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Power Priors for Leveraging Historical Data: Looking Back and Looking Forward☆

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Abstract

Historical data or real-world data are often available in clinical trials, genetics, health care, psychology, environmental health, engineering, economics, and business. The power priors have emerged as a useful class of informative priors for a variety of situations in which historical data are available. In this paper, an overview of the development of the power priors is provided. Various variations of the power priors are derived under a binomial regression model and a normal linear regression model. The development of software on the power priors is also briefly reviewed. Throughout this paper, the data from the Kociba study and the National Toxicology Program study as well as the data from the Alzheimer's Disease Neuroimaging Initiative (ADNI) study are used to demonstrate the derivations of the power priors and their variations. Detailed analyses of the data from these studies are carried out to further demonstrate the usefulness of the power priors and their variations in these real applications. Finally, the directions of future research on the power priors are discussed.

Keywords *Bayesian design of clinical trials; borrowing-by-parts power priors; discounting parameters; informative priors; partial borrowing power priors; propensity score based power priors*

1 Introduction

Historical data are often available in genetics, health care, psychology, environmental health, engineering, economics, business, and clinical trials. In medical devices, historical data are often available from previous trials for the control device only. In pediatric rare cancer studies, data from adult patients may be available. In rare disease settings, an efficacious standard of care (S) is already on the market. Thus, historical data are available from the treatment of S. The early developments and applications of power priors include Berry (1991), Eddy et al. (1992), Lin (1993), Berry and Hardwick (1994), Spiegelhalter et al. (1994), Berry and Stangl (1996), Chen et al. (1998), Ibrahim et al. (1998), Chen et al. (1999a), Chen et al. (1999b), and Ibrahim and Chen (2000b). An early review paper discussing the formalization of the power prior as a general prior for various classes of regression models is Ibrahim and Chen (2000a).

☆Data used in preparation of this article were obtained from the Alzheimer's Disease Neuroimaging Initiative database (adni.loni.usc.edu). As such, the investigators within the ADNI contributed to the design and implementation of ADNI and/or provided data but did not participate in analysis or writing of this report. A complete listing of ADNI investigators can be found at: http://adni.loni.usc.edu/wp-content/uploads/how_to_apply/ADNI_Acknowledgement_List.pdf

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The power priors have emerged as a useful class of informative priors for a variety of situations in which historical data are available. The power priors began to attract increased attention after the publication of Ibrahim and Chen (2000a). The class of power priors raise the likelihood of the historical data to a power, enabling analysts to adjust the influence of historical data through the discounting parameter in the power. Subsequent research has expanded on the foundational concepts and applied power priors in diverse contexts. The theoretical justification of the power prior is given in Ibrahim et al. (2003), and Chen and Ibrahim (2006) established the relationship between the power priors and hierarchical models. Spiegelhalter et al. (2004), De Santis (2006), De Santis (2007), Berry et al. (2012), and Ibrahim et al. (2012a) explored and illustrated the use of the power priors in epidemiological studies and clinical trials, in which historical data are often available. Chen et al. (2011), Ibrahim et al. (2012b), Chen et al. (2014a), and Chen et al. (2014b) advanced methodological developments of the power priors for the Bayesian design of clinical trials. When it is desirable to treat the discounting parameter as random, the normalized power priors and their variations introduced an additional normalizing term to adhere to the likelihood principle (Duan, 2005; Duan et al., 2006; Neuenschwander et al., 2009; Hobbs et al., 2011). The review article of Ibrahim and Chen (2000a) is also cited in United States Food and Drug Administration (US FDA) guidance for the use of Bayesian statistics in the design and analysis of medical device clinical trials (US Food and Drug Administration, 2010). A comprehensive exposition of the power prior and its applications up to 2015 is provided in Ibrahim et al. (2015a).

