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Data Science in Action

Interval Forecasting in Time Series Analysis: Application to COVID-19 and Beyond

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Abstract

Forecasting is essential for optimizing resource allocation, particularly during crises such as the unprecedented COVID-19 pandemic. This paper focuses on developing an algorithm for generating k-step-ahead interval forecasts for autoregressive time series. Unlike conventional methods that assume a fixed distribution, our approach utilizes kernel distribution estimation to accommodate the unknown distribution of prediction errors. This flexibility is crucial in real-world data, where deviations from normality are common, and neglecting these deviations can result in inaccurate predictions and unreliable confidence intervals. We evaluate the performance of our method through simulation studies on various autoregressive time series models. The results show that the proposed approach performs robustly, even with small sample sizes, as low as 50 observations. Moreover, our method outperforms traditional linear model-based prediction intervals and those derived from the empirical distribution function, particularly when the underlying data distribution is non-normal. This highlights the algorithm's flexibility and accuracy for interval forecasting in non-Gaussian contexts. We also apply the method to log-transformed weekly COVID-19 case counts from lower-middle-income countries, covering the period from June 1, 2020, to March 13, 2022.

Keywords *autoregressive time series; empirical cumulative distribution function; kernel density estimation; prediction interval*

1 Introduction

The COVID-19 outbreak in late 2019 marked the first time pandemic data was collected and shared globally in real-time. Advances in modern technology enabled unprecedented access to COVID-19 data, allowing for the timely extraction of critical information. Throughout the pandemic, numerous time series datasets, such as those shown in Figure 1, were frequently collected and updated. Accurate forecasting based on these datasets allowed governments and healthcare systems to anticipate demand, allocate resources, and make informed decisions to mitigate the virus's impact. Several attempts have been made to forecast COVID-19-related time series, including studies by Petropoulos et al. (2022) and Abbasimehr et al. (2022). However, most studies have focused on point forecasting, often at the cost of computational efficiency. While point forecasts are useful for predicting future values, they are unlikely to precisely match future observations. A key limitation of point forecasts is that they do not account for the uncertainty inherent in forecasting. Throughout this paper, we use the terms 'forecasting' and 'prediction'

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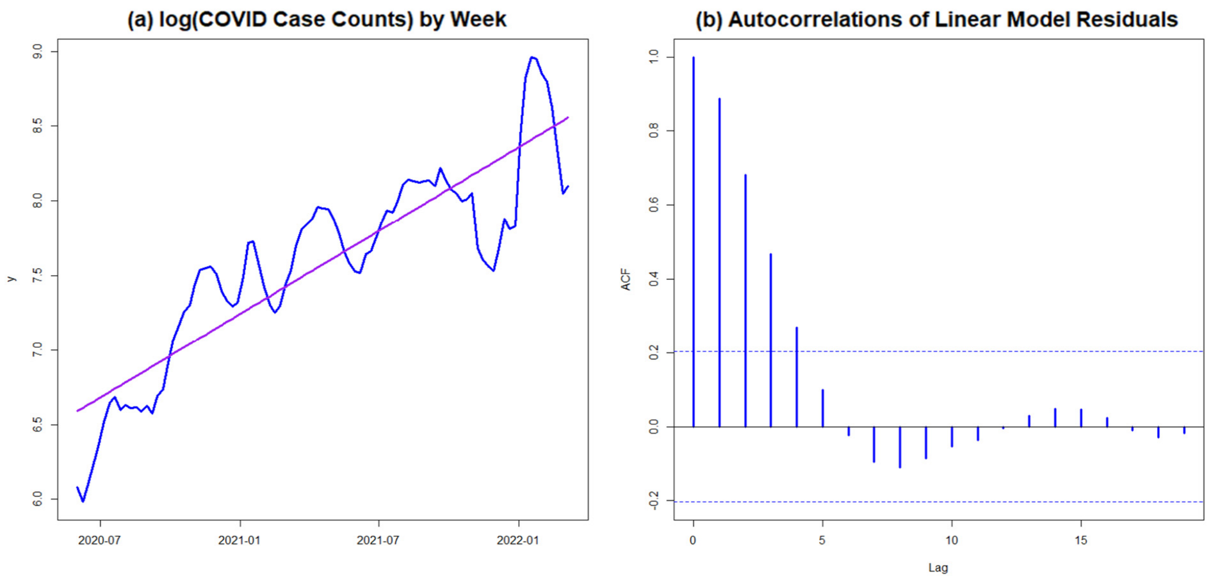

Figure 1: (a) scatter plot for $\{y_t\}$, (b) ACF of linear model residuals $\{\hat{w}_t\}$.

interchangeably. In contrast, interval forecasts provide a range within which future observations are expected to fall, based on a specified prediction level, such as 95%. Interval predictions offer the flexibility to balance forecast precision with associated uncertainty by adjusting the prediction level. This flexibility supports more informed decision-making, as it considers both best-case and worst-case scenarios, offering a more objective assessment of data variability.

Forecasting based on the intrinsic correlation structure is a key focus in time series analysis. A well-known approach involves autoregressive moving-average (ARMA) modeling for stationary time series, where both the first and second moments remain constant. If the underlying process that generates the time series can be modeled by an ARMA model, point forecasts can be derived using the model's parameter estimates. For more details, readers can refer to standard textbooks, such as Shumway and Stoffer (2010) and Cryer and Chan (2008). To handle forecasting for more complex nonstationary time series, many computationally intensive methods have been proposed. For example, Inoue et al. (2017) employed a nonparametric approach for predicting time series with structural changes. While nonparametric methods offer greater flexibility, this advantage comes at a cost—they typically require a large number of observations and significant computational power to achieve a desired level of accuracy.

Most time series forecasting methodologies have focused on point forecasting, while interval forecasting has received comparatively less attention. For an ARMA time series with a normal distribution, constructing a forecast interval is straightforward. However, real-world data often deviate from the assumptions of stationarity, linearity, or normality. Chatfield (1993) and Gooijer and Hyndman (2006) provide comprehensive summaries of various methods for general time series forecasting. For instance, Cerqueira et al. (2020) conducted an empirical comparison of performance estimation methods for time series forecasting and found significant differences in their effectiveness. Several empirical plug-in methods, particularly those based on computationally intensive techniques, have been developed to address the challenge of unknown distributions. The primary drawback of these methods is that they often require substantial computational time.

