ \ln(1 + x_1{}^{\alpha}) \ e^{1-(1+x_1{}^{\alpha})^{\theta}}}{1 - e^{1-(1+x_1{}^{\alpha})^{\theta}}} - (1 + x_m{}^{\alpha})^{\theta} \ \ln(1 + x_m{}^{\alpha}) + \sum_{i=2}^{m} \frac{e^{1-(1+x_i{}^{\alpha})^{\theta}} \ (1 + x_i{}^{\alpha})^{\theta} \ \ln(1 + x_i{}^{\alpha}) - e^{1-(1+x_{i-1}{}^{\alpha})^{\theta}} \ (1 + x_{i-1}{}^{\alpha})^{\theta} \ \ln(1 + x_{i-1}{}^{\alpha})}{\left[e^{1-(1+x_{i-1}{}^{\alpha})^{\theta}} - e^{1-(1+x_i{}^{\alpha})^{\theta}}\right]}\right) - \sum_{i=1}^{m} R_i \ (1 + x_i{}^{\alpha})^{\theta} \ \ln(1 + x_i{}^{\alpha}), \tag{14}$$

and

$$\frac{\partial \ln L_{MPS}}{\partial \alpha} = \left(\frac{\theta x_1{}^{\alpha} \ \ln x_1 \, (1 + x_1{}^{\alpha})^{\ \theta-1} \ e^{1-(1+x_1{}^{\alpha})^{\theta}}}{1 - e^{1-(1+x_1{}^{\alpha})^{\theta}}} - \theta x_m{}^{\alpha} \ \ln x_m \, (1 + x_m{}^{\alpha})^{\ \theta-1} + \sum_{i=2}^{m} \frac{\theta x_{i-1}{}^{\alpha} \ \ln x_{i-1} \, (1 + x_{i-1}{}^{\alpha})^{\ \theta-1} \ e^{1-(1+x_{i-1}{}^{\alpha})^{\theta}} - \theta x_i{}^{\alpha} \ \ln x_i \, (1 + x_i{}^{\alpha})^{\ \theta-1} e}{\left[e^{1-(1+x_{i-1}{}^{\alpha})^{\theta}} - e^{1-(1+x_i{}^{\alpha})^{\theta}}\right]}\right) - \sum_{i=1}^{m} \theta x_i{}^{\alpha} \ R_i \ \ln x_i \, (1 + x_i{}^{\alpha})^{\ \theta-1}. \tag{15}$$

The MPS $\hat{\alpha}$, $\hat{\theta}$ can be obtained by solving simultaneously the likelihood equations

$$\left.\frac{\partial L_{MPS}}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0 \ , \qquad \left.\frac{\partial L_{MPS}}{\partial \alpha}\right|_{\alpha=\hat{\alpha}} = 0.$$

But the equation (14) and (15) have to be performed numerically using a nonlinear optimization algorithm.

3 Bayesian estimation of the GPW distribution

In this section we consider the Bayesian estimation of the unknown parameters α and θ. The Bayes estimates is considered under the assumption that the random variables α and θ have an independent gamma distribution is a conjugate prior to the GPW distributions. Assumed that $\theta \sim Gamma(a,b)$ and $\alpha \sim Gamma(c,d)$, then, the joint prior density function of α and θ can be written as follows:

$$g(\alpha,\theta) \propto \ \theta^{a-1}\ e^{-\frac{\theta}{b}}\ \alpha^{c-1}\ e^{-\frac{\alpha}{d}}\ ; \qquad a,b,c, and\ \ d\ > 0, \tag{16}$$

here all the hyper parameters $a,b,c \ \ and \ \ d$ are known and non-negative.

Based on the likelihood function (5) and the joint prior distribution density (16), the joint posterior distribution of α and θ is

$$g(\alpha,\theta|x) \ = K\theta^{m+a-1}\alpha^{m+c-1}e^{-\alpha d-\theta b}\ \ e^{\sum_{i=1}^{m}\left(1-(1+x_i{}^{\alpha})^{\theta}\right)^{R_i+1}}\prod_{i=1}^{m}\left(x_i{}^{\alpha-1}(1+x_i{}^{\alpha})^{\theta-1}\right)\ , \tag{17}$$

here the normalizing constant K is

$$K_{ML} = \left[\iint_0^{\infty} \theta^{m+a-1}\alpha^{m+c-1}e^{-\alpha d-\theta b}\ \ e^{\sum_{i=1}^{m}\left(1-(1+x_i{}^{\alpha})^{\theta}\right)^{R_i+1}}\prod_{i=1}^{m}\left(x_i{}^{\alpha-1}(1+x_i{}^{\alpha})^{\theta-1}\right)d\theta d\alpha\right]^{-1}. \tag{18}$$

In the method of MCMC can be used to generate samples from the posterior distribution density function (17) and in turn to compute the Bayes estimates of the unknown parameters. To generate samples from (17), we can rewrite the posterior distribution density (17) as

$$g(\alpha,\theta|x) \ \ \propto g_1(\alpha|\theta,x)\ \ g_2(\theta|\alpha,x), \tag{19}$$

where

$$g_1(\alpha|\theta,x) \propto \alpha^{m+c-1}e^{-\alpha d}\ \ e^{\sum_{i=1}^{m}\left(1-(1+x_i{}^{\alpha})^{\theta}\right)^{R_i+1}}\prod_{i=1}^{m}\left(x_i{}^{\alpha-1}(1+x_i{}^{\alpha})^{\theta-1}\right),$$

and

$$g_2(\theta|\alpha,x) \propto \theta^{m+a-1}e^{-\theta b}\ \ e^{\sum_{i=1}^{m}\left(1-(1+x_i{}^{\alpha})^{\theta}\right)^{R_i+1}}\prod_{i=1}^{m}\left((1+x_i{}^{\alpha})^{\theta-1}\right).$$

Since the conditional posterior distributions do not have simple forms in perspective of sampling, we use the Metropolis-Hastings algorithm. To generate samples from the conditional posterior density distributions, we use Markov chain Monte Carlo (MCMC). For more information about the Metropolis-Hastings algorithm see Metropolis et al (1953) and Nassar et al (2018). For more information about Bayesian estimation see Mahanta et al (2018) and Hanagal & Kamble (2016).

Almetwaly and Almongy (2018) discussed squares error (SE) of loss function is A very well-known symmetric loss function which is define as $L(\hat{\delta}_{SE},\delta_{SE})=\left(\hat{\delta}-\delta\right)^2$, after generating the parameters

$$\tilde{\alpha}=\sum_{i=1}^{M}\frac{\alpha^{(i)}}{M},\qquad \tilde{\gamma}=\sum_{i=1}^{M}\frac{\gamma^{(i)}}{M},\qquad \tilde{\theta}=\sum_{i=1}^{M}\frac{\theta^{(i)}}{M},$$

where M is the number of periods in the MCMC process.

4 Simulation study

In this section, we study a Monte Carlo simulation to compare the performance of the MLE method, MPS method and Bayesian estimation. The data were generated from the GPW Distribution for life time of different values of α and θ.

Monte Carlo experiments were carried out based on the data generated form GPW distribution for the following parameter values for α and θ are taken: case is $\alpha=4$; $\theta=1.5$, case is $\alpha=1.5$; $\theta=4$, case is $\alpha=1.5$; $\theta=0.5$ and case is $\alpha=1.5$; $\theta=1.5$ and for different sample size $n\ =20,\ 30, 50, 70, 100$ and 150, with 1000 replications. Different ratio of effective sample sizes $r=\frac{m}{n}$ and sets of different sample schemes are considered as following

Scheme 1: $R_1=R_2=\cdots=R_{\mathrm{m}-1}=0, and\ \ R_{\mathrm{m}}=n-m$. It is type-II scheme

Scheme 2: $R_1=n-m\ \ and\ \ R_2=R_3=\cdots=R_{\mathrm{m}-1}=0$.

Scheme 3: $R_1=R_2=\cdots=R_{\mathrm{m}-1}=1, and\ \ R_{\mathrm{m}}=n-2m-1$.

The bias, mean square errors (MSE) and **relative efficiency (RE)** of the estimates derived are calculated using the following formulae:

$Bias=\hat{\delta}-\delta,\ MSE=Mean(\hat{\delta}-\delta)^2$ and

$$RE1 = \frac{MSE(MLE)}{MSE(MPS)}, \qquad RE2 = \frac{MSE(MLE)}{MSE(MCMC)},$$

where $\hat{\delta}$ is the estimated value of $\delta = (\theta, \alpha)$.

We could define the best scheme as the scheme, which minimizes the mean squared error (MSE) of the estimator.

For all the above considered choices graph of MSE is plotted and attached in complete sample

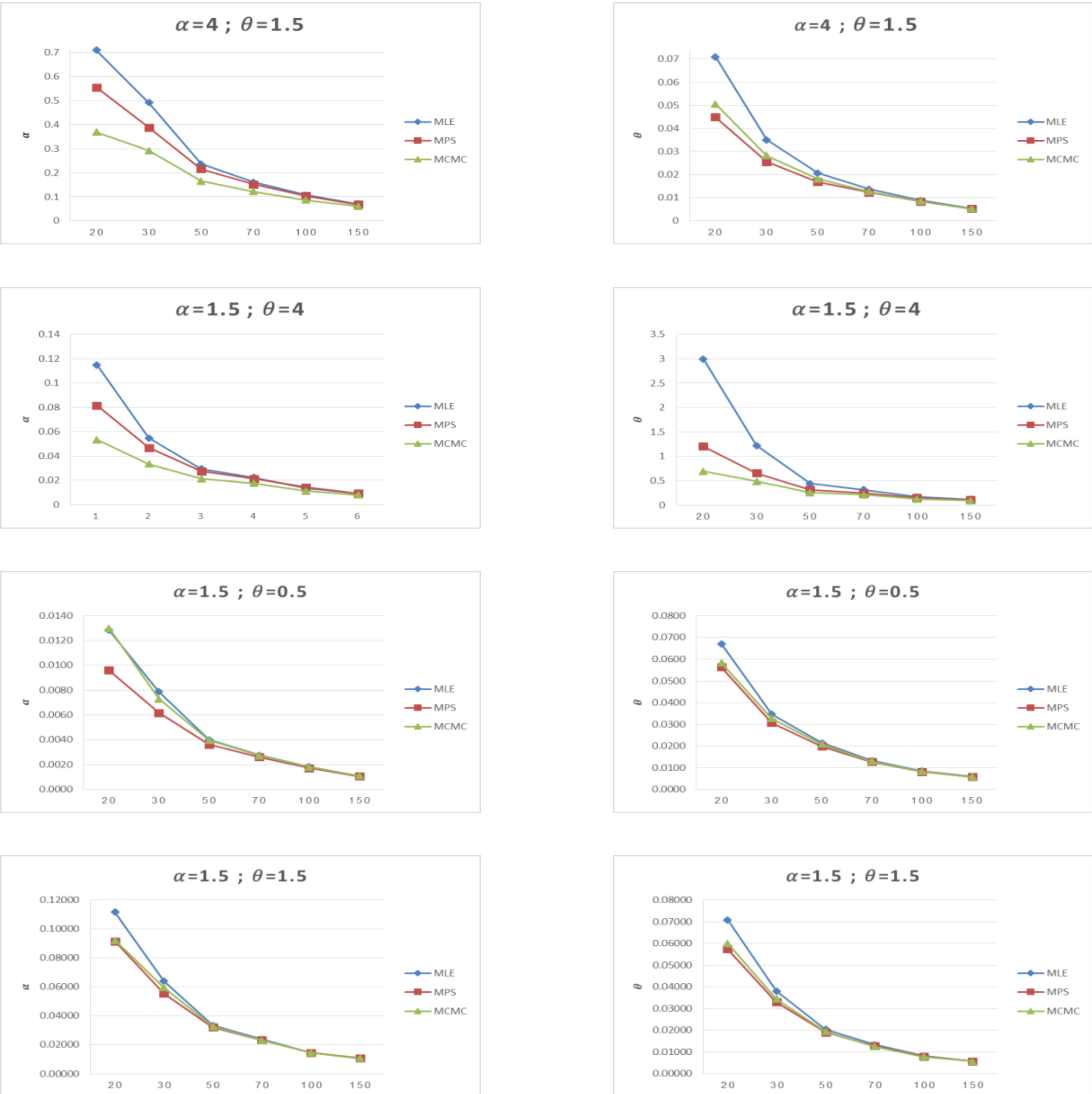

Figure 2: MSE of the estimates for different parameters with variation of sample size (n)