Real world data (RWD)/external data have played an increasing role in drug development, especially after the release of the 21st Century Cures Act (US Government Publishing Office, 2016). The US FDA released several guidance documents on leveraging existing clinical data (US Food and Drug Administration, 2016), the framework for Real-World Evidence (RWE) program (US Food and Drug Administration, 2018), and use of RWD/RWE in drug development (US Food and Drug Administration, 2021a,b, 2023). Also, owing to modern advances in computer technology, electronic health record (EHR) data and other related external medical and trial data have become more widely available in digital format. All of these have led to a third wave of methodological developments and novel applications in using power priors. Recently published vast literature on this topic includes the theoretical and computational development of the normalized power prior or the partial-borrowing normalized power prior (Banbeta et al., 2019; Carvalho and Ibrahim, 2021; Ye et al., 2022; Han et al., 2023a,b; Pawel et al., 2023a); adaptive or dynamic borrowing power priors (Gravestock and Held, 2017; Pan et al., 2017; Liu, 2018; Nikolakopoulos et al., 2018; Psioda and Ibrahim, 2018; Gravestock and Held, 2019; Ollier et al., 2020; Thompson et al., 2021; Sawamoto et al., 2022; Han et al., 2023a; Hickey et al., 2023; Baumann et al., 2024); propensity score based power priors (US Government Publishing Office, 2016; Lin et al., 2019; Wang et al., 2019; Li et al., 2020; Bennett et al., 2021; Baron et al., 2022; Li et al., 2022a; Lu et al., 2022; Wang et al., 2022; Baron et al., 2024); and Bayesian design of clinical trials and sample size re-estimation using power priors (Hees and Kieser, 2017; Psioda et al., 2018; Brakenhoff et al., 2019; Feißt et al., 2020; Kopp-Schneider et al., 2020; Nagase et al., 2020; Wiesenfarth and Calderazzo, 2020; Duan et al., 2021; Huang et al., 2022; Kopp-Schneider et al., 2023). Power priors and their extensions have also recently used or applied in behavioral and cognitive neuroscience (Egbon et al., 2023; Mezzetti et al., 2023); causal inference (Li et al., 2022b); clinical trials (Warasi et al., 2016; van Rosmalen et al., 2018; Li and Yuan, 2020; Pateras et al., 2021; Chao et al., 2022; Yu et al., 2022); diagnostic accuracy studies and tests (Bai et al., 2021; Wilson et al., 2022); energy engineering (Lorencin and Pantos, 2017); judicial studies

(Pandya et al., 2023); and meta-analysis and replication studies (Zhang et al., 2019; Pawel et al., 2023b). Other development and applications of power priors are further explored and discussed in Sections 3 and 7.

The remainder of this paper is organized as follows. Section 2 presents data from the Kociba, National Toxicology Program (NTP), and Alzheimer's Disease Neuroimaging Initiative (ADNI) studies to motivate and demonstrate the formulations of the power priors. Section 3 expands on the development of the power priors, and is followed by detailed derivation of different variants of the power priors under binomial and normal linear regression model settings in Section 4. Section 5 serves as a review on available software regarding the power priors. In-depth analyses of the Kociba and NTP data and the ADNI data are carried out in Section 6. We conclude the paper with a brief discussion of the directions of future research relating to power priors.

2 The Motivating Case Studies

2.1 Kociba and NTP Study on the Benchmark Approach in Toxicology

The benchmark approach is a useful tool in toxicology. The benchmark dose (BMD) is defined as the dose of an environmental toxicant that corresponds to a prescribed change in response compared with the background response level. For toxicological data from the current study, let y_i denote the number of the adverse responses out of m_i animals tested at dose level x_i for $i = 1, \ldots, n$. Let $\boldsymbol{y} = (y_1, \ldots, y_n)'$, $\boldsymbol{n} = (m_1, \ldots, m_n)'$, $\boldsymbol{x}_i = (1, x_i)'$, and $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)'$. We further let $D = \{\boldsymbol{y}, \boldsymbol{X}, \boldsymbol{n}\}$ denote the data from the current study. A binomial regression model is assumed for y_i with a probability mass function (pmf) given by

$$\begin{aligned} f(y_i|\boldsymbol{\beta}, \boldsymbol{x}_i, m_i) &= \binom{m_i}{y_i} p_i^{y_i}(1-p_i)^{m_i-y_i}, \\ \text{logit}(p_i) &= \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i, \end{aligned} \tag{1}$$

where $0 < p_i < 1$ is the adverse response rate at dose level x_i for $i = 1, \ldots, n$ and $\boldsymbol{\beta} = (\beta_0, \beta_1)'$.