The pioneering work of Wang et al. (2014) addresses this challenge by introducing oracle-efficient estimation for a distribution function. Their approach is based on kernel density estimation (KDE), and it possesses the oracle property, meaning that the prediction intervals derived from the estimated model coefficients are asymptotically as efficient as those constructed using true parameter values. Kong et al. (2018) extended this approach to autoregressive time series (AR), proposing a prediction interval based on kernel estimation of the cumulative distribution function, which also retains oracle efficiency. Their semiparametric kernel distribution estimation method offers several key advantages, making it particularly well-suited for COVID-19 data. First, it does not require the assumption of normality and can be applied to non-normally distributed data—an important consideration for COVID-19 case counts, which deviate from normality. Second, it performs well with relatively small sample sizes; their simulation studies show that it works effectively with as few as 50 observations. Finally, they demonstrated that their method is computationally efficient, taking significantly less time than the bootstrap, a widely known computationally intensive method.

At the onset of the COVID-19 outbreak, no comparable historical datasets were available. However, even limited data proved valuable to society. The advantages of KDE, combined with the unique characteristics of COVID-19 time series, motivate our proposal for constructing a prediction interval (PI). Specifically, we extend the KDE approach to autoregressive time series with a linear trend and propose an algorithm to construct forecast intervals for such data. The remainder of the article is structured as follows: In Section 2, we introduce the details of the algorithm. A linear model is fitted using linear regression with time series errors, and a k-step-ahead PI is constructed from the prediction residuals. This residual-based approach has been applied in the analysis of time series with regressors, as in Pierce (1971) for regression models and Shao and Yang (2017) for nonparametric models. The residuals are used to estimate the distribution of the prediction errors via kernel estimation. In Section 3, we present simulation studies for several AR models, comparing different PI construction methods in terms of coverage frequencies, and the means and standard deviations of PI widths. In Section 4, we apply the algorithm to the log-transformed COVID-19 case count data from lower-middle-income countries. Finally, in Section 5, we conclude with a summary and final remarks.

2 Construction of Forecasting Intervals

We start with the following simple linear model for the dependent variable y and a fixed independent variable x:

$$y_t = \beta_0 + \beta_1 x_t + w_t. \tag{2.1}$$

When time series exhibits a linear trend, x_t is time t. Unlike a classic linear model, the random error terms $\{w_t\}$ in (2.1) are an autoregressive time series with order p (AR(p))

$$w_t = \sum_{i=1}^{p} \phi_i w_{t-i} + \epsilon_t, \tag{2.2}$$

where $\{\epsilon_t\}$ is white noise or a sequence of independent and identically distributed (i.i.d) random variables with $\mathbb{E}(\epsilon_t) = 0$ and $\mathbb{E}\left(\epsilon_t^2\right) = \sigma^2$. In addition, the time series $\{w_t\}$ is causal; that is, there exists a sequence of constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$w_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}.$$

These typical conditions on AR(p) time series $\{w_t\}$ ensure that Yule-Walker or maximum likelihood estimators for $\{\phi_i\}_{i=1}^p$ exist and possess certain asymptotic properties, such as consistency and normality. Interested readers can refer to, for example, Brockwell and Davis (1991).

From (2.1) and (2.2), we obtain the following combined model:

$$y_t = b_0 + b_1 x_t + \sum_{i=1}^{p} \phi_i y_{t-i} + \epsilon_t, \tag{2.3}$$

where $\{b_0, b_1\}$ are functions of the model coefficients $\{\beta_0, \beta_1, \phi_1, \dots, \phi_p\}$. Model (2.3) resembles a classic linear regression with explanatory variables $\{x_t, y_{t-1}, \dots, y_{t-p}\}$ and independent and identically distributed error terms $\{\epsilon_t\}$. Although a key difference between model (2.1) and model (2.3) is the inclusion of lagged explanatory variables $\{y_{t-1}, \dots, y_{t-p}\}$, we can still leverage the linear relationship between the dependent variable y_t and the explanatory variables to construct an out-of-sample prediction interval for y_{n+k}.

Using the true values of model coefficients $\{b_0, b_1, \phi_1, \dots, \phi_p\}$, a k-step-ahead prediction $\tilde{y}_{t+k}^{[k]}$ for y_{t+k} is defined by

$$\tilde{y}_{t+k}^{[k]} = b_0 + b_1 x_{t+k} + \sum_{i=1}^{k-1} \phi_i \tilde{y}_{t+k-i}^{[k-i]} + \sum_{i=k}^{p} \phi_i y_{t+k-i}, \tag{2.4}$$

and the k-step-ahead prediction errors or residuals $\left\{\tilde{\epsilon}_{t+k}^{[k]}\right\}_{t=k+1}^{n-k}$ can be calculated from y_{t+k} and $\tilde{y}_{t+k}^{[k]}$,

$$\tilde{\epsilon}_{t+k}^{[k]} = y_{t+k} - \tilde{y}_{t+k}^{[k]}. \tag{2.5}$$

Consider one-step-ahead prediction. When $k = 1$, (2.4) is simplified to

$$\tilde{y}_{t+1}^{[1]} = b_0 + b_1 x_{t+1} + \sum_{i=1}^{p} \phi_i y_{t+1-i}.$$

Thus, $\tilde{\epsilon}_{t+1}^{[1]} = \epsilon_{t+1}$. Hereafter, we ignore the subscript $[k]$ when $k = 1$. When $k = 2$, the two-step-ahead prediction $\tilde{y}_{t+2}^{[2]}$ relies on the one-step-ahead prediction $\tilde{y}_{t+1}$ as follows:

$$\begin{aligned}
\tilde{y}_{t+2}^{[2]} &= b_0 + b_1 x_{t+2} + \phi_1 \tilde{y}_{t+1} + \sum_{i=2}^{p} \phi_i y_{t+2-i} \\
&= b_0 + b_1 x_{t+2} + \phi_1 \left\{ b_0 + b_1 x_{t+1} + \sum_{i=1}^{p} \phi_i y_{t+1-i} \right\} + \sum_{i=2}^{p} \phi_i y_{t+2-i} \\
&= b_0(1 + \phi_1) + b_1(x_{t+2} + \phi_1 x_{t+1}) + \sum_{i=1}^{p-1} (\phi_1 \phi_i + \phi_{i+1}) y_{t+1-i} + \phi_1 \phi_p y_{t+1-p}.
\end{aligned}$$

A two-step-ahead prediction error $\tilde{\epsilon}_{t+2}^{[2]}$ is calculated by

$$\tilde{\epsilon}_{t+2}^{[2]} = y_{t+2} - \tilde{y}_{t+2}^{[2]} = \epsilon_{t+2} + \phi_1 \epsilon_{t+1}. \tag{2.6}$$

Thus, unlike one-step-ahead prediction errors, $\left\{\tilde{\epsilon}_{t+2}^{[2]}\right\}$ is not an independent sequence. If $\{\epsilon_t\}$ is independent and normally distributed with mean zero and variance σ^2, $\left\{\tilde{\epsilon}_{t+2}^{[2]}\right\}$ is also normally distributed with mean zero and variance $(1 + \phi_1^2)\sigma^2$.