Table 1: Parameter estimation of GPW in complete sample when $\alpha = 4$; $\theta = 1.5$

$\alpha = 4$; $\theta = 1.5$									
		MLE		MPS		Bays MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0741	0.0710	-0.0300	0.0449	0.0432	0.0505	1.5813	1.4059
	$\hat{\alpha}$	0.2655	0.7084	-0.2439	0.5546	0.0131	0.3687	1.2773	1.9213
30	$\hat{\theta}$	0.0472	0.0351	-0.0264	0.0255	0.0301	0.0282	1.3765	1.2447
	$\hat{\alpha}$	0.2301	0.4906	-0.1487	0.3871	0.0567	0.2914	1.2674	1.6836
50	$\hat{\theta}$	0.0284	0.0207	-0.0205	0.0169	0.0188	0.0182	1.2249	1.1374
	$\hat{\alpha}$	0.1161	0.2358	-0.1420	0.2153	0.0177	0.1645	1.0952	1.4334
70	$\hat{\theta}$	0.0139	0.0137	-0.0230	0.0123	0.0076	0.0127	1.1138	1.0787
	$\hat{\alpha}$	0.0889	0.1609	-0.1114	0.1514	0.0208	0.1215	1.0627	1.3243
100	$\hat{\theta}$	0.0058	0.0088	-0.0220	0.0084	0.0013	0.0085	1.0476	1.0353
	$\hat{\alpha}$	0.0646	0.1077	-0.0880	0.1035	0.0169	0.0865	1.0406	1.2451
150	$\hat{\theta}$	0.0074	0.0055	-0.0130	0.0052	0.0042	0.0053	1.0577	1.0377
	$\hat{\alpha}$	0.0478	0.0708	-0.0630	0.0688	0.0172	0.0603	1.0291	1.1741

Table 2: Parameter estimation of GPW in scheme 1 when $\alpha = 4$; $\theta = 1.5$

scheme 1									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.9181	3.8723	0.4631	1.8354	0.4712	0.5395	2.1098	7.1777
	$\hat{\alpha}$	1.2465	6.3948	-0.4534	2.4099	0.1926	0.5630	2.6536	11.3581
30	$\hat{\theta}$	0.7432	5.3159	0.0719	0.7520	0.4193	0.4837	7.0693	10.9890
	$\hat{\alpha}$	0.6687	2.3626	-0.4827	1.3369	0.3030	0.6444	1.7672	3.6665
50	$\hat{\theta}$	0.4161	1.3059	0.0533	0.3635	0.2496	0.2501	3.5931	5.2207
	$\hat{\alpha}$	0.4173	1.1228	-0.3541	0.7797	0.1506	0.3992	1.4400	2.8124
70	$\hat{\theta}$	0.2151	0.3168	-0.0112	0.1386	0.2428	0.2343	2.2860	1.3520
	$\hat{\alpha}$	0.2581	0.6412	-0.3181	0.5335	0.2131	0.3862	1.2018	1.6601
100	$\hat{\theta}$	0.1256	0.1378	-0.0326	0.0757	0.2037	0.0553	1.8212	2.4922
	$\hat{\alpha}$	0.1728	0.3713	-0.2596	0.3452	0.1483	0.2069	1.0757	1.7946
150	$\hat{\theta}$	0.0928	0.1025	-0.0192	0.0673	0.1563	0.0436	1.5221	2.3489
	$\hat{\alpha}$	0.1281	0.2533	-0.1813	0.2375	0.1455	0.2052	1.0665	1.2346

$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.8823	4.9408	0.1728	1.0649	0.0059	0.1213	4.6397	40.7256
	$\hat{\alpha}$	0.8195	3.3104	-0.4978	1.6401	-0.2352	0.8831	2.0184	3.7487
30	$\hat{\theta}$	0.3649	0.8591	0.0427	0.2491	0.0417	0.1609	3.4486	5.3387
	$\hat{\alpha}$	0.5271	1.5397	-0.4245	0.9760	-0.1316	0.6760	1.5776	2.2776
50	$\hat{\theta}$	0.1924	0.2936	0.0102	0.1302	0.0467	0.1237	2.2558	2.3732
	$\hat{\alpha}$	0.2504	0.6726	-0.3649	0.5936	-0.0560	0.5600	1.1331	1.2012
70	$\hat{\theta}$	0.1287	0.1350	-0.0005	0.0749	0.0268	0.0866	1.8019	1.5591
	$\hat{\alpha}$	0.2147	0.4297	-0.2566	0.3746	-0.1052	0.2558	1.1469	1.6797
100	$\hat{\theta}$	0.0875	0.0790	-0.0056	0.0522	0.0610	0.0419	1.5139	1.8862
	$\hat{\alpha}$	0.1467	0.2986	-0.2044	0.2818	-0.0514	0.2505	1.0597	1.1919
150	$\hat{\theta}$	0.0419	0.0411	-0.0224	0.0316	0.0058	0.0303	1.2992	1.3529
	$\hat{\alpha}$	0.0720	0.1758	-0.1787	0.1442	-0.0447	0.1400	1.2192	1.2553
$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.2097	0.2961	0.0656	0.1263	0.1406	0.1519	2.3449	1.9496
	$\hat{\alpha}$	0.4839	1.5436	-0.4642	1.0620	0.0715	0.6586	1.4535	2.3439
30	$\hat{\theta}$	0.1540	0.1998	0.0555	0.1055	0.1007	0.0982	1.8936	2.0358
	$\hat{\alpha}$	0.3187	0.8507	-0.3672	0.6820	-0.0192	0.3836	1.2474	2.2175
50	$\hat{\theta}$	0.0848	0.0613	0.0274	0.0396	0.0509	0.0410	1.5466	1.4953
	$\hat{\alpha}$	0.1949	0.4152	-0.2606	0.3728	0.0462	0.2127	1.1137	1.9521
70	$\hat{\theta}$	0.0505	0.0401	0.0094	0.0295	0.0373	0.0345	1.3592	1.1625
	$\hat{\alpha}$	0.1288	0.2859	-0.2171	0.2791	0.0162	0.2195	1.0243	1.3026
100	$\hat{\theta}$	0.0338	0.0256	0.0044	0.0207	0.0262	0.0244	1.2378	1.0486
	$\hat{\alpha}$	0.1002	0.2035	-0.1558	0.1978	0.0227	0.1329	1.0286	1.5307
150	$\hat{\theta}$	0.0258	0.0149	0.0047	0.0126	0.0187	0.0136	1.1834	1.0946
	$\hat{\alpha}$	0.0518	0.1189	-0.1301	0.1135	0.0485	0.1114	1.0482	1.0675

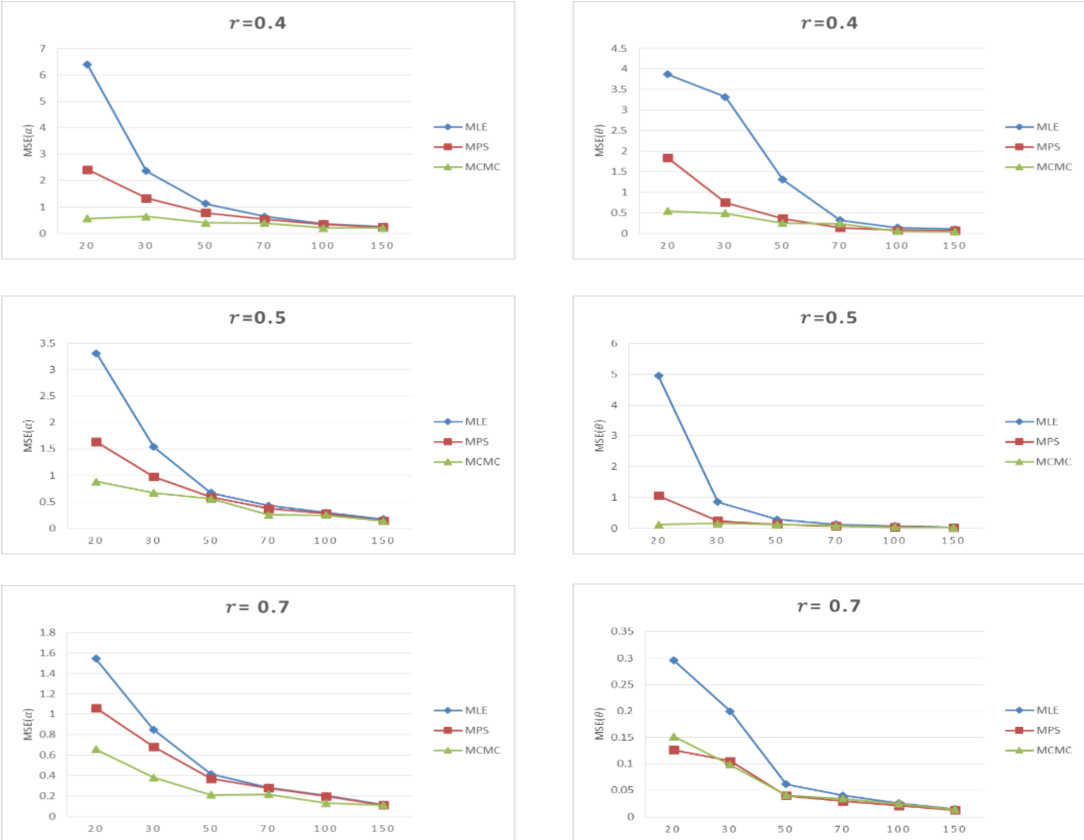

Figure 3: MSE of the estimates for different parameters with variations of sample size and effective sample size in scheme 1

Table 3: Parameter estimation of GPW in scheme 2 when $\alpha = 4$; $\theta = 1.5$

