Appendix

In the appendix, we present the details of the EM algorithm for our model estimation.

A Forward-backward procedure

In order to compute $\gamma_i^{\langle r,t\rangle}$ and $\xi_{ij}^{\langle r,t\rangle}$, we use the following forward-backward procedure. The forward recursion is established as follows. Let $\delta_i^{\langle r,t\rangle} = P(Y^{\langle r,1\rangle},\dots,Y^{\langle r,t\rangle},X^{\langle r,t\rangle}=i|\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})$, $i=0,1$, denote the probability of observing $Y^{\langle r,1\rangle},\dots,Y^{\langle r,t\rangle}$ and being the hidden state i at time t for rth loan ID. Given $\hat{\boldsymbol{\pi}},\hat{\mathscr{F}}$ and $\mathbf{Z}$, we can get $\delta_i^{\langle r,t\rangle}, i=0,1$, at each time t recursively:

$$\begin{aligned}
\delta_i^{\langle r,1\rangle} =& P(Y^{\langle r,1\rangle},X^{\langle r,1\rangle}=i|\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}}) = P(Y^{\langle r,1\rangle}|X^{\langle r,1\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})P(X^{\langle r,1\rangle}=i|\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}}) = \hat{b}_{iY^{\langle r,1\rangle}}^{\langle r,t\rangle}\hat{\pi}_i\\
&\dots\\
\delta_i^{\langle r,t\rangle} =& P(Y^{\langle r,1\rangle},\dots,Y^{\langle r,t\rangle},X^{\langle r,t\rangle}=i|\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
=&\sum_{j=0}^{1}(P(Y^{\langle r,t\rangle},X^{\langle r,t\rangle}=i|X^{\langle r,t-1\rangle}=j,Y^{\langle r,1\rangle},\dots,Y^{\langle r,t-1\rangle},\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
&P(Y^{\langle r,1\rangle},\dots,Y^{\langle r,t-1\rangle},X^{\langle r,t-1\rangle}=j|\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
=&\sum_{j=0}^{1}P(Y^{\langle r,t\rangle}|X^{\langle r,t\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})P(X^{\langle r,t\rangle}=i|X^{\langle r,t-1\rangle}=j,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\delta_j^{\langle r,t-1\rangle}\\
=&\hat{b}_{iY^{\langle r,t\rangle}}^{\langle r,t\rangle}\sum_{j=0}^{1}\delta_j^{\langle r,t-1\rangle}\hat{a}_{ji}^{\langle r,t-1\rangle}.
\end{aligned}$$

Analogously, the backward recursion is established as follows. Let

$$\tau_i^{\langle r,t\rangle} = P(Y^{\langle r,t+1\rangle},\dots,Y^{\langle r,T_r\rangle}|X^{\langle r,t\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})$$

denote the probability of the ending partial sequence $Y^{\langle r,t+1\rangle},\dots,Y^{\langle r,T_r\rangle}$ given hidden state i at time t for rth loan ID. We calculate $\tau_i^{\langle r,t\rangle}$ recursively as follows,

$$\begin{aligned}
\tau_i^{\langle r,T_r\rangle} &= 1\\
&\dots\\
\tau_i^{\langle r,t\rangle} &= P(Y^{\langle r,t+1\rangle},\dots,Y^{\langle r,T_r\rangle}|X^{\langle r,t\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
&=\sum_{j=0}^{1}P(Y^{\langle r,t+1\rangle},\dots,Y^{\langle r,T_r\rangle},X^{\langle r,t+1\rangle}=j|X^{\langle r,t\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
&=\sum_{j=0}^{1}P(Y^{\langle r,t+1\rangle},\dots,Y^{\langle r,T_r\rangle}|X^{\langle r,t+1\rangle}=j,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})P(X^{\langle r,t+1\rangle}=j|X^{\langle r,t\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
&=\sum_{j=0}^{1}P(Y^{\langle r,t+1\rangle}|Y^{\langle r,t+2\rangle},X^{\langle r,t+1\rangle}=j,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\tau_j^{\langle r,t+1\rangle}P(X^{\langle r,t+1\rangle}=j|X^{\langle r,t\rangle}=i,\mathbf{Z},\hat{\boldsymbol{\pi}},\hat{\mathscr{F}})\\
&=\sum_{j=0}^{1}\hat{b}_{jY^{\langle r,t+1\rangle}}^{\langle r,t+1\rangle}\tau_j^{\langle r,t+1\rangle}\hat{a}_{ij}^{\langle r,t\rangle}.
\end{aligned}$$