A $(1-\alpha)\%$ prediction interval for y_{n+k} can be constructed from the cumulative distribution function $F^{[k]}$ of $\left\{\tilde{\epsilon}_{t+k}^{[k]}\right\}$. Let $q_{\alpha/2}^{[k]}$ and $q_{1-\alpha/2}^{[k]}$ be respectively $\alpha/2$-th and $q_{1-\alpha/2}$-th quantiles such that $q_\alpha = \inf\{z : F^{[k]}(z) \geqslant \alpha\}$. Then,

$$\left(\tilde{y}_{n+k}^{[k]} + q_{\alpha/2}^{[k]}, \quad \tilde{y}_{n+k}^{[k]} + q_{1-\alpha/2}^{[k]}\right)$$

is a $(1-\alpha)\%$ prediction interval. However, in general, it is complicated to find the exact distribution of $\left\{\tilde{\epsilon}_{t+k}^{[k]}\right\}$. We intend to estimate the density function nonparametrically and in particular, utilize the kernel density estimator as follows:

$$\widetilde{F}_{n,h}^{[k]}(z) = \int_{-\infty}^{z} \frac{1}{nh} \sum_{t=1}^{n} K\left(\frac{u - \tilde{\epsilon}_{t+k}^{[k]}}{h}\right) du, \quad z \in R, \tag{2.7}$$

where h is a bandwidth and the kernel function K satisfies the conditions: a symmetric bounded density function and

$$K(z) \geqslant 0, \quad \int K(z)dz = 1, \quad \int zK(z)dz = 0.$$

There has been extensive research on kernel estimators, and interested readers can refer to, such as Fan and Gijbels (1996). Let $\tilde{q}_{\alpha/2}^{[k]}$ and $\tilde{q}_{1-\alpha/2}^{[k]}$ be respectively $\alpha/2$-th and $q_{1-\alpha/2}$-th quantiles of $\widetilde{F}_{n,h}^{[k]}$. Then an approximate $(1-\alpha)\%$ PI is

$$\left(\tilde{y}_{n+k}^{[k]} + \tilde{q}_{\alpha/2}^{[k]}, \quad \tilde{y}_{n+k}^{[k]} + \tilde{q}_{1-\alpha/2}^{[k]}\right). \tag{2.8}$$

However, using (2.8) presents another challenge: both the prediction $\tilde{y}_{t+k}^{[k]}$ and residual $\tilde{\epsilon}_{t+k}^{[k]}$ depend on unknown model coefficients. A common approach in statistics to address this issue is the plug-in method, where estimates are substituted for the unknowns. This leads to our proposed prediction algorithm for constructing out-of-sample prediction intervals. We elaborate it as follows: in the first step, model (2.3) is fitted, and the ordinary least squares estimates $\{\hat{b}_0, \hat{b}_1, \hat{\phi}_i, 1 \leqslant i \leqslant p\}$ are calculated; in the second step, a point prediction $\hat{y}_{t+k}^{[k]}$ is obtained from equation (2.4) with $\{b_0, b_1, \phi_i, 1 \leqslant i \leqslant p\}$ replaced by the estimates $\{\hat{b}_0, \hat{b}_1, \hat{\phi}_i, 1 \leqslant i \leqslant p\}$, and residuals $\left\{\hat{\epsilon}_{t+k}^{[k]}\right\}$ are calculated from (2.5) with $\tilde{y}_{t+k}^{[k]}$ replaced by $\hat{y}_{t+k}^{[k]}$; in the third step, the KDE $\widehat{F}_{n,h}^{[k]}$ is computed from in (2.7) with $\tilde{\epsilon}_{t+k}^{[k]}$ replaced by $\hat{\epsilon}_{t+k}^{[k]}$, and a $(1-\alpha)\%$ PI is constructed as follows:

$$\left(\hat{y}_{n+k}^{[k]} + \hat{q}_{\alpha/2}^{[k]}, \quad \hat{y}_{n+k}^{[k]} + \hat{q}_{1-\alpha/2}^{[k]}\right), \tag{2.9}$$

where $\hat{q}_{\alpha}^{[k]}$ is the αth quantile of the $\widehat{F}_{n,h}^{[k]}(z)$.

3 Simulation Studies

In this section, we study the performance of one-step-ahead ($k = 1$) and two-step-ahead ($k = 2$) 95% PI's of several time series with a linear trend using the computing platform R (2024). We follow Kong et al. (2018) to choose white noise distributions and model parameters. Specifically, we use two distributions for white noise $\{\epsilon_1, \dots, \epsilon_n\}$: the standard normal distribution $N(0, 1)$ and kurtotic distribution $2/3N(0, 1) + 1/3N(0, (0.1)^2)$. The linear model parameters are

$(\beta_0, \beta_1) = (6.5, 0.02)$, which are the estimates in the COVID-19 case count data to be analyzed in the next section. We consider autoregressive models with two different orders: AR(1) with $\phi_1 = -0.8, -0.6, -0.2, 0.2, 0.6, 0.8$ and AR(2) with $(\phi_1, \phi_2) = (-0.8, -0.4)$, $(0.2, 0.1)$, $(0.8, -0.4)$, $(0.2, -0.1)$, $(0.1, -0.05)$, $(-0.1, 0.05)$, $(1.4, -0.56)$. We simulate four sample sizes: $n = 50, 100, 500, 1000$ with 500 sample paths each.

In addition to the coverage frequencies, we also present the mean and the standard deviation of the widths for the proposed algorithm in Section 2. Moreover, for each model, we include another two PI's in the simulation studies for the purpose of comparison. The first one is the model PI which is calculated from model (2.4) with $\{\hat{b}_0, \hat{b}_1, \hat{\phi}_i, 1 \leqslant i \leqslant p\}$. It is clear that this PI is obtained based on the assumption that the error terms are independent and identically normally distributed, and moreover $\hat{y}_{t+k-i}^{[k-i]}$ on the right-hand-side of the model (2.4) are treated as observed values. The second one is the empirical PI which is computed from the empirical cumulative distribution function $\widehat{F}_n^{[k]}$:

$$\widehat{F}_n^{[k]}(z) = n^{-1} \sum_{t=k}^{n} I\left(\hat{\epsilon}_t^{[k]} \leqslant z\right), \tag{3.1}$$

where $I\left(\hat{\epsilon}_t^{[k]} \leqslant z\right)$ is 1 if $\hat{\epsilon}_t^{[k]} \leqslant z$ is true and 0 otherwise. Zhong (2024) has discussed how to use a quantile estimator that is derived from the empirical CDF of prediction residuals of autoregressive moving-average time series to establish multi-step-ahead prediction intervals for future observations.