Scheme 2									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.3036	0.8340	0.2448	0.3456	0.1097	0.1929	2.4130	4.3234
	$\hat{\alpha}$	0.5686	1.9617	-0.4753	1.2479	0.0001	0.4155	1.5720	4.7208
30	$\hat{\theta}$	0.1241	0.1698	0.1275	0.1165	0.0659	0.0948	1.4585	1.7917
	$\hat{\alpha}$	0.3281	0.9581	-0.4601	0.8138	0.1115	0.6375	1.1773	1.5030
50	$\hat{\theta}$	0.0993	0.0889	0.1104	0.0712	0.0423	0.0477	1.2483	1.8647
	$\hat{\alpha}$	0.2209	0.5507	-0.3511	0.5063	-0.0530	0.3612	1.0877	1.5246
70	$\hat{\theta}$	0.0539	0.0420	0.0690	0.0363	-0.0002	0.0275	1.1549	1.5287
	$\hat{\alpha}$	0.1436	0.3667	-0.3036	0.3703	-0.0758	0.1757	0.9901	2.0874
100	$\hat{\theta}$	0.0292	0.0233	0.0437	0.0211	0.0411	0.0203	1.1039	1.1436
	$\hat{\alpha}$	0.1016	0.2330	-0.2464	0.2315	0.1051	0.1453	1.0066	1.6038
150	$\hat{\theta}$	0.0203	0.0176	0.0320	0.0164	0.0098	0.0149	1.0694	1.1781
	$\hat{\alpha}$	0.0818	0.1633	-0.1735	0.1617	0.0321	0.1470	1.0099	1.1112

$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.2001	0.3360	0.1857	0.1946	0.0549	0.1279	1.7265	2.6271
	$\hat{\alpha}$	0.4497	1.4361	-0.4802	1.0163	-0.1062	0.7639	1.4131	1.8800
30	$\hat{\theta}$	0.0996	0.1178	0.1118	0.0876	0.0285	0.0736	1.3443	1.5999
	$\hat{\alpha}$	0.3209	0.8277	-0.4009	0.6689	-0.1568	0.5514	1.2374	1.5011
50	$\hat{\theta}$	0.0660	0.0539	0.0814	0.0457	-0.0099	0.0403	1.1785	1.3372
	$\hat{\alpha}$	0.1433	0.4069	-0.3468	0.4024	-0.1630	0.4017	1.0113	1.0130
70	$\hat{\theta}$	0.0384	0.0308	0.0542	0.0276	-0.0143	0.0228	1.1185	1.3504
	$\hat{\alpha}$	0.1417	0.2788	-0.2433	0.2773	-0.0304	0.2703	1.0055	1.0315
100	$\hat{\theta}$	0.0270	0.0203	0.0404	0.0188	0.0097	0.0140	1.0768	1.4445
	$\hat{\alpha}$	0.0964	0.2026	-0.1959	0.2016	0.0092	0.1869	1.0048	1.0838
150	$\hat{\theta}$	0.0123	0.0115	0.0228	0.0108	0.0054	0.0110	1.0574	1.0393
	$\hat{\alpha}$	0.0485	0.1261	-0.1655	0.1202	-0.0243	0.1010	1.0491	1.2478
$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0990	0.1097	0.1162	0.0816	0.1372	0.0973	1.3449	1.1268
	$\hat{\alpha}$	0.3558	1.0581	-0.4366	0.8308	0.0259	0.4297	1.2736	2.4622
30	$\hat{\theta}$	0.0774	0.0765	0.0949	0.0620	0.0203	0.0428	1.2340	1.7875
	$\hat{\alpha}$	0.2376	0.6074	-0.3475	0.5378	-0.0508	0.3563	1.1294	1.7050
50	$\hat{\theta}$	0.0449	0.0305	0.0614	0.0276	0.0231	0.0175	1.1029	1.7424
	$\hat{\alpha}$	0.1534	0.3241	-0.2433	0.3059	-0.0021	0.2165	1.0594	1.4966
70	$\hat{\theta}$	0.0246	0.0201	0.0390	0.0186	0.0064	0.0161	1.0799	1.2476
	$\hat{\alpha}$	0.1015	0.2254	-0.2024	0.2219	-0.0056	0.1915	1.0158	1.1770
100	$\hat{\theta}$	0.0145	0.0135	0.0262	0.0128	0.0042	0.0120	1.0534	1.1205
	$\hat{\alpha}$	0.0796	0.1616	-0.1460	0.1610	0.0407	0.1171	1.0032	1.3794
150	$\hat{\theta}$	0.0146	0.0085	0.0231	0.0083	0.0123	0.0076	1.0265	1.1267
	$\hat{\alpha}$	0.0403	0.0974	-0.1213	0.0930	-0.0512	0.0808	1.0477	1.2065

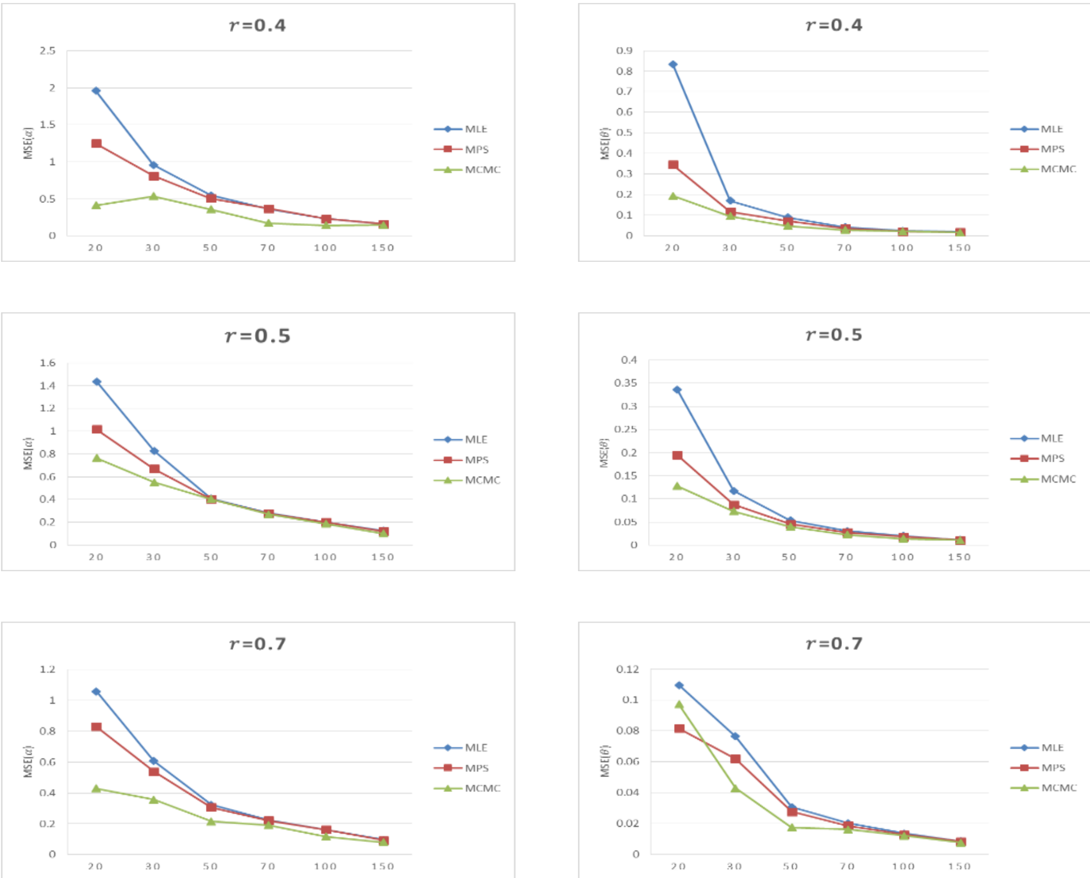

Figure 4: MSE of the estimates for different parameters with variations of sample size and effective sample size in scheme 2

Table 4: Parameter estimation of GPW in scheme 3 when $\alpha = 4$; $\theta = 1.5$

scheme 3									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	2.5697	14.1096	0.3914	9.3542	0.5324	0.6970	1.5084	20.2439
	$\hat{\alpha}$	1.0581	4.8069	-0.4161	2.0531	0.2901	0.6399	2.3413	7.5120
30	$\hat{\theta}$	0.5132	2.2454	0.0852	0.5032	0.3505	0.4190	4.4623	5.3590
	$\hat{\alpha}$	0.5700	1.8242	-0.4356	1.1330	0.3534	0.6886	1.6101	2.6494
50	$\hat{\theta}$	0.3123	0.6954	0.0707	0.2581	0.1809	0.2165	2.6945	3.2124
	$\hat{\alpha}$	0.3576	0.8862	-0.3134	0.6517	0.1329	0.4749	1.3599	1.8660
70	$\hat{\theta}$	0.1655	0.1988	0.0123	0.1026	0.1896	0.0933	1.9378	2.1312
	$\hat{\alpha}$	0.2217	0.5170	-0.2791	0.4460	0.1132	0.3679	1.1592	1.4051
100	$\hat{\theta}$	0.0969	0.0914	-0.0103	0.0562	0.1623	0.0434	1.6263	2.1073
	$\hat{\alpha}$	0.1494	0.3038	-0.2254	0.2875	0.1494	0.2304	1.0569	1.3186
150	$\hat{\theta}$	0.0711	0.0682	-0.0039	0.0490	0.0332	0.0347	1.3930	1.9654

	$\hat{\alpha}$	0.1108	0.2056	-0.1559	0.1955	0.0270	0.1693	1.0513	1.2141
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.4928	2.6705	0.1895	0.5833	0.1544	0.1839	4.5780	14.5228
	$\hat{\alpha}$	0.6374	2.1995	-0.4159	1.2566	-0.2162	0.5384	1.7503	4.0850
30	$\hat{\theta}$	0.2266	0.3555	0.0856	0.1649	0.0167	0.0955	2.1558	3.7213
	$\hat{\alpha}$	0.4194	1.0609	-0.3372	0.7338	-0.2427	0.4225	1.4457	2.5111
50	$\hat{\theta}$	0.1266	0.1343	0.0507	0.0823	0.0407	0.0699	1.6312	1.9203
	$\hat{\alpha}$	0.1901	0.4574	-0.2913	0.4272	-0.0807	0.3053	1.0706	1.4982
70	$\hat{\theta}$	0.0829	0.0676	0.0321	0.0475	-0.0068	0.0365	1.4254	1.8539
	$\hat{\alpha}$	0.1690	0.2977	-0.1948	0.2682	-0.0913	0.2349	1.1099	1.2674
100	$\hat{\theta}$	0.0565	0.0407	0.0212	0.0316	-0.0181	0.0301	1.2864	1.3506
	$\hat{\alpha}$	0.1130	0.2062	-0.1549	0.1991	-0.0808	0.1388	1.0359	1.4861
150	$\hat{\theta}$	0.0274	0.0220	0.0035	0.0185	-0.0030	0.0218	1.1871	1.0098
	$\hat{\alpha}$	0.0572	0.1228	-0.1331	0.1218	0.0070	0.0958	1.0079	1.2810