Similarly, the historical toxicological data comprise n_0 adverse binomial responses (y_{0i}, m_{0i}, x_{0i}) tested at dose level x_{0i} for $i = 1, \ldots, n_0$. Let $\boldsymbol{y}_0 = (y_{01}, \ldots, y_{0n_0})$, $\boldsymbol{n}_0 = (m_{01}, \ldots, m_{0n_0})'$, $\boldsymbol{x}_{0i} = (1, x_{0i})'$, and $\boldsymbol{X}_0 = (\boldsymbol{x}_{01}, \ldots, \boldsymbol{x}_{0n})'$. We further let $D_0 = \{\boldsymbol{y}_0, \boldsymbol{X}_0, \boldsymbol{n}_0\}$ denote the data from the historical study. Again, a binomial regression model is assumed for y_{0i} with a probability mass function given by

$$\begin{aligned} f(y_{0i}|\boldsymbol{\beta}, \boldsymbol{x}_{0i}, m_{0i}) &= \binom{m_{0i}}{y_{0i}} p_{0i}^{y_{0i}}(1-p_{0i})^{m_{0i}-y_{0i}}, \\ \text{logit}(p_{0i}) &= \beta_0 + \beta_1 x_{0i}, \end{aligned} \tag{2}$$

where $0 < p_{0i} < 1$ is the adverse response rate at dose level x_{0i} for $i = 1, \ldots, n_0$.

The Kociba study (Kociba et al., 1978) is a lifetime feeding study of both female and male Sprague Dawley rats, with 50 rats tested in each group at doses of 0, 1, 10, and 100 ng/kg/day. Inferences derived from the Kociba study have been widely used as the basis for risk assessments for 2,3,7,8-tetrachlorodibenzodioxin (TCDD). The NTP study (National Toxicology Program, 1982) comprises groups of 50 male rats, 50 female rats, and 50 male mice that received TCDD as a suspension in 9:1 corn oil-acetone by gavage twice each week to achieve doses of 0, 10, 50, or 500 ng/kg/week over two years. In this analysis, the data were from liver tumor (neoplastic

Table 1: Kociba and NTP data.

	Kociba			NTP		
i	y_{0i}	m_{0i}	x_{0i}	y_i	m_i	x_i
1	9	86	0	5	75	0
2	3	50	1	1	49	1.4
3	18	50	10	3	50	7.1
4	34	48	100	12	49	71

Table 2: Maximum Likelihood Estimates of $\boldsymbol{\beta}$ for the Kociba and NTP data.

	Kociba			NTP		
Parameter	Estimate	SE	p-value	Estimate	SE	p-value
β_0	-1.768	0.209	< 0.001	-2.973	0.355	< 0.001
β_1	0.027	0.004	< 0.001	0.026	0.007	< 0.001

nodule) incidences of female rats from both studies. Similar to Shao and Small (2011), Shi et al. (2021), and Shi et al. (2022), we treat the Kociba data as the historical data and the NTP data as the current data, which are shown in Table 1.

We fit the binomial regression models in (1) and (2) to the NTP and Kociba data, respectively. The maximum likelihood estimates (MLE), standard errors (SE), and p-values of $\boldsymbol{\beta}$ are given in Table 2. From this table, we see that the MLEs ($\hat{\beta}_0$'s) of β_0 are -1.768 and -2.973, respectively, for the Kociba and NTP data while the MLEs ($\hat{\beta}_1$'s) of β_1 are 0.027 and 0.026, respectively, for the Kociba and NTP data. Thus, the MLEs of β_0 (intercept) are quite different while the MLEs of β_1 (slope) are quite similar.

2.2 Alzheimer's Disease Neuroimaging Initiative (ADNI) Study

The following is quoted from https://adni.loni.usc.edu/wp-content/uploads/how_to_apply/ADNI_Manuscript_Citations.pdf:

> "Data used in the preparation of this article were obtained from the Alzheimer's Disease Neuroimaging Initiative (ADNI) database (adni.loni.usc.edu). The ADNI was launched in 2003 as a public-private partnership, led by Principal Investigator Michael W. Weiner, MD. The primary goal of ADNI has been to test whether serial magnetic resonance imaging (MRI), positron emission tomography (PET), other biological markers, and clinical and neuropsychological assessment can be combined to measure the progression of mild cognitive impairment (MCI) and early Alzheimer's disease (AD)."

The ADNI database comprises data from approximately 1800 participants aged 55 or above. The cohort in the initial five-year study starting in 2004 is called ADNI–1. The cohorts corresponding to the extended studies beginning in 2009 and 2011 are called ADNI–GO and ADNI–2, respectively. See https://adni.loni.usc.edu/about/ for the more detailed description of these study cohorts. The data include three primary diagnosed disease states: Cognitive Normal (CN), Mild Cognitive Impairment (MCI), and Alzheimer's Disease (AD). Two cognitive measurements, the Alzheimer's Disease Assessment Scale (ADAS) and the Mini-Mental State Exam (MMSE), were assessed over time. ADAS is a rating scale to assess the severity of cognitive and non-cognitive dysfunction from mild to severe AD. A higher ADAS score indicates worse performance. The MMSE score is derived based on the number of correctly completed items, ranging from 0 to 30. A lower MMSE score indicates a poorer cognitive condition.