In Tables 2–9, which present the simulation results, the three prediction interval construction methods are labeled as follows: 'KDE' for PIs from kernel density estimation, 'Normal' for PIs from model (2.4) with normal quantiles, and 'EM' for PIs from the empirical distribution function (3.1). In most cases, the KDE PIs demonstrate greater robustness and superiority over the other methods, with coverage frequencies closer to the nominal 95% level and the smallest mean and standard deviation of PI widths, especially when the sample size is small. For one-step-ahead predictions, the performance of KDE and EM becomes very similar as the sample size increases. However, the coverage frequencies tend to deviate further from 95% at $k = 2$. The relatively poor performance of 'Normal' and 'EM' PIs at small sample sizes is expected. The 'Normal' PIs rely on the assumption that the data are independent and identically normally distributed, or based on misspecified models, while the 'EM' PIs are constructed using the empirical distribution function, which exhibits sudden jumps at each data point in the sample.

4 Application to COVID-19 Case Counts

In this section, we construct 95% prediction intervals for the weekly log-transformed COVID-19 case counts from lower-middle-income countries between June 1, 2020, and March 13, 2022. These countries faced more challenges during the pandemic, including limited access to healthcare resources. Thus, making accurate forecasting is essential for effective policy planning in these countries.

The log transformation reduces variability and better meets the assumptions of autoregressive models. The data set is downloaded from the website *Our World in Data* (https://ourworldindata.org/). More detailed information about the data set can be found from Shao et al. (2024). The observations are marked by blue in Figure 1 (a). The scatter plot exhibits a pronounced upward linear trend, which suggests that a linear model (2.1) with week as a covariate be a candidate for model fitting. Unlike a simulation study, the true AR order p is unknown. To identify the most appropriate p, we fit a linear model (2.1) and obtain the ordinary least squares estimates for the model parameters: $\hat{\beta}_0 = 6.570$ and $\hat{\beta}_1 = 0.021$. Moreover, we calculate the residuals $\{\hat{w}_t, 1 \leqslant t \leqslant 93\}$ and notice strong correlations from their sample autocorrelation plot in Figure 1 (b). The correlations at lags 1–4 are significant and beyond the 95% intervals. After fitting several autoregressive models, we conclude that AR(2) is the most appropriate model—The sample autocorrelation plot in Figure 2 (a) of the AR(2) residual sequence does not demonstrate any significant correlation. The Q-Q plot of the residuals in Figure 2 (b) exhibits a non-normal pattern with some dots outside the confidence band.

We apply the proposed algorithm to construct 95% prediction intervals at $k = 1, 2$. Firstly we fit the model (2.3) to the data and obtain the estimates: $\hat{\beta}_0 = 1.027$, $\hat{\beta}_1 = 0.003$, $\hat{\phi}_1 = 1.421$, $\hat{\phi}_2 = -0.573$. We calculate one-step-ahead prediction residuals $\{\hat{\epsilon}_t\}$ and two-step-ahead prediction residuals $\left\{\hat{\epsilon}^{[2]}_{t+2}\right\}$, and use these residuals in computing $\widehat{F}^{[1]}_{n,h}(z)$ and $\widehat{F}^{[2]}_{n,h}(z)$ which are respectively plotted in Figures 3 (a) and (b). In addition to the KDE, we also include the PI from empirical cumulative function in Table 1.

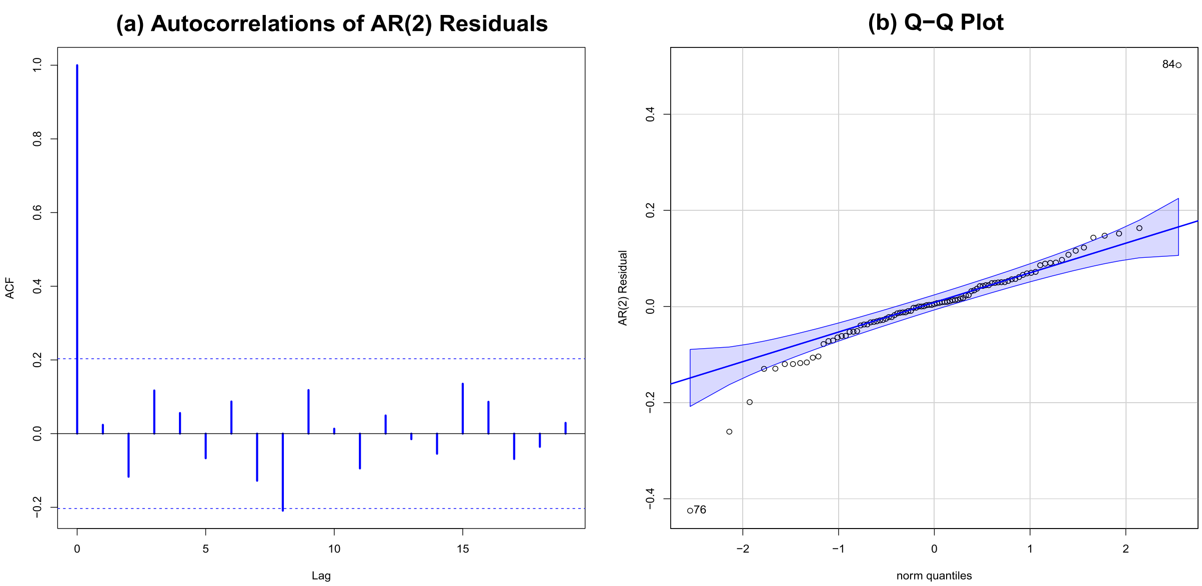

Figure 2: (a) ACF of AR2 residuals $\{e_t\}$, (b) Q-Q plot of $\{e_t\}$.

Table 1: 95% prediction intervals for COVID-19 case count time series.