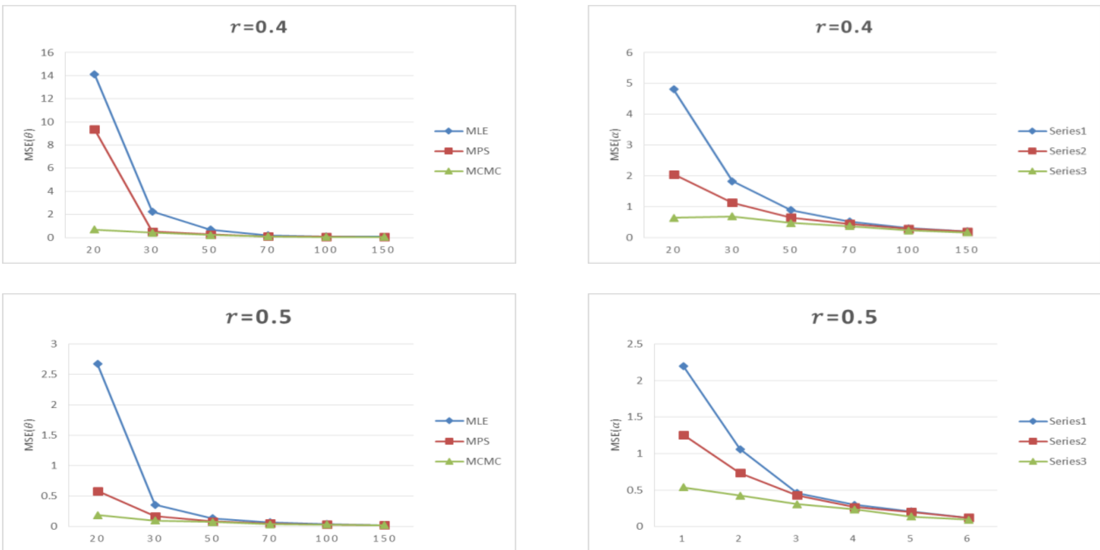

Figure 5: MSE of the estimates for different parameters with variations of sample size and effective sample size in scheme 3

Table 5: Parameter estimation of GPW in complete sample when $\alpha = 1.5$; $\theta = 4$

$\alpha = 1.5$; $\theta = 4$									
		MLE		MPS		Bays MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.6767	2.9968	-0.2103	1.2034	0.1761	0.6957	2.4903	4.3076
	$\hat{\alpha}$	0.1245	0.1150	-0.0680	0.0816	0.0456	0.0534	1.4093	2.1536
30	$\hat{\theta}$	0.3922	1.2186	-0.2146	0.6531	0.1368	0.4834	1.8659	2.5209
	$\hat{\alpha}$	0.0661	0.0546	-0.0729	0.0468	0.0236	0.0332	1.1667	1.6446
50	$\hat{\theta}$	0.2101	0.4436	-0.1884	0.3139	0.0803	0.2529	1.4132	1.7541
	$\hat{\alpha}$	0.0409	0.0294	-0.0547	0.0273	0.0175	0.0212	1.0769	1.3868
70	$\hat{\theta}$	0.1603	0.3170	-0.1478	0.2428	0.0745	0.2109	1.3056	1.5031
	$\hat{\alpha}$	0.0321	0.0224	-0.0425	0.0212	0.0165	0.0175	1.0566	1.2800
100	$\hat{\theta}$	0.0815	0.1660	-0.1479	0.1520	0.0275	0.1227	1.0921	1.3529
	$\hat{\alpha}$	0.0123	0.0133	-0.0442	0.0121	0.0027	0.0111	1.0717	1.1982
150	$\hat{\theta}$	0.0692	0.1159	-0.1013	0.1055	0.0329	0.0926	1.0986	1.2516
	$\hat{\alpha}$	0.0133	0.0089	-0.0288	0.0088	0.0067	0.0077	1.0114	1.1558

Table 6: Parameter estimation of GPW in scheme 1 sample when $\alpha = 1.5$; $\theta = 4$

scheme 1									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	3.6761	19.1503	2.4784	11.4017	0.0080	0.5224	1.6796	36.6585
	$\hat{\alpha}$	0.5022	1.0279	-0.1602	0.3565	0.0551	0.0559	2.8832	18.3947
30	$\hat{\theta}$	2.6701	10.4972	0.5702	7.8786	-0.0324	0.4716	1.3324	22.2572
	$\hat{\alpha}$	0.2691	0.3732	-0.1798	0.1966	-0.0152	0.0299	1.8985	12.4741
50	$\hat{\theta}$	3.0629	9.7645	0.1840	6.1017	0.1623	0.8930	1.6003	10.9347
	$\hat{\alpha}$	0.1670	0.1747	-0.1341	0.1152	0.0169	0.0320	1.5160	5.4604
70	$\hat{\theta}$	1.4050	4.9344	-0.1533	3.4706	0.1900	0.8219	1.4218	6.0035
	$\hat{\alpha}$	0.1034	0.0989	-0.1215	0.0788	-0.0080	0.0185	1.2547	5.3536
100	$\hat{\theta}$	0.7846	4.3253	-0.2436	1.6757	0.0861	0.7085	2.5812	6.1045
	$\hat{\alpha}$	0.0689	0.0565	-0.0999	0.0511	-0.0319	0.0184	1.1062	3.0660
150	$\hat{\theta}$	0.5702	2.8780	-0.1503	1.4621	0.3729	1.0939	1.9684	2.6308
	$\hat{\alpha}$	0.0512	0.0387	-0.0700	0.0354	0.0290	0.0192	1.0927	2.0183

$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	3.2665	12.2826	1.2240	10.0211	0.1479	0.5662	1.2257	21.6914
	$\hat{\alpha}$	0.3315	0.5311	-0.1810	0.2418	0.0119	0.0397	2.1965	13.3767
30	$\hat{\theta}$	2.1835	9.2084	0.0746	6.9535	0.1999	0.5932	1.3243	15.5241
	$\hat{\alpha}$	0.2117	0.2413	-0.1570	0.1423	0.0139	0.0303	1.6957	7.9683
50	$\hat{\theta}$	1.3142	6.9808	-0.1488	3.5681	0.1763	0.6731	1.9564	10.3717
	$\hat{\alpha}$	0.1009	0.1032	-0.1372	0.0865	-0.0033	0.0236	1.1938	4.3720
70	$\hat{\theta}$	0.8706	5.1934	-0.1418	1.9637	0.0874	0.7119	2.6447	7.2951
	$\hat{\alpha}$	0.0859	0.0657	-0.0969	0.0549	0.0007	0.0252	1.1971	2.6109
100	$\hat{\theta}$	0.5739	2.5409	-0.1368	1.2916	0.2688	0.5994	1.9672	4.2390
	$\hat{\alpha}$	0.0588	0.0453	-0.0775	0.0415	0.0170	0.0171	1.0923	2.6474
150	$\hat{\theta}$	0.2869	1.1596	-0.1931	0.7602	0.3045	0.6991	1.5253	1.6587
	$\hat{\alpha}$	0.0291	0.0265	-0.0683	0.0262	0.0136	0.0170	1.0125	1.5631
$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	2.0784	33.6040	0.0127	5.6110	0.2247	0.8638	5.9890	38.9023
	$\hat{\alpha}$	0.1944	0.2364	-0.1652	0.1505	0.0342	0.0564	1.5705	4.1935
30	$\hat{\theta}$	1.2632	7.0251	-0.0233	3.6150	0.2264	0.7504	1.9433	9.3613
	$\hat{\alpha}$	0.1274	0.1280	-0.1318	0.0964	0.0256	0.0361	1.3283	3.5498
50	$\hat{\theta}$	0.6131	2.2628	-0.1088	0.9945	0.2922	0.8547	2.2754	2.6476
	$\hat{\alpha}$	0.0774	0.0614	-0.0945	0.0528	0.0521	0.0308	1.1640	1.9926
70	$\hat{\theta}$	0.3915	1.4165	-0.1300	0.8009	0.3076	0.7123	1.7687	1.9887
	$\hat{\alpha}$	0.0511	0.0421	-0.0792	0.0397	0.0461	0.0258	1.0609	1.6288
100	$\hat{\theta}$	0.2759	0.8329	-0.1001	0.5517	0.2115	0.4954	1.5099	1.6814
	$\hat{\alpha}$	0.0397	0.0297	-0.0568	0.0281	0.0194	0.0208	1.0560	1.4288
150	$\hat{\theta}$	0.1665	0.4108	-0.0961	0.3109	0.2376	0.3013	1.3214	1.3634
	$\hat{\alpha}$	0.0207	0.0172	-0.0480	0.0172	0.0302	0.0169	1.0038	1.0167

Table 7: Parameter estimation of GPW in scheme 2 sample when $\alpha = 1.5$; $\theta = 4$

scheme 2									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	3.1193	31.0689	2.5922	8.7140	0.1247	0.6538	3.5654	47.5202
	$\hat{\alpha}$	0.2298	0.2916	-0.1603	0.1721	0.0792	0.0486	1.6944	5.9997
30	$\hat{\theta}$	1.0423	8.1588	0.0335	2.1809	0.2961	0.7822	3.7410	10.4308
	$\hat{\alpha}$	0.1330	0.1359	-0.1558	0.1079	0.1197	0.0588	1.2594	2.3115
50	$\hat{\theta}$	0.6596	2.7813	0.0352	1.1395	0.2163	0.6841	2.4408	4.0659
	$\hat{\alpha}$	0.0879	0.0760	-0.1174	0.0662	0.0562	0.0439	1.1480	1.7313
70	$\hat{\theta}$	0.3778	1.0594	-0.0545	0.5591	0.2477	0.5017	1.8948	2.1118
	$\hat{\alpha}$	0.0575	0.0497	-0.1010	0.0478	0.0770	0.0421	1.0387	1.1801
100	$\hat{\theta}$	0.2276	0.4945	-0.0838	0.3148	0.1507	0.2961	1.5709	1.6702
	$\hat{\alpha}$	0.0403	0.0309	-0.0814	0.0292	0.0447	0.0232	1.0582	1.3337
150	$\hat{\theta}$	0.1709	0.3609	-0.0501	0.2607	0.1387	0.2395	1.3840	1.5065
	$\hat{\alpha}$	0.0321	0.0215	-0.0562	0.0213	0.0359	0.0171	1.0101	1.2568
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	1.7995	9.7234	0.2735	5.5227	0.2447	0.7621	1.7606	12.7581
	$\hat{\alpha}$	0.1820	0.2110	-0.1606	0.1374	0.1011	0.0684	1.5363	3.0831
30	$\hat{\theta}$	0.8461	4.1851	0.0174	1.4362	0.3110	0.8094	2.9140	5.1708
	$\hat{\alpha}$	0.1273	0.1155	-0.1328	0.0879	0.0967	0.0532	1.3136	2.1705
50	$\hat{\theta}$	0.4275	1.4289	-0.0554	0.7164	0.2426	0.7052	1.9945	2.0261
	$\hat{\alpha}$	0.0576	0.0548	-0.1161	0.0535	0.0766	0.0387	1.0238	1.4183
70	$\hat{\theta}$	0.3091	0.7377	-0.0432	0.4373	0.2306	0.3988	1.6869	1.8497
	$\hat{\alpha}$	0.0555	0.0373	-0.0792	0.0355	0.0573	0.0341	1.0530	1.0938
100	$\hat{\theta}$	0.2096	0.4378	-0.0452	0.3037	0.1301	0.2961	1.4413	1.4786
	$\hat{\alpha}$	0.0374	0.0267	-0.0638	0.0262	0.0238	0.0242	1.0190	1.1022
150	$\hat{\theta}$	0.1065	0.2335	-0.0723	0.1862	0.0917	0.1523	1.2545	1.5338
	$\hat{\alpha}$	0.0196	0.0164	-0.0539	0.0162	0.0182	0.0147	1.0131	1.1140