The state posterior probability $\gamma_i^{\langle r,t\rangle}$ can be expressed in terms of δ and τ. In particular, according to the Bayes' theorem, we have

$$\gamma_i^{\langle r,t\rangle} = P(X^{\langle r,t\rangle} = i|\mathbf{Y}, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) = \frac{P(X^{\langle r,t\rangle} = i, \mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})}{P(\mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})},$$

where $P(X^{\langle r,t\rangle} = i, \mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})$ can be rewritten as $P(Y^{\langle r,t+1\rangle}, \ldots, Y^{\langle r,T_r\rangle}|X^{\langle r,t\rangle} = i, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})$ $\times P(Y^{\langle r,1\rangle}, \ldots, Y^{\langle r,t\rangle}, X^{\langle r,t\rangle} = i|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) = \tau_i^{\langle r,t\rangle}\delta_i^{\langle r,t\rangle}$. Herein, $P(\mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) = \sum_{i=0}^{1} P(\mathbf{Y}, X^{\langle r,T_r\rangle} = i|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) = \sum_{i=0}^{1} \delta_i^{\langle r,T_r\rangle}$. Thus, $\gamma_i^{\langle r,t\rangle}$ can be obtained by

$$\gamma_i^{\langle r,t\rangle} = \frac{\tau_i^{\langle r,t\rangle}\delta_i^{\langle r,t\rangle}}{\sum_{i=0}^{1} \delta_i^{\langle r,T_r\rangle}}. \tag{12}$$

Analogously, the transition posterior probability $\xi_{ij}^{\langle r,t-1\rangle}$ can be expressed in terms of δ and τ as follows.

$$\xi_{ij}^{\langle r,t-1\rangle} = P(X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i|\mathbf{Y}, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) = \frac{P(X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i, \mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})}{P(\mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})},$$

where the denominator can be rewritten as

$$\begin{aligned} &P(X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i, \mathbf{Y}|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) && (13)\\ =&P(Y^{\langle r,t\rangle}, \ldots, Y^{\langle r,T_r\rangle}|X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) && (14)\\ &\times P(Y^{\langle r,1\rangle}, \ldots, Y^{\langle r,t-1\rangle}, X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}}) && (15) \end{aligned}$$

The first term of the right-hand-side (15) can be rewritten as

$$\begin{aligned} &P(Y^{\langle r,t\rangle}, \ldots, Y^{\langle r,T_r\rangle}|X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})\\ &= P(Y^{\langle r,t+1\rangle}, \ldots, Y^{\langle r,T_r\rangle}|X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})P(Y^{\langle r,t\rangle}|X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})\\ &= \tau_j^{\langle r,t\rangle}\hat{b}_{jY^{\langle r,t\rangle}}^{\langle r,t\rangle}. \end{aligned}$$

The second term of the right-hand-side (15) can be rewritten as

$$\begin{aligned} &P(Y^{\langle r,1\rangle}, \ldots, Y^{\langle r,t-1\rangle}, X^{\langle r,t\rangle} = j, X^{\langle r,t-1\rangle} = i|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})\\ &= P(X^{\langle r,t\rangle} = j|X^{\langle r,t-1\rangle} = i, Y^{\langle r,1\rangle}, \ldots, Y^{\langle r,t-1\rangle}, \mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})P(Y^{\langle r,1\rangle}, \ldots, Y^{\langle r,t-1\rangle}, X^{\langle r,t-1\rangle} = i|\mathbf{Z}, \hat{\boldsymbol{\pi}}, \hat{\mathscr{F}})\\ &= \hat{a}_{ij}^{\langle r,t-1\rangle}\delta_i^{\langle r,t-1\rangle}. \end{aligned}$$

Therefore, $\xi_{ij}^{\langle r,t-1\rangle}$ can be updated by

$$\xi_{ij}^{\langle r,t-1\rangle} = \frac{\tau_j^{\langle r,t\rangle}\hat{b}_{jY^{\langle r,t\rangle}}^{\langle r,t\rangle}\hat{a}_{ij}^{\langle r,t-1\rangle}\delta_i^{\langle r,t-1\rangle}}{\sum_{i=0}^{1} \delta_i^{\langle r,T_r\rangle}}. \tag{16}$$