	Point	KDE		EM	
	Prediction	95% PI	Width	95% PI	Width
$k = 1$	8.194	(7.992, 8.354)	0.362	(8.016, 8.343)	0.328
$k = 2$	8.301	(7.818, 8.760)	0.942	(7.880, 8.729)	0.849

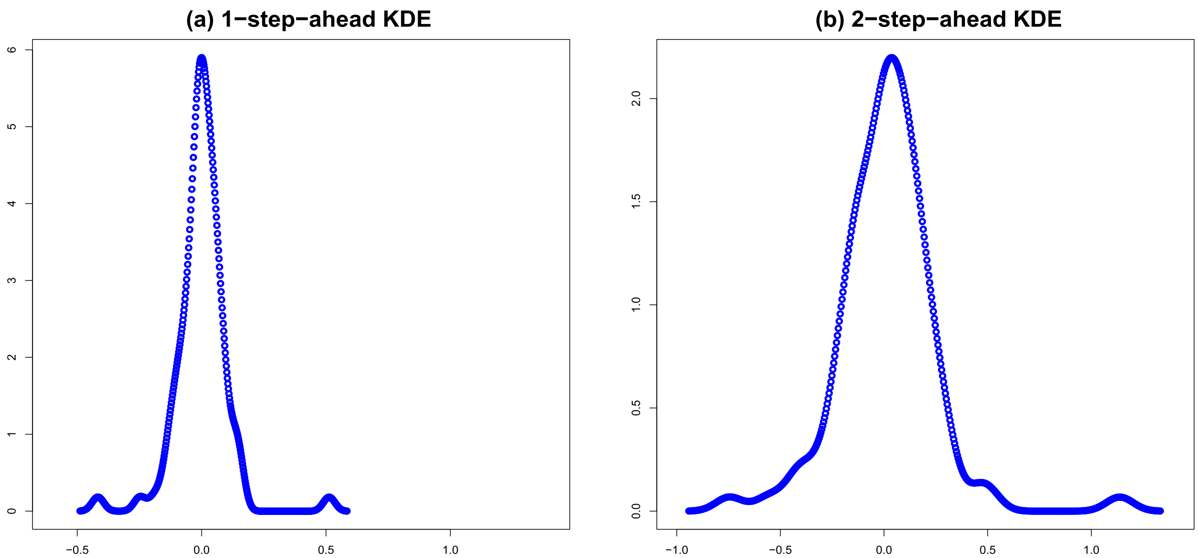

Figure 3: Kernel density estimates.

5 Concluding Remarks

The COVID-19 pandemic presented both opportunities and challenges for data scientists seeking to extract actionable insights to help society understand and combat the disease. Forecasting played a crucial role in optimizing resource allocation, especially during crises like the unprecedented COVID-19 pandemic. In response, we developed an algorithm for constructing k-step-ahead interval forecasts for time series models, which are widely used in time series analysis due to their ability to capture temporal dependencies in data.

The strength of the proposed algorithm lies in its ability to incorporate the correlation structure of the data, a critical factor for accurate forecasting. This flexibility is particularly valuable in real-world scenarios, where deviations from normality are common, and failing to account for these deviations can result in inaccurate predictions and unreliable confidence intervals.

Our method addresses several limitations of existing forecasting techniques by explicitly considering the non-normality and correlated structure inherent in time series data, offering a powerful tool for resource optimization and decision-making during critical periods like the COVID-19 pandemic. Other computationally intensive methods, such as bootstrap techniques extended to ARMA models by Pan and Politis (2016), for example, can also address issues related to unknown distributions. Further simulation studies are needed to compare these alternative methods.

Supplementary Material

Interested readers can refer to https://github.com/qinshao/JSD_IF.git for the data and R code.

Table 2: One-step-ahead prediction intervals for AR(1) with $N(0,1)$.

		Coverage Frequency			Mean Width			SD Width		
ϕ_1	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
−0.8	50	0.986	0.976	0.95	5.001	4.760	4.030	0.630	0.541	0.501
	100	0.956	0.950	0.932	4.505	4.360	3.930	0.417	0.356	0.360
	400	0.968	0.972	0.952	4.090	4.104	3.926	0.160	0.165	0.188
	800	0.948	0.946	0.932	4.002	4.041	3.912	0.106	0.118	0.132
−0.6	50	0.964	0.954	0.912	4.609	4.503	3.895	0.516	0.519	0.488
	100	0.962	0.960	0.940	4.306	4.317	3.921	0.326	0.333	0.332
	400	0.956	0.956	0.952	4.044	4.106	3.921	0.148	0.166	0.188
	800	0.948	0.952	0.944	3.975	4.042	3.913	0.099	0.114	0.126
−0.2	50	0.940	0.930	0.898	4.248	4.112	3.567	0.461	0.455	0.445
	100	0.934	0.942	0.912	4.092	4.117	3.750	0.302	0.327	0.338
	400	0.946	0.952	0.944	3.953	4.042	3.865	0.139	0.162	0.181
	800	0.942	0.948	0.934	3.941	4.021	3.895	0.100	0.118	0.134
0.2	50	0.956	0.946	0.890	4.219	4.102	3.568	0.453	0.463	0.449
	100	0.940	0.938	0.910	4.074	4.104	3.744	0.299	0.326	0.340
	400	0.944	0.956	0.944	3.958	4.043	3.865	0.138	0.160	0.184
	800	0.958	0.964	0.956	3.941	4.019	3.895	0.099	0.119	0.136
0.6	50	0.938	0.918	0.862	4.439	4.307	3.739	0.539	0.558	0.509
	100	0.948	0.950	0.916	4.252	4.267	3.869	0.332	0.341	0.347
	400	0.944	0.958	0.938	4.017	4.081	3.902	0.137	0.153	0.177
	800	0.950	0.950	0.944	3.978	4.041	3.915	0.105	0.120	0.134
0.8	50	0.962	0.952	0.898	4.749	4.590	3.920	0.617	0.603	0.510
	100	0.968	0.962	0.948	4.452	4.357	3.941	0.398	0.359	0.368
	400	0.958	0.962	0.950	4.088	4.096	3.907	0.160	0.170	0.191
	800	0.954	0.956	0.956	3.998	4.036	3.907	0.108	0.124	0.138

Table 3: Two-step-ahead prediction intervals for AR(1) with $N(0,1)$.