$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.9310	4.8583	0.0192	1.6136	0.3169	0.7454	3.0109	6.5175
	$\hat{\alpha}$	0.1402	0.1465	-0.1435	0.1091	0.1087	0.0786	1.3426	1.8633
30	$\hat{\theta}$	0.6236	2.7244	0.0115	1.1568	0.1707	0.5956	2.3551	4.5742
	$\hat{\alpha}$	0.0934	0.0826	-0.1134	0.0692	0.0529	0.0508	1.1934	1.6272
50	$\hat{\theta}$	0.3430	0.7533	-0.0222	0.4168	0.1470	0.4817	1.8072	1.5638
	$\hat{\alpha}$	0.0599	0.0432	-0.0781	0.0391	0.0693	0.0393	1.1058	1.1012
70	$\hat{\theta}$	0.2126	0.4738	-0.0508	0.3213	0.1434	0.3039	1.4747	1.5592
	$\hat{\alpha}$	0.0393	0.0296	-0.0653	0.0290	0.0322	0.0188	1.0182	1.5687
100	$\hat{\theta}$	0.1459	0.2824	-0.0420	0.2160	0.1469	0.1907	1.3074	1.4807
	$\hat{\alpha}$	0.0303	0.0208	-0.0468	0.0204	0.0419	0.0183	1.0173	1.1347
150	$\hat{\theta}$	0.0945	0.1504	-0.0378	0.1236	0.1180	0.1062	1.2168	1.4167
	$\hat{\alpha}$	0.0156	0.0124	-0.0393	0.0124	0.0362	0.0108	1.0015	1.1446

Table 8: Parameter estimation of GPW in scheme 3 sample when $\alpha = 1.5$; $\theta = 4$

scheme 3									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	7.1150	29.2712	3.7847	17.6073	0.0654	0.5495	1.6624	53.2667
	$\hat{\alpha}$	0.4273	0.7707	-0.1469	0.3037	0.0460	0.0432	2.5376	17.8202
30	$\hat{\theta}$	4.9733	14.2867	0.4031	8.1060	-0.0686	0.6776	1.7625	21.0827
	$\hat{\alpha}$	0.2291	0.2852	-0.1598	0.1648	-0.0266	0.0465	1.7303	6.1358
50	$\hat{\theta}$	2.1431	10.4151	0.1645	8.4821	0.3183	0.8373	1.2279	12.4387
	$\hat{\alpha}$	0.1433	0.1367	-0.1168	0.0952	0.0061	0.0273	1.4358	5.0127
70	$\hat{\theta}$	1.0566	7.2828	-0.0991	2.4599	0.2727	0.9322	2.9606	7.8124
	$\hat{\alpha}$	0.0890	0.0790	-0.1049	0.0651	0.0319	0.0298	1.2125	2.6459
100	$\hat{\theta}$	0.6078	2.7251	-0.1738	1.2393	0.1629	0.7365	2.1988	3.7002
	$\hat{\alpha}$	0.0598	0.0459	-0.0854	0.0420	0.0008	0.0213	1.0933	2.1574
150	$\hat{\theta}$	0.4394	1.8162	-0.1043	1.0453	0.1516	0.5994	1.7375	3.0301
	$\hat{\alpha}$	0.0444	0.0310	-0.0592	0.0288	0.0145	0.0170	1.0790	1.8258

$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	2.7825	19.0146	2.8344	12.7835	0.1879	0.7267	1.4874	26.1656
	$\hat{\alpha}$	0.2578	0.3447	-0.1439	0.1802	0.0249	0.0475	1.9131	7.2559
30	$\hat{\theta}$	1.7874	8.5057	0.1413	4.6927	0.2384	0.7970	1.8126	10.6720
	$\hat{\alpha}$	0.1679	0.1604	-0.1177	0.1033	0.0522	0.0432	1.5526	3.7121
50	$\hat{\theta}$	0.7941	4.8326	-0.0340	1.8856	0.2243	0.8456	2.5629	5.7153
	$\hat{\alpha}$	0.0765	0.0671	-0.1037	0.0594	0.0390	0.0306	1.1299	2.1932
70	$\hat{\theta}$	0.5494	2.0905	-0.0282	1.0632	0.3165	0.7748	1.9663	2.6982
	$\hat{\alpha}$	0.0675	0.0436	-0.0684	0.0375	0.0417	0.0228	1.1628	1.9148
100	$\hat{\theta}$	0.3622	1.1125	-0.0412	0.6921	0.2109	0.6552	1.6075	1.6980
	$\hat{\alpha}$	0.0450	0.0298	-0.0547	0.0278	0.0429	0.0224	1.0711	1.3299
150	$\hat{\theta}$	0.1851	0.5376	-0.0886	0.3988	0.1035	0.3564	1.3482	1.5083
	$\hat{\alpha}$	0.0233	0.0175	-0.0474	0.0173	0.0111	0.0127	1.0141	1.3836

Table 9: Parameter estimation of GPW in complete sample when $\alpha = 1.5$; $\theta = 0.5$

n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0758	0.0671	0.0999	0.0564	0.0421	0.0584	1.1904	1.1494
	$\hat{\alpha}$	0.0383	0.0128	-0.0407	0.0096	0.0479	0.0125	1.3396	1.0078
30	$\hat{\theta}$	0.0311	0.0347	0.0533	0.0306	0.0115	0.0327	1.1306	1.0606
	$\hat{\alpha}$	0.0247	0.0079	-0.0326	0.0062	0.0313	0.0073	1.2781	1.0797
50	$\hat{\theta}$	0.0243	0.0214	0.0397	0.0199	0.0127	0.0208	1.0764	1.0282
	$\hat{\alpha}$	0.0148	0.0040	-0.0229	0.0036	0.0192	0.0040	1.1062	1.0081
70	$\hat{\theta}$	0.0158	0.0133	0.0279	0.0127	0.0072	0.0129	1.0513	1.0324
	$\hat{\alpha}$	0.0086	0.0028	-0.0198	0.0026	0.0116	0.0027	1.0567	1.0137
100	$\hat{\theta}$	0.0088	0.0085	0.0181	0.0082	0.0030	0.0083	1.0364	1.0217
	$\hat{\alpha}$	0.0083	0.0018	-0.0127	0.0017	0.0102	0.0018	1.0542	1.0111
150	$\hat{\theta}$	0.0102	0.0059	0.0167	0.0058	0.0066	0.0058	1.0180	1.0114
	$\hat{\alpha}$	0.0050	0.0011	-0.0099	0.0011	0.0062	0.0011	1.0351	1.0076

Table 10: Parameter estimation of GPW in scheme 1 sample when $\alpha = 1.5$; $\theta = 0.5$

scheme 1									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0816	0.1138	0.0387	0.0908	0.0142	0.0183	1.2529	6.2291
	$\hat{\alpha}$	0.4728	1.0089	0.2343	0.6149	0.0073	0.0267	1.6407	37.7515
30	$\hat{\theta}$	0.0345	0.0322	0.0135	0.0321	0.0170	0.0164	1.0040	1.9602
	$\hat{\alpha}$	0.2508	0.3625	0.1147	0.2615	0.0267	0.0370	1.3863	9.8072
50	$\hat{\theta}$	0.0325	0.0180	0.0227	0.0152	0.0395	0.0114	1.1860	1.5773
	$\hat{\alpha}$	0.1555	0.1655	0.0804	0.1346	0.0150	0.0290	1.2299	5.7042
70	$\hat{\theta}$	0.0165	0.0083	0.0103	0.0075	0.0269	0.0071	1.1145	1.1753
	$\hat{\alpha}$	0.0963	0.0955	0.0450	0.0825	-0.0059	0.0241	1.1577	3.9605
100	$\hat{\theta}$	0.0082	0.0048	0.0041	0.0045	0.0211	0.0058	1.0737	0.8351
	$\hat{\alpha}$	0.0641	0.0550	0.0292	0.0494	-0.0026	0.0258	1.1138	2.1307
150	$\hat{\theta}$	0.0057	0.0035	0.0031	0.0034	0.0063	0.0033	1.0477	1.0676
	$\hat{\alpha}$	0.0475	0.0378	0.0247	0.0351	-0.0049	0.0166	1.0773	2.2698
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0335	0.0430	0.0162	0.0369	0.0451	0.0190	1.1665	2.2612
	$\hat{\alpha}$	0.3312	0.6548	0.1383	0.3888	0.0184	0.0294	1.6842	22.3068
30	$\hat{\theta}$	0.0153	0.0203	0.0067	0.0169	0.0287	0.0136	1.2011	1.4876
	$\hat{\alpha}$	0.2098	0.2627	0.0942	0.1916	0.0255	0.0292	1.3710	8.9944
50	$\hat{\theta}$	0.0156	0.0094	0.0104	0.0084	0.0121	0.0086	1.1173	1.0902
	$\hat{\alpha}$	0.0990	0.1124	0.0365	0.0947	0.0265	0.0254	1.1868	4.4335
70	$\hat{\theta}$	0.0060	0.0058	0.0026	0.0054	0.0046	0.0045	1.0757	1.2824
	$\hat{\alpha}$	0.0844	0.0701	0.0406	0.0607	0.0158	0.0207	1.1541	3.3934
100	$\hat{\theta}$	0.0045	0.0041	0.0021	0.0039	0.0333	0.0058	1.0522	0.6949
	$\hat{\alpha}$	0.0579	0.0486	0.0280	0.0441	0.0047	0.0211	1.1029	2.3080
150	$\hat{\theta}$	0.0018	0.0023	0.0002	0.0023	0.0129	0.0019	1.0328	1.2473
	$\hat{\alpha}$	0.0276	0.0284	0.0082	0.0269	-0.0049	0.0197	1.0547	1.4400

$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0027	0.0199	-0.0023	0.0163	0.0317	0.0108	1.2178	1.8394
	$\hat{\alpha}$	0.2397	0.4715	0.0757	0.2755	0.0125	0.0242	1.7114	19.5184
30	$\hat{\theta}$	0.0061	0.0129	0.0020	0.0113	0.0325	0.0095	1.1421	1.3616
	$\hat{\alpha}$	0.1456	0.1917	0.0488	0.1440	-0.0179	0.0301	1.3318	6.3646
50	$\hat{\theta}$	0.0038	0.0070	0.0012	0.0064	0.0115	0.0043	1.0814	1.6338
	$\hat{\alpha}$	0.0863	0.0853	0.0319	0.0714	0.0175	0.0287	1.1946	2.9682
70	$\hat{\theta}$	0.0004	0.0047	-0.0015	0.0044	0.0161	0.0042	1.0553	1.1007
	$\hat{\alpha}$	0.0580	0.0575	0.0202	0.0510	-0.0035	0.0263	1.1278	2.1874
100	$\hat{\theta}$	-0.0011	0.0034	-0.0025	0.0033	0.0155	0.0033	1.0371	1.0345
	$\hat{\alpha}$	0.0450	0.0412	0.0189	0.0378	-0.0151	0.0213	1.0916	1.9390
150	$\hat{\theta}$	0.0024	0.0022	0.0014	0.0022	0.0014	0.0022	1.0272	1.0096
	$\hat{\alpha}$	0.0239	0.0240	0.0069	0.0228	0.0033	0.0187	1.0539	1.2850