We consider the changes in ADAS and MMSE at 24 months from the baseline as our response variables. We consider only the CN patients (control group) and the MCI patients (exposure group) at the baseline (bl). The baseline covariates in our analysis include ADAS (ADAS_bl) or MMSE (MMSE_bl), age (in years), sex (coded as 'Female' = 1, 'Male' = 0), race (coded as 'White' = 1, 'Other' = 0), marital status (coded as 'Married' = 1, 'Other' = 0), education (in years), apolipoprotein epsilon 4 (APOE4) allele count, and Rey's Auditory Verbal Learning Test (RAVLT) forgetting percentage. We consider ADNI–GO2 (a combination of ADNI–GO and ADNI–2) as the current study and ADNI–1 as the historical study. Our goal is to assess the (exposure) effect of MCI on the change in ADAS or MMSE at 24 months from the baseline adjusting for the baseline covariates. A summary of the outcome variables and the baseline covariates for ADNI–1 and ADNI–GO2 is shown in Table 3.

In terms of the response variables, the mean and standard deviation of the change in ADAS are −0.09 and 3.17 for CN patients, and 2.87 and 5.78 for MCI patients, respectively, in ADNI–1; and −0.44 and 2.81 for CN patients and 0.81 and 4.41 for MCI patients, respectively, in ADNI–GO2. The mean and standard deviation of the change in MMSE are −0.07 and 1.29 for CN patients and −1.76 and 3.54 for MCI patients, respectively, in ADNI–1; and −0.18 and 1.53

Table 3: Summary of the ADNI data, where each entry shows mean (standard deviation) for a continuous variable and frequency (percentage) for a discrete variable.

Var	Type	ADNI–1 ($n_1 = 505$)	ADNI–GO2 ($n_2 = 531$)
ADAS_bl	continuous	9.16 (4.63)	8.09 (4.11)
MMSE_bl	continuous	27.93 (1.78)	28.39 (1.62)
Gender	Male	301 (59.6%)	286 (53.86%)
	Female	204 (40.4%)	245 (46.14%)
APOE4	0	284 (56.24%)	313 (58.95%)
	1	178 (35.25%)	175 (32.96%)
	2	43 (8.51%)	43 (8.1%)
Race	1	476 (94.26%)	489 (92.09%)
	0	29 (5.74%)	42 (7.91%)
Marital	1	392 (77.62%)	390 (73.45%)
	0	113 (22.38%)	141 (26.55%)
RAVLT	continuous	54.12 (34.24)	49.41 (31.82)
Age	continuous	75.24 (6.40)	71.97 (7.14)
Education	continuous	15.92 (2.86)	16.38 (2.58)
Group	CN	202 (40%)	157 (29.57%)
	MCI	303 (60%)	374 (70.43%)

for CN patients and −0.92 and 2.46 for MCI patients, respectively, in ADNI–GO2.

For the data from the current study, we let y_i denote the change in ADAS or MMSE at 24 months from the baseline, $z_i = 1$ if MCI at baseline, and $z_i = 0$ if CN at baseline. Let $\boldsymbol{x}_i^*$ denote a q-dimensional vector of baseline covariates, including an intercept. The regression model we consider is given by

$$y_i = \gamma z_i + \boldsymbol{\beta}_1' \boldsymbol{x}_i^* + \varepsilon_i, \tag{3}$$

where $\varepsilon_i \sim N(0, \sigma^2)$ independently for $i = 1, \ldots, n$, and $\boldsymbol{\beta}_1$ is a q-dimensional vector of the regression coefficients. Let $\boldsymbol{x}_i = (z_i, (\boldsymbol{x}_i^*)')'$ and $\boldsymbol{\beta} = (\gamma, \boldsymbol{\beta}_1')'$. Then, the probability density function (pdf) of y_i is given by

$$f(y_i|\boldsymbol{x}_i, \boldsymbol{\beta}, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right\}. \tag{4}$$

Let $\boldsymbol{y} = (y_1, \ldots, y_n)'$ and $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)'$. Also let $D = \{\boldsymbol{y}, \boldsymbol{X}, n\}$ denote the current data. Similarly, let y_{0i}, z_{0i}, and $\boldsymbol{x}_{0i}^*$ denote the change in ADAS or MMSE at 24 months from the baseline, the exposure group indicator, and