		Coverage Frequency			Mean Width			SD Width		
ϕ_1	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
−0.8	50	0.872	0.930	0.890	4.212	5.263	4.556	0.437	0.686	0.661
	100	0.878	0.934	0.916	4.066	5.289	4.816	0.288	0.483	0.478
	400	0.874	0.942	0.934	3.961	5.195	4.966	0.138	0.233	0.252
	800	0.846	0.936	0.926	3.937	5.151	4.989	0.098	0.171	0.186
−0.6	50	0.922	0.946	0.912	4.194	4.820	4.173	0.436	0.620	0.592
	100	0.898	0.948	0.920	4.037	4.798	4.363	0.286	0.421	0.425
	400	0.916	0.958	0.950	3.959	4.735	4.529	0.141	0.214	0.238
	800	0.912	0.966	0.954	3.934	4.689	4.542	0.101	0.160	0.176
−0.2	50	0.946	0.946	0.906	4.214	4.252	3.671	0.458	0.510	0.492
	100	0.954	0.956	0.944	4.057	4.201	3.826	0.300	0.327	0.324
	400	0.922	0.934	0.924	3.944	4.124	3.937	0.136	0.174	0.187
	800	0.964	0.972	0.966	3.944	4.109	3.982	0.103	0.126	0.143
0.2	50	0.938	0.932	0.880	4.192	4.161	3.596	0.461	0.494	0.475
	100	0.958	0.966	0.924	4.055	4.177	3.802	0.300	0.349	0.366
	400	0.928	0.940	0.922	3.956	4.134	3.956	0.135	0.164	0.183
	800	0.946	0.958	0.946	3.929	4.090	3.962	0.101	0.119	0.132
0.6	50	0.864	0.896	0.848	4.199	4.653	4.030	0.463	0.670	0.633
	100	0.904	0.944	0.920	4.059	4.701	4.278	0.313	0.451	0.450
	400	0.908	0.950	0.946	3.953	4.709	4.506	0.146	0.238	0.256
	800	0.912	0.956	0.948	3.942	4.697	4.551	0.101	0.153	0.170
0.8	50	0.826	0.892	0.842	4.158	5.001	4.351	0.464	0.746	0.702
	100	0.854	0.934	0.902	4.017	5.089	4.632	0.280	0.488	0.489
	400	0.856	0.940	0.924	3.947	5.161	4.938	0.148	0.261	0.278
	800	0.846	0.952	0.944	3.938	5.137	4.977	0.105	0.181	0.196

Table 4: One-step-ahead prediction intervals for AR(2) with $N(0,1)$.

		Coverage Frequency			Mean Width			SD Width		
(ϕ_1, ϕ_2)	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
$(-0.8, -0.4)$	50	0.960	0.940	0.898	4.278	4.038	3.490	0.455	0.437	0.436
	100	0.946	0.940	0.920	4.082	4.061	3.696	0.314	0.335	0.341
	400	0.952	0.952	0.946	3.958	4.038	3.861	0.144	0.168	0.187
	800	0.932	0.946	0.934	3.938	4.013	3.887	0.106	0.122	0.137
$(0.2, 0.1)$	50	0.940	0.934	0.888	4.288	4.037	3.493	0.442	0.432	0.432
	100	0.948	0.950	0.912	4.078	4.057	3.693	0.309	0.326	0.341
	400	0.952	0.956	0.942	3.946	4.028	3.851	0.145	0.163	0.178
	800	0.956	0.958	0.948	3.935	4.014	3.887	0.096	0.109	0.121
$(0.8, -0.4)$	50	0.940	0.916	0.878	4.263	4.023	3.486	0.459	0.464	0.459
	100	0.956	0.946	0.904	4.078	4.054	3.682	0.295	0.315	0.327
	400	0.954	0.956	0.944	3.957	4.031	3.854	0.143	0.164	0.181
	800	0.952	0.954	0.946	3.935	4.010	3.882	0.105	0.121	0.133
$(0.2, -0.1)$	50	0.958	0.938	0.880	4.283	4.032	3.480	0.477	0.464	0.457
	100	0.962	0.958	0.936	4.094	4.068	3.703	0.301	0.327	0.345
	400	0.954	0.956	0.942	3.956	4.031	3.856	0.145	0.166	0.185
	800	0.948	0.952	0.946	3.942	4.017	3.891	0.100	0.119	0.133
$(0.1, -0.05)$	50	0.938	0.918	0.876	4.241	3.998	3.446	0.459	0.453	0.442
	100	0.958	0.956	0.918	4.108	4.086	3.718	0.297	0.314	0.331
	400	0.958	0.960	0.944	3.967	4.045	3.870	0.140	0.158	0.183
	800	0.940	0.944	0.934	3.940	4.018	3.894	0.099	0.117	0.129
$(-0.1, 0.05)$	50	0.932	0.926	0.882	4.248	4.000	3.442	0.476	0.460	0.446
	100	0.948	0.950	0.926	4.078	4.058	3.688	0.289	0.309	0.338
	400	0.972	0.968	0.964	3.968	4.050	3.873	0.140	0.163	0.184
	800	0.950	0.952	0.948	3.936	4.011	3.886	0.095	0.110	0.122
$(1.4, -0.56)$	50	0.952	0.930	0.876	4.260	3.998	3.457	0.467	0.440	0.433
	100	0.966	0.952	0.938	4.100	4.069	3.704	0.320	0.346	0.358
	400	0.958	0.956	0.946	3.964	4.035	3.858	0.143	0.164	0.178
	800	0.954	0.954	0.952	3.935	4.007	3.882	0.099	0.113	0.127

Table 5: Two-step-ahead prediction intervals for AR(2) with $N(0,1)$.