Table 11: Parameter estimation of GPW in scheme 2 sample when $\alpha = 1.5$; $\theta = 0.5$

scheme 2									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0342	0.0347	-0.0156	0.0221	0.0365	0.0168	1.5735	2.0721
	$\hat{\alpha}$	0.2736	0.6302	-0.0110	0.2949	0.0086	0.0275	2.1375	22.9338
30	$\hat{\theta}$	0.0149	0.0199	-0.0215	0.0150	0.0311	0.0118	1.3305	1.6862
	$\hat{\alpha}$	0.1468	0.2367	-0.0459	0.1563	0.0005	0.0269	1.5137	8.7903
50	$\hat{\theta}$	0.0166	0.0125	-0.0101	0.0101	0.0124	0.0078	1.2423	1.6088
	$\hat{\alpha}$	0.0956	0.1295	-0.0364	0.0993	-0.0155	0.0236	1.3050	5.4807
70	$\hat{\theta}$	0.0095	0.0089	-0.0113	0.0077	0.0199	0.0058	1.1539	1.5285
	$\hat{\alpha}$	0.0635	0.0881	-0.0395	0.0734	-0.0113	0.0233	1.1999	3.7739
100	$\hat{\theta}$	0.0044	0.0060	-0.0120	0.0055	0.0232	0.0051	1.0848	1.1876
	$\hat{\alpha}$	0.0440	0.0563	-0.0372	0.0497	-0.0192	0.0216	1.1331	2.6066
150	$\hat{\theta}$	0.0011	0.0042	-0.0110	0.0040	0.0054	0.0027	1.0497	1.5351
	$\hat{\alpha}$	0.0377	0.0415	-0.0232	0.0377	0.0122	0.0174	1.1010	2.3806

$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0201	0.0255	-0.0163	0.0177	0.0280	0.0129	1.4434	1.9771
	$\hat{\alpha}$	0.2255	0.5284	-0.0321	0.2454	0.0146	0.0248	2.1536	21.2793
30	$\hat{\theta}$	0.0087	0.0172	-0.0189	0.0135	0.0123	0.0076	1.2684	2.2609
	$\hat{\alpha}$	0.1471	0.2128	-0.0319	0.1409	0.0197	0.0238	1.5106	8.9370
50	$\hat{\theta}$	0.0127	0.0101	-0.0082	0.0086	0.0214	0.0075	1.1792	1.3530
	$\hat{\alpha}$	0.0696	0.1081	-0.0495	0.0874	-0.0385	0.0238	1.2360	4.5470
70	$\hat{\theta}$	0.0020	0.0070	-0.0143	0.0063	0.0136	0.0051	1.0988	1.3590
	$\hat{\alpha}$	0.0684	0.0720	-0.0258	0.0593	0.0097	0.0276	1.2139	2.6045
100	$\hat{\theta}$	0.0020	0.0054	-0.0111	0.0050	0.0138	0.0033	1.0659	1.6118
	$\hat{\alpha}$	0.0482	0.0527	-0.0246	0.0465	0.0123	0.0203	1.1351	2.6002
150	$\hat{\theta}$	0.0017	0.0033	-0.0080	0.0032	0.0226	0.0042	1.0349	0.7776
	$\hat{\alpha}$	0.0224	0.0325	-0.0314	0.0307	-0.0390	0.0212	1.0578	1.5318
$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0047	0.0208	-0.0170	0.0159	0.0290	0.0106	1.3052	1.9527
	$\hat{\alpha}$	0.2078	0.4605	-0.0223	0.2387	-0.0088	0.0274	1.9293	16.7917
30	$\hat{\theta}$	0.0064	0.0134	-0.0119	0.0111	0.0244	0.0063	1.2053	2.1298
	$\hat{\alpha}$	0.1251	0.1893	-0.0302	0.1326	-0.0089	0.0282	1.4271	6.7072
50	$\hat{\theta}$	0.0037	0.0080	-0.0101	0.0071	0.0281	0.0062	1.1224	1.2791
	$\hat{\alpha}$	0.0769	0.0890	-0.0255	0.0721	-0.0305	0.0308	1.2349	2.8872
70	$\hat{\theta}$	0.0006	0.0056	-0.0107	0.0052	0.0117	0.0042	1.0658	1.3339
	$\hat{\alpha}$	0.0530	0.0612	-0.0261	0.0531	0.0062	0.0297	1.1534	2.0645
100	$\hat{\theta}$	-0.0014	0.0042	-0.0106	0.0040	0.0073	0.0029	1.0343	1.4212
	$\hat{\alpha}$	0.0430	0.0454	-0.0180	0.0405	0.0059	0.0240	1.1210	1.8891
150	$\hat{\theta}$	0.0021	0.0028	-0.0050	0.0027	0.0086	0.0024	1.0381	1.1439
	$\hat{\alpha}$	0.0232	0.0277	-0.0212	0.0262	-0.0109	0.0188	1.0593	1.4735

Table 12: Parameter estimation of GPW in scheme 3 sample when $\alpha = 1.5$; $\theta = 0.5$

scheme 3									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0640	0.0740	0.0213	0.0717	0.0329	0.0150	1.0321	4.9304
	$\hat{\alpha}$	0.4193	0.8603	0.1833	0.5052	0.0173	0.0354	1.7029	24.3085
30	$\hat{\theta}$	0.0265	0.0258	0.0041	0.0212	0.0606	0.0212	1.2170	1.2141
	$\hat{\alpha}$	0.2223	0.3130	0.0844	0.2217	0.0095	0.0361	1.4117	8.6685
50	$\hat{\theta}$	0.0273	0.0151	0.0159	0.0127	0.0395	0.0115	1.1908	1.3078
	$\hat{\alpha}$	0.1382	0.1430	0.0611	0.1151	-0.0490	0.0291	1.2432	4.9105
70	$\hat{\theta}$	0.0141	0.0078	0.0065	0.0069	0.0198	0.0075	1.1206	1.0297
	$\hat{\alpha}$	0.0856	0.0839	0.0324	0.0720	0.0035	0.0209	1.1654	4.0219
100	$\hat{\theta}$	0.0068	0.0046	0.0016	0.0043	0.0195	0.0046	1.0773	1.0005
	$\hat{\alpha}$	0.0571	0.0487	0.0206	0.0436	-0.0133	0.0231	1.1181	2.1076
150	$\hat{\theta}$	0.0045	0.0033	0.0011	0.0032	0.0016	0.0025	1.0502	1.3434
	$\hat{\alpha}$	0.0425	0.0334	0.0185	0.0308	-0.0104	0.0155	1.0812	2.1549
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0269	0.0294	-0.0012	0.0203	0.0261	0.0123	1.4527	2.3908
	$\hat{\alpha}$	0.2826	0.5656	0.0519	0.2970	0.0013	0.0299	1.9045	18.8986
30	$\hat{\theta}$	0.0114	0.0176	-0.0094	0.0138	0.0181	0.0103	1.2772	1.7081
	$\hat{\alpha}$	0.1823	0.2252	0.0293	0.1509	0.0020	0.0191	1.4920	11.7787
50	$\hat{\theta}$	0.0139	0.0087	-0.0020	0.0073	0.0201	0.0074	1.1917	1.1784
	$\hat{\alpha}$	0.0826	0.0952	-0.0127	0.0763	-0.0212	0.0235	1.2476	4.0444
70	$\hat{\theta}$	0.0046	0.0056	-0.0078	0.0050	0.0207	0.0059	1.1221	0.9477
	$\hat{\alpha}$	0.0722	0.0596	-0.0004	0.0491	0.0120	0.0258	1.2135	2.3094
100	$\hat{\theta}$	0.0037	0.0040	-0.0063	0.0037	0.0155	0.0043	1.0841	0.9426
	$\hat{\alpha}$	0.0494	0.0416	-0.0053	0.0364	-0.0062	0.0232	1.1443	1.7955
150	$\hat{\theta}$	0.0018	0.0023	-0.0058	0.0022	0.0092	0.0029	1.0486	0.8157
	$\hat{\alpha}$	0.0233	0.0245	-0.0158	0.0229	-0.0058	0.0198	1.0719	1.2376

Table 13: Parameter estimation of GPW in complete sample when $\alpha = 1.5$; $\theta = 1.5$

n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.05628	0.07084	0.08245	0.05734	0.02963	0.05997	1.23538	1.18125
	$\hat{\alpha}$	0.10755	0.11165	-0.13226	0.09139	0.08526	0.09222	1.22169	1.21072
30	$\hat{\theta}$	0.03206	0.03798	0.05399	0.03293	0.01127	0.03440	1.15360	1.10431
	$\hat{\alpha}$	0.07993	0.06388	-0.09402	0.05538	0.06967	0.05952	1.15357	1.07336
50	$\hat{\theta}$	0.02336	0.02038	0.03902	0.01905	0.01259	0.01947	1.07001	1.04691
	$\hat{\alpha}$	0.04239	0.03333	-0.07048	0.03212	0.03866	0.03272	1.03784	1.01882
70	$\hat{\theta}$	0.01651	0.01332	0.02864	0.01269	0.00797	0.01255	1.04964	1.06090
	$\hat{\alpha}$	0.02839	0.02385	-0.05638	0.02316	0.02704	0.02298	1.02986	1.03804
100	$\hat{\theta}$	0.01066	0.00822	0.01991	0.00796	0.00516	0.00783	1.03211	1.04999
	$\hat{\alpha}$	0.01444	0.01458	-0.04768	0.01451	0.01488	0.01449	1.00477	1.00634
150	$\hat{\theta}$	0.00903	0.00586	0.01556	0.00575	0.00527	0.00578	1.01992	1.01446
	$\hat{\alpha}$	0.01573	0.01072	-0.02877	0.01068	0.01535	0.01070	1.00374	1.00195

Table 14: Parameter estimation of GPW in scheme 1 sample when $\alpha = 1.5$; $\theta = 1.5$

scheme 1									
r=0.4									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	1.0806	6.6608	0.4632	0.8409	0.0163	0.2299	7.9209	28.9726
	$\hat{\alpha}$	0.4676	0.9002	-0.1700	0.3389	-0.0049	0.1051	2.6562	8.5652
30	$\hat{\theta}$	0.7433	5.3164	0.0719	0.7518	0.0271	0.2109	7.0712	25.2081
	$\hat{\alpha}$	0.2508	0.3323	-0.1810	0.1880	0.0114	0.0412	1.7673	8.0655
50	$\hat{\theta}$	0.4160	1.3045	0.0533	0.3634	0.0218	0.0576	3.5901	22.6476
	$\hat{\alpha}$	0.1565	0.1579	-0.1328	0.1097	-0.0021	0.0196	1.4399	8.0541
70	$\hat{\theta}$	0.2147	0.3166	-0.0111	0.1385	0.0185	0.0312	2.2851	10.1513
	$\hat{\alpha}$	0.0966	0.0901	-0.1192	0.0750	-0.0136	0.0194	1.2006	4.6424
100	$\hat{\theta}$	0.1255	0.1377	-0.0326	0.0757	0.0190	0.0244	1.8205	5.6416
	$\hat{\alpha}$	0.0648	0.0522	-0.0974	0.0485	0.0008	0.0194	1.0753	2.6968
150	$\hat{\theta}$	0.0928	0.1025	-0.0192	0.0673	-0.0219	0.0223	1.5236	4.6007
	$\hat{\alpha}$	0.0480	0.0356	-0.0680	0.0334	0.0145	0.0148	1.0666	2.4031

r=0.5									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.8824	2.9451	0.1728	1.0648	0.0040	0.0998	2.7658	29.5100
	$\hat{\alpha}$	0.3073	0.4656	-0.1867	0.2307	0.0238	0.0483	2.0184	9.6398
30	$\hat{\theta}$	0.3649	0.8593	0.0427	0.2492	0.0264	0.0325	3.4487	26.4589
	$\hat{\alpha}$	0.1977	0.2165	-0.1592	0.1372	0.0221	0.0275	1.5778	7.8628
50	$\hat{\theta}$	0.1924	0.2937	0.0102	0.1302	0.0039	0.0298	2.2559	9.8621
	$\hat{\alpha}$	0.0939	0.0946	-0.1368	0.0835	0.0061	0.0248	1.1330	3.8166
70	$\hat{\theta}$	0.1287	0.1350	-0.0005	0.0750	0.0211	0.0278	1.8005	4.8505
	$\hat{\alpha}$	0.0805	0.0604	-0.0962	0.0527	0.0205	0.0242	1.1466	2.5000
100	$\hat{\theta}$	0.0875	0.0790	-0.0056	0.0522	-0.0054	0.0240	1.5145	3.2961
	$\hat{\alpha}$	0.0550	0.0420	-0.0766	0.0396	-0.0110	0.0196	1.0599	2.1423
150	$\hat{\theta}$	0.0419	0.0411	-0.0224	0.0316	0.0021	0.0207	1.2995	1.9874
	$\hat{\alpha}$	0.0270	0.0247	-0.0670	0.0239	0.0053	0.0147	1.0343	1.6812
r=0.7									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.2096	0.2961	0.0656	0.1263	0.0041	0.0389	2.3448	7.6084
	$\hat{\alpha}$	0.1815	0.2171	-0.1741	0.1493	0.0260	0.0260	1.4535	8.3438
30	$\hat{\theta}$	0.1540	0.1999	0.0556	0.1055	-0.0076	0.0316	1.8940	6.3345
	$\hat{\alpha}$	0.1195	0.1196	-0.1377	0.0959	0.0359	0.0288	1.2473	4.1588
50	$\hat{\theta}$	0.0848	0.0613	0.0274	0.0396	-0.0052	0.0257	1.5469	2.3896
	$\hat{\alpha}$	0.0731	0.0584	-0.0978	0.0524	-0.0078	0.0261	1.1138	2.2336
70	$\hat{\theta}$	0.0505	0.0401	0.0094	0.0295	0.0011	0.0186	1.3591	2.1604
	$\hat{\alpha}$	0.0483	0.0402	-0.0814	0.0393	0.0069	0.0260	1.0243	1.5440
100	$\hat{\theta}$	0.0338	0.0256	0.0044	0.0207	0.0018	0.0175	1.2378	1.4583
	$\hat{\alpha}$	0.0375	0.0286	-0.0584	0.0278	0.0132	0.0219	1.0283	1.3068
150	$\hat{\theta}$	0.0258	0.0149	0.0047	0.0126	-0.0044	0.0114	1.1833	1.3109
	$\hat{\alpha}$	0.0194	0.0167	-0.0488	0.0164	0.0139	0.0161	1.0221	1.0375

Table 15: Parameter estimation of GPW in scheme 2 sample when $\alpha = 1.5$; $\theta = 1.5$

scheme 2									
$r = 0.4$									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.3037	0.8342	0.2448	0.3457	-0.0078	0.0289	2.4135	28.8779
	$\hat{\alpha}$	0.2132	0.2759	-0.1783	0.1755	0.0148	0.0320	1.5719	8.6248
30	$\hat{\theta}$	0.1242	0.1699	0.1275	0.1165	0.0040	0.0326	1.4586	5.2038
	$\hat{\alpha}$	0.1230	0.1347	-0.1725	0.1144	0.0160	0.0286	1.1773	4.7123
50	$\hat{\theta}$	0.0993	0.0889	0.1104	0.0712	0.0091	0.0239	1.2484	3.7274
	$\hat{\alpha}$	0.0828	0.0774	-0.1317	0.0712	-0.0201	0.0221	1.0878	3.5080
70	$\hat{\theta}$	0.0539	0.0420	0.0690	0.0363	-0.0015	0.0204	1.1549	2.0572
	$\hat{\alpha}$	0.0539	0.0516	-0.1139	0.0511	-0.0406	0.0216	1.0096	2.3838
100	$\hat{\theta}$	0.0292	0.0233	0.0437	0.0211	-0.0002	0.0169	1.1037	1.3758
	$\hat{\alpha}$	0.0381	0.0328	-0.0924	0.0324	0.0053	0.0218	1.0124	1.5062
150	$\hat{\theta}$	0.0203	0.0176	0.0320	0.0164	0.0011	0.0130	1.0695	1.3550
	$\hat{\alpha}$	0.0307	0.0230	-0.0651	0.0219	-0.0006	0.0164	1.0501	1.4018
$r = 0.5$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.2001	0.3359	0.1857	0.1946	0.0141	0.0316	1.7263	10.6446
	$\hat{\alpha}$	0.1686	0.2020	-0.1801	0.1429	0.0113	0.0288	1.4131	7.0021
30	$\hat{\theta}$	0.0996	0.1178	0.1118	0.0876	0.0203	0.0224	1.3444	5.2499
	$\hat{\alpha}$	0.1203	0.1164	-0.1504	0.0941	0.0241	0.0272	1.2374	4.2777
50	$\hat{\theta}$	0.0660	0.0539	0.0814	0.0457	0.0052	0.0194	1.1786	2.7707
	$\hat{\alpha}$	0.0537	0.0572	-0.1300	0.0508	0.0048	0.0355	1.1264	1.6099
70	$\hat{\theta}$	0.0384	0.0308	0.0543	0.0276	0.0037	0.0166	1.1184	1.8578
	$\hat{\alpha}$	0.0532	0.0392	-0.0912	0.0390	-0.0050	0.0190	1.0057	2.0638
100	$\hat{\theta}$	0.0270	0.0203	0.0404	0.0188	-0.0142	0.0125	1.0770	1.6170
	$\hat{\alpha}$	0.0362	0.0285	-0.0734	0.0278	0.0176	0.0191	1.0263	1.4937
150	$\hat{\theta}$	0.0122	0.0115	0.0228	0.0108	-0.0054	0.0085	1.0573	1.3446
	$\hat{\alpha}$	0.0182	0.0177	-0.0621	0.0167	0.0163	0.0152	1.0607	1.1686

$r = 0.7$									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.0990	0.1097	0.1162	0.0815	0.0141	0.0238	1.3449	4.6178
	$\hat{\alpha}$	0.1334	0.1488	-0.1637	0.1168	0.0449	0.0292	1.2735	5.0921
30	$\hat{\theta}$	0.0774	0.0765	0.0949	0.0620	0.0131	0.0256	1.2340	2.9857
	$\hat{\alpha}$	0.0891	0.0854	-0.1303	0.0756	0.0329	0.0231	1.1294	3.6996
50	$\hat{\theta}$	0.0449	0.0305	0.0614	0.0276	-0.0073	0.0239	1.1032	1.2733
	$\hat{\alpha}$	0.0575	0.0456	-0.0912	0.0430	-0.0265	0.0191	1.0592	2.3883
70	$\hat{\theta}$	0.0246	0.0201	0.0390	0.0186	0.0089	0.0158	1.0798	1.2703
	$\hat{\alpha}$	0.0381	0.0317	-0.0759	0.0302	0.0162	0.0227	1.0493	1.3974
100	$\hat{\theta}$	0.0146	0.0135	0.0262	0.0128	0.0216	0.0125	1.0551	1.0826
	$\hat{\alpha}$	0.0300	0.0228	-0.0548	0.0228	0.0177	0.0118	1.0010	1.9308
150	$\hat{\theta}$	0.0146	0.0085	0.0231	0.0083	-0.0302	0.0071	1.0264	1.2062
	$\hat{\alpha}$	0.0151	0.0137	-0.0455	0.0135	-0.0081	0.0126	1.0160	1.0892

Table 16: Parameter estimation of GPW in scheme 3 sample when $\alpha = 1.5$; $\theta = 1.5$

scheme 3									
r=0.4									
		MLE		MPS		MCMC			
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	1.0571	2.7010	0.3914	0.6610	-0.0148	0.0371	4.0861	72.8176
	$\hat{\alpha}$	0.3968	0.6759	-0.1560	0.2887	0.0317	0.0332	2.3412	20.3674
30	$\hat{\theta}$	0.5132	2.2457	0.0852	0.5033	-0.0067	0.0843	4.4623	26.6393
	$\hat{\alpha}$	0.2137	0.2566	-0.1633	0.1593	0.0201	0.0247	1.6102	10.3854
50	$\hat{\theta}$	0.3123	0.6951	0.0707	0.2583	0.0095	0.0275	2.6910	25.3177
	$\hat{\alpha}$	0.1341	0.1246	-0.1175	0.0916	-0.0083	0.0244	1.3596	5.0978
70	$\hat{\theta}$	0.1656	0.1988	0.0123	0.1026	0.0065	0.0330	1.9378	6.0258
	$\hat{\alpha}$	0.0831	0.0727	-0.1047	0.0627	-0.0026	0.0192	1.1592	3.7856
100	$\hat{\theta}$	0.0969	0.0914	-0.0104	0.0561	0.0150	0.0214	1.6285	4.2681
	$\hat{\alpha}$	0.0560	0.0427	-0.0846	0.0404	-0.0098	0.0186	1.0568	2.2927
150	$\hat{\theta}$	0.0711	0.0682	-0.0039	0.0490	0.0070	0.0195	1.3926	3.4905
	$\hat{\alpha}$	0.0416	0.0289	-0.0585	0.0275	0.0130	0.0187	1.0512	1.5443

r=0.5									
n		Bias	MSE	Bias	MSE	Bias	MSE	RE1	RE2
20	$\hat{\theta}$	0.4928	2.6699	0.1895	0.5829	0.0124	0.0475	4.5804	56.2084
	$\hat{\alpha}$	0.2390	0.3093	-0.1560	0.1767	0.0020	0.0274	1.7503	11.2697
30	$\hat{\theta}$	0.2266	0.3555	0.0856	0.1649	0.0236	0.0268	2.1557	13.2749
	$\hat{\alpha}$	0.1573	0.1492	-0.1265	0.1032	0.0126	0.0275	1.4459	5.4281
50	$\hat{\theta}$	0.1266	0.1343	0.0507	0.0823	0.0017	0.0195	1.6312	6.8692
	$\hat{\alpha}$	0.0713	0.0643	-0.1092	0.0601	-0.0064	0.0202	1.0705	3.1766
70	$\hat{\theta}$	0.0829	0.0676	0.0321	0.0475	0.0094	0.0184	1.4252	3.6684
	$\hat{\alpha}$	0.0634	0.0419	-0.0731	0.0377	0.0029	0.0185	1.1098	2.2661
100	$\hat{\theta}$	0.0565	0.0407	0.0212	0.0317	-0.0021	0.0148	1.2864	2.7535
	$\hat{\alpha}$	0.0424	0.0290	-0.0581	0.0280	0.0281	0.0207	1.0359	1.4017
150	$\hat{\theta}$	0.0274	0.0220	0.0034	0.0185	-0.0105	0.0206	1.1874	1.0685
	$\hat{\alpha}$	0.0214	0.0173	-0.0499	0.0170	-0.0082	0.0171	1.0157	1.0089