		Coverage Frequency			Mean Width			SD Width		
(ϕ_1, ϕ_2)	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
$(-0.8, -0.4)$	50	0.818	0.930	0.878	4.278	5.957	5.151	0.485	1.120	1.027
	100	0.800	0.922	0.890	4.081	5.813	5.275	0.289	0.824	0.777
	400	0.814	0.932	0.914	3.954	5.596	5.345	0.143	0.387	0.387
	800	0.768	0.920	0.908	3.935	5.553	5.388	0.103	0.275	0.278
$(0.2, 0.1)$	50	0.940	0.938	0.886	4.204	4.134	3.558	0.438	0.481	0.477
	100	0.952	0.952	0.938	4.074	4.159	3.775	0.297	0.338	0.347
	400	0.930	0.94	0.934	3.950	4.082	3.906	0.134	0.167	0.184
	800	0.962	0.964	0.956	3.935	4.055	3.927	0.097	0.120	0.132
$(0.8, -0.4)$	50	0.860	0.958	0.926	4.259	6.403	5.557	0.475	0.978	0.914
	100	0.890	0.982	0.976	4.106	6.476	5.891	0.302	0.769	0.734
	400	0.894	0.996	0.994	3.968	6.434	6.151	0.138	0.376	0.393
	800	0.914	0.990	0.986	3.934	6.365	6.164	0.099	0.277	0.286
$(0.2, -0.1)$	50	0.932	0.936	0.876	4.204	4.312	3.718	0.468	0.558	0.543
	100	0.950	0.952	0.924	4.044	4.281	3.885	0.295	0.391	0.389
	400	0.936	0.962	0.946	3.968	4.262	4.078	0.130	0.185	0.204
	800	0.946	0.958	0.944	3.935	4.210	4.082	0.101	0.140	0.152
$(0.1, -0.05)$	50	0.960	0.954	0.922	4.266	4.241	3.652	0.467	0.507	0.496
	100	0.944	0.948	0.912	4.080	4.189	3.811	0.298	0.357	0.364
	400	0.942	0.954	0.940	3.954	4.101	3.919	0.143	0.175	0.196
	800	0.914	0.932	0.920	3.939	4.073	3.946	0.105	0.126	0.138
$(-0.1, 0.05)$	50	0.938	0.926	0.892	4.240	4.209	3.634	0.453	0.496	0.475
	100	0.940	0.940	0.922	4.055	4.171	3.789	0.309	0.358	0.369
	400	0.946	0.954	0.944	3.953	4.095	3.912	0.141	0.173	0.192
	800	0.95	0.956	0.952	3.940	4.075	3.941	0.103	0.128	0.140
$(1.4, -0.56)$	50	0.848	0.986	0.976	4.254	8.732	7.553	0.435	1.399	1.279
	100	0.892	1.000	1.000	4.072	9.044	8.230	0.297	1.012	0.970
	400	0.906	0.998	0.998	3.953	8.997	8.610	0.142	0.507	0.523
	800	0.890	1.000	1.000	3.941	8.979	8.693	0.097	0.358	0.386

Table 6: One-step-ahead prediction intervals for AR(1) with kurtotic.

		Coverage Frequency			Mean Width			SD Width		
ϕ_1	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
−0.8	50	0.950	0.942	0.916	4.228	4.165	3.647	0.630	0.590	0.545
	100	0.968	0.968	0.950	3.852	3.891	3.618	0.404	0.386	0.38
	400	0.948	0.962	0.960	3.397	3.64	3.566	0.154	0.182	0.191
	800	0.930	0.954	0.948	3.301	3.592	3.546	0.108	0.127	0.136
−0.6	50	0.954	0.942	0.914	3.827	3.877	3.454	0.539	0.587	0.551
	100	0.944	0.956	0.944	3.580	3.803	3.556	0.334	0.382	0.382
	400	0.944	0.950	0.952	3.307	3.623	3.548	0.158	0.186	0.195
	800	0.944	0.962	0.962	3.271	3.606	3.558	0.104	0.133	0.139
−0.2	50	0.934	0.932	0.908	3.448	3.497	3.143	0.455	0.502	0.474
	100	0.934	0.938	0.928	3.333	3.563	3.353	0.308	0.359	0.352
	400	0.920	0.946	0.940	3.241	3.575	3.495	0.149	0.187	0.198
	800	0.922	0.950	0.950	3.230	3.582	3.536	0.107	0.127	0.130
0.2	50	0.938	0.926	0.912	3.406	3.450	3.108	0.479	0.504	0.481
	100	0.940	0.948	0.930	3.348	3.586	3.371	0.327	0.371	0.367
	400	0.940	0.952	0.948	3.249	3.583	3.507	0.160	0.194	0.202
	800	0.920	0.946	0.944	3.219	3.573	3.528	0.104	0.129	0.137
0.6	50	0.938	0.928	0.904	3.708	3.742	3.343	0.533	0.582	0.565
	100	0.932	0.942	0.928	3.535	3.767	3.520	0.338	0.386	0.386
	400	0.932	0.948	0.942	3.319	3.628	3.554	0.156	0.187	0.195
	800	0.920	0.944	0.942	3.264	3.598	3.550	0.111	0.132	0.139
0.8	50	0.938	0.928	0.896	4.011	4.009	3.511	0.616	0.627	0.568
	100	0.958	0.960	0.934	3.743	3.832	3.544	0.401	0.410	0.387
	400	0.928	0.946	0.944	3.382	3.614	3.539	0.161	0.183	0.193
	800	0.944	0.958	0.956	3.305	3.602	3.557	0.108	0.133	0.139

Table 7: Two-step-ahead prediction intervals for AR(1) with kurtotic.

		Coverage Frequency			Mean Width			SD Width		
ϕ_1	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
−0.8	50	0.902	0.958	0.928	3.448	4.461	3.901	0.433	0.696	0.660
	100	0.890	0.954	0.934	3.345	4.464	4.102	0.284	0.450	0.451
	400	0.846	0.938	0.932	3.233	4.389	4.240	0.154	0.256	0.270
	800	0.886	0.956	0.952	3.224	4.379	4.279	0.108	0.178	0.190
−0.6	50	0.900	0.936	0.904	3.408	4.049	3.575	0.459	0.655	0.631
	100	0.908	0.952	0.940	3.306	4.072	3.764	0.295	0.454	0.450
	400	0.904	0.960	0.950	3.232	4.039	3.913	0.154	0.232	0.247
	800	0.912	0.954	0.954	3.216	4.026	3.946	0.112	0.170	0.178
−0.2	50	0.940	0.948	0.926	3.410	3.606	3.233	0.454	0.516	0.482
	100	0.936	0.952	0.940	3.300	3.610	3.373	0.306	0.363	0.358
	400	0.924	0.946	0.942	3.239	3.645	3.557	0.155	0.193	0.197
	800	0.898	0.922	0.922	3.216	3.634	3.584	0.110	0.140	0.148
0.2	50	0.920	0.926	0.902	3.442	3.566	3.201	0.482	0.537	0.516
	100	0.924	0.936	0.926	3.302	3.615	3.394	0.312	0.359	0.357
	400	0.932	0.952	0.950	3.242	3.647	3.564	0.146	0.191	0.202
	800	0.904	0.934	0.932	3.219	3.626	3.575	0.105	0.139	0.146
0.6	50	0.888	0.914	0.876	3.436	3.887	3.414	0.482	0.636	0.587
	100	0.920	0.946	0.928	3.325	4.001	3.698	0.296	0.423	0.417
	400	0.912	0.964	0.962	3.238	4.021	3.895	0.157	0.235	0.248
	800	0.900	0.958	0.954	3.217	4.019	3.936	0.103	0.157	0.166
0.8	50	0.848	0.906	0.864	3.426	4.196	3.680	0.456	0.701	0.658
	100	0.858	0.924	0.912	3.319	4.316	3.973	0.321	0.530	0.521
	400	0.866	0.934	0.930	3.230	4.363	4.212	0.149	0.262	0.279
	800	0.874	0.940	0.934	3.215	4.353	4.248	0.100	0.179	0.192

Table 8: One-step-ahead prediction intervals for AR(2) with kurtotic.