In complete sample: It is observed that as sample size increases for fixed values of α and θ the MSE of the estimates decreases in all the three considered methods, and also the relative efficiency displays the range of changes that occur as a result of sample size changes and different parameters for GPW distribution, but MPS method performs better than other two considered method except some cases. In the small sample, we note how positive the use of the MPS method is, but the method of MCMC is the best method, see Figure 2.

From the simulation results, we observe that the Biases, MSEs and RE for all the estimators decrease when the sample size n and effective sample size m increase in most cases. We note that the MPS is comparable to other estimation procedures discussed here in most situations. We observed that the Bayes estimates using MCMC method with respect to the non-informative prior (Gamma) are quite close to the MPS. In most cases, the Bayes estimates using MCMC method perform better than those by using MPS and MLE estimates using. For fixed n when r increase the MSE decrease for MLE and Bayes estimate. Also, when r increases while n remain fixed, the MSE may decrease, because sample increase under censoring (the number of observed failures m is predetermined). Comparing the performance of the three censoring schemes, we found that Scheme 1 is better than schemes 2 and 3 except some cases.

We suggest the use of MCMC in consideration of bias, MSE and RE in general. Sometimes, the Bayes estimation for GPW distribution is not as reliable when parameters are likely to be more than 1, therefore, if one wants a reliable estimation procedure and knowing that parameters are likely to be more than 1, then we would suggest the use of MPS method. Due to the fact that the relationships between the performance of estimator for GPW distribution and the censoring scheme are depending on the choice of estimation procedure, we focus our discussion on comparing different types of progressive censoring scheme, type-II censoring scheme and complete censoring scheme based on the MPS method, which is the estimation methods we recommend based on the simulation results.

5 Application

In this section, we have given an application of GPW distribution using real data set to illustrate that GPW distribution provides significant improvements over. These data are from Soliman et al (2013) concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 k.v. (minutes). The 19 times to breakdown are (0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, and 72.89)

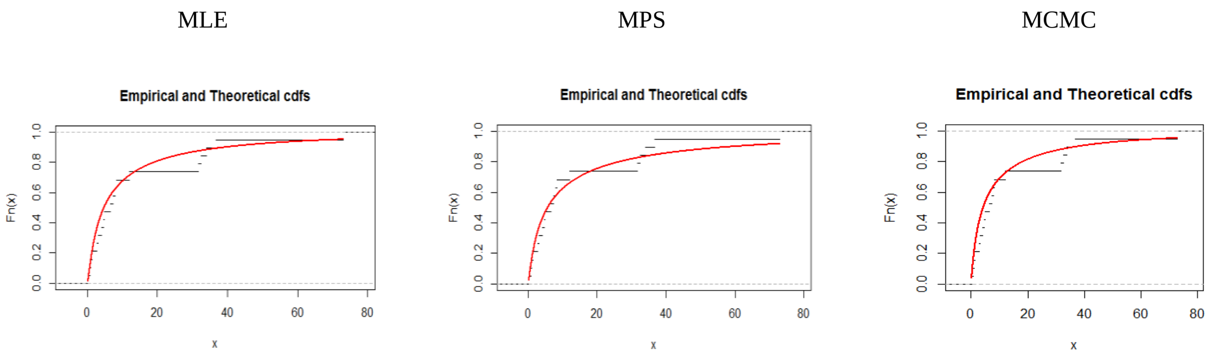

Figure 6: Plot the Maximum Distance between the Empirical and Theoretical CDF

Table 17: The MLEs, KS-test AIC, CAIC, BIC and HQIC values by MLE method

	GPW	weibull	GR	GE
$\hat{\alpha}$	1.5205 (0.56697)	14.05531 (5.84714)	0.25396 (0.06418)	0.05349 (0.01804)
$\hat{\theta}$	0.21354 (0.0833)	0.68138 (0.13603)	0.02514 (0.00596)	0.60258 (0.19413)
D (P-value)	0.1530 (0.709)	0.18148 (0.5019)	0.23627 (0.204)	0.14851 (0.6923)
AIC	141.5324	141.5866	143.2034	141.5618
BIC	143.4213	143.4755	145.0922	143.4507
HQIC	141.8521	141.9063	143.523	141.8815
CAIC	142.2824	142.3366	143.9534	142.3118

Table 18: The MLEs, KS-test AIC, CAIC, BIC and HQIC values by MPS method

	GPW	weibull	GR	GE
$\hat{\alpha}$	1.2780 (0.48254)	12.88514 (5.11719)	0.22071 (0.0565)	0.03245 (0.0112)
$\hat{\theta}$	0.22831 (0.0895)	0.57081 (0.1256)	0.0182 (0.0050)	0.5612 (0.0409)
D (P-value)	0.13464 (0.8367)	0.14288 (0.782)	0.25228 (0.1492)	0.25719 (0.1234)
AIC	150.9144	151.0389	152.6904	153.1618
BIC	152.8033	152.9278	154.5793	155.0507
HQIC	151.2341	151.3586	153.0101	154.4815
CAIC	151.6644	151.7889	153.4404	154.9118

From Table 17 and 18, they are clear that, the GPW distribution has the smallest value among AIC, CAIC, BIC and HQIC. So, the GPW distribution provides a better fit than the other tested distributions.

Table 19: Parameter estimation of real data for GPW in complete sample

	MLE	MPS	MCMC
$\hat{\alpha}$	1.5205 (0.56697)	1.2780 (0.48254)	1.1296 (0.2327)
$\hat{\theta}$	0.21354 (0.0833)	0.22831 (0.0895)	0.2895 (0.0806)
D (P-value)	0.1530 (0.709)	0.13464 (0.8367)	0.1904 (0.4417)

In application on real data, the data is fitted to GPW distribution and the K-S statistics, between the fitted and the empirical distribution is also calculated and estimates of the parameter using MLE method, MPS method and Bayesian estimation method by MCMC are calculated in Table 19 and Figure 6. For the above data set. We notice that K-S distance (D) through MPS is smaller than K-S distance through MLE and MCMC methods, and also we notice the P-value through MPS method is larger than MLE and MCMC. So based on estimates, standard deviation and K-S statistics, for the considered data MPS method fits better as compared to MLE and MCMC. The result of these estimators shows that MPS method serve better than MLE method in this data.

Table 20: Parameter estimation of real data for GPW in scheme1

	MLE	MPS	MCMC
$r = 0.4$			
$\hat{\theta}$	0.1868 (0.0873)	0.2091 (0.0954)	0.2714 (0.0951)
$\hat{\alpha}$	1.3652 (0.5862)	1.0966 (0.4986)	1.0183 (0.1806)
$r = 0.5$			
$\hat{\theta}$	0.2028 (0.0872)	0.2231 (0.0942)	0.2602 (0.0690)
$\hat{\alpha}$	1.4895 (0.5865)	1.2447 (0.5070)	1.0973 (0.2248)
$r = 0.7$			
$\hat{\theta}$	0.2181 (0.0861)	0.2368 (0.0919)	0.2943 (0.0927)
$\hat{\alpha}$	1.5601 (0.5702)	1.3394 (0.4931)	1.1414 (0.2256)

Table 21: Parameter estimation of real data for GPW in scheme2

	MLE	MPS	MCMC
$r = 0.4$			
$\hat{\theta}$	0.4688 (0.1604)	0.4347 (0.1505)	0.5423 (0.0755)
$\hat{\alpha}$	1.8013 (0.5831)	1.5203 (0.5101)	1.5108 (0.1410)
$r = 0.5$			
$\hat{\theta}$	0.3939 (0.1355)	0.3867 (0.1324)	0.4754 (0.0746)
$\hat{\alpha}$	1.7799 (0.5801)	1.5385 (0.5102)	1.5318 (0.1576)
$r = 0.7$			
$\hat{\theta}$	0.3086 (0.1076)	0.3170 (0.1073)	0.3724 (0.0740)
$\hat{\alpha}$	1.7215 (0.5686)	1.4999 (0.4970)	1.3763 (0.1902)

Table 22: Parameter estimation of real data for GPW in scheme 3

	MLE	MPS	MCMC
$r = 0.4$			
$\hat{\theta}$	0.2129 (0.0888)	0.2283 (0.0941)	0.2609 (0.0525)
$\hat{\alpha}$	1.6122 (0.6187)	1.3449 (0.5393)	1.2927 (0.1437)
$r = 0.5$			
$\hat{\theta}$	0.2601 (0.0802)	0.2414 (0.0927)	0.2034 (0.0721)
$\hat{\alpha}$	1.5190 (0.5522)	1.4599 (0.5375)	1.4402 (0.5158)

In application in real data, we note extent the results of the simulation are consistent with the results of the practical application, where the sequence of results. Hence, the higher the r value is, the greater the efficiency of the estimate. The MPS method is considered better than MLE method in all schemes. But the best method is Bayesian estimation by using MCMC in all different samples size. In general model, we can use MPS as alternative method of MLE method in all cases and in all schemes. The Bayesian method still is the best method, under using suitable prior distribution.

6 Conclusion

In this paper, the estimation problem of the unknown parameters of the GPW distribution based on progressive type-II censoring scheme was discussed. A comparison had been done between the proposed estimators (maximum likelihood estimator, Maximum Product Spacing estimator, and Bayesian estimation) on the basis of Monte Carlo Simulation study. The Bayesian estimation based on squares error of loss function under the assumption of independent gamma priors was introduced by use MCMC. The performance of the different estimator's optimal censoring schemes is compared based on simulation study to determine the optimal censoring schemes by using MSE, Bias and RE. Finally, a real data set has been considered to illustrate the practical utility of the paper and show how the scheme works in practice. It was observed that Bayesian estimation with respect to the gamma priors behave quite better for GPW distribution, where Bias and MSE decrease than another methods. We can use MPS as alternative method of MLE method in all cases and in all schemes.

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