		Coverage Frequency			Mean Width			SD Width		
(ϕ_1, ϕ_2)	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
$(-0.8, -0.4)$	50	0.938	0.924	0.888	3.504	3.438	3.071	0.476	0.483	0.462
	100	0.932	0.946	0.932	3.332	3.514	3.292	0.334	0.382	0.371
	400	0.920	0.952	0.946	3.231	3.553	3.478	0.149	0.185	0.198
	800	0.948	0.964	0.960	3.225	3.575	3.530	0.102	0.132	0.138
$(0.2, 0.1)$	50	0.938	0.908	0.878	3.493	3.424	3.067	0.489	0.504	0.485
	100	0.920	0.926	0.904	3.324	3.507	3.293	0.318	0.364	0.363
	400	0.926	0.934	0.932	3.244	3.573	3.503	0.148	0.186	0.195
	800	0.916	0.940	0.932	3.224	3.570	3.525	0.111	0.131	0.140
$(0.8, -0.4)$	50	0.944	0.932	0.910	3.460	3.386	3.024	0.494	0.509	0.481
	100	0.938	0.946	0.936	3.359	3.552	3.338	0.340	0.380	0.373
	400	0.934	0.952	0.946	3.245	3.570	3.493	0.161	0.194	0.203
	800	0.918	0.940	0.938	3.217	3.567	3.522	0.105	0.130	0.136
$(0.2, -0.1)$	50	0.928	0.922	0.902	3.453	3.393	3.035	0.468	0.491	0.471
	100	0.936	0.940	0.930	3.334	3.525	3.312	0.317	0.360	0.354
	400	0.920	0.948	0.940	3.236	3.563	3.485	0.149	0.185	0.193
	800	0.924	0.952	0.950	3.228	3.576	3.530	0.108	0.128	0.131
$(0.1, -0.05)$	50	0.944	0.916	0.892	3.485	3.414	3.055	0.503	0.510	0.495
	100	0.928	0.942	0.932	3.343	3.534	3.316	0.321	0.363	0.356
	400	0.940	0.946	0.948	3.228	3.566	3.496	0.154	0.183	0.193
	800	0.940	0.962	0.958	3.229	3.580	3.531	0.104	0.131	0.134
$(-0.1, 0.05)$	50	0.934	0.920	0.874	3.474	3.409	3.056	0.517	0.500	0.474
	100	0.958	0.960	0.950	3.361	3.550	3.329	0.333	0.372	0.366
	400	0.932	0.962	0.960	3.244	3.579	3.506	0.152	0.181	0.190
	800	0.928	0.954	0.952	3.223	3.564	3.520	0.105	0.128	0.137
$(1.4, -0.56)$	50	0.948	0.934	0.902	3.471	3.402	3.039	0.467	0.466	0.458
	100	0.904	0.920	0.898	3.316	3.508	3.285	0.325	0.375	0.365
	400	0.924	0.946	0.938	3.236	3.562	3.488	0.147	0.181	0.189
	800	0.930	0.948	0.946	3.229	3.584	3.541	0.108	0.138	0.144

Table 9: Two-step-ahead prediction intervals for AR(2) with kurtotic.

		Coverage Frequency			Mean Width			SD Width		
(ϕ_1, ϕ_2)	n	Normal	KDE	EM	Normal	KDE	EM	Normal	KDE	EM
$(-0.8, -0.4)$	50	0.844	0.924	0.892	3.477	4.915	4.268	0.462	0.980	0.885
	100	0.786	0.920	0.900	3.339	4.874	4.467	0.325	0.730	0.69
	400	0.782	0.914	0.906	3.232	4.679	4.502	0.149	0.361	0.371
	800	0.790	0.924	0.914	3.216	4.654	4.539	0.101	0.252	0.259
$(0.2, 0.1)$	50	0.932	0.928	0.912	3.488	3.562	3.185	0.499	0.553	0.534
	100	0.924	0.932	0.922	3.323	3.598	3.380	0.321	0.365	0.363
	400	0.93	0.954	0.952	3.246	3.617	3.536	0.145	0.189	0.196
	800	0.912	0.932	0.932	3.221	3.599	3.551	0.104	0.135	0.142
$(0.8, -0.4)$	50	0.874	0.964	0.938	3.470	5.298	4.618	0.473	1.024	0.947
	100	0.900	0.982	0.964	3.321	5.371	4.933	0.309	0.711	0.689
	400	0.872	0.990	0.986	3.245	5.433	5.241	0.155	0.370	0.377
	800	0.864	0.980	0.974	3.219	5.389	5.263	0.110	0.272	0.284
$(0.2, -0.1)$	50	0.918	0.916	0.888	3.441	3.692	3.274	0.475	0.560	0.525
	100	0.926	0.946	0.928	3.324	3.710	3.456	0.304	0.407	0.398
	400	0.912	0.948	0.944	3.234	3.715	3.617	0.155	0.202	0.210
	800	0.930	0.968	0.962	3.217	3.696	3.637	0.112	0.145	0.151
$(0.1, -0.05)$	50	0.954	0.950	0.930	3.463	3.612	3.223	0.461	0.539	0.516
	100	0.924	0.942	0.922	3.352	3.643	3.411	0.292	0.369	0.364
	400	0.920	0.944	0.942	3.235	3.611	3.531	0.155	0.200	0.210
	800	0.932	0.960	0.958	3.226	3.616	3.568	0.108	0.135	0.143
$(-0.1, 0.05)$	50	0.932	0.946	0.916	3.418	3.547	3.160	0.490	0.521	0.500
	100	0.930	0.946	0.934	3.320	3.620	3.384	0.306	0.38	0.369
	400	0.928	0.952	0.944	3.248	3.633	3.556	0.151	0.194	0.200
	800	0.922	0.948	0.946	3.219	3.611	3.563	0.109	0.141	0.149
$(1.4, -0.56)$	50	0.870	0.994	0.982	3.477	7.420	6.496	0.508	1.425	1.322
	100	0.910	1.000	1.000	3.313	7.618	7.033	0.321	0.985	0.978
	400	0.902	1.000	1.000	3.252	7.764	7.530	0.146	0.497	0.522
	800	0.890	1.000	0.998	3.225	7.749	7.585	0.101	0.345	0.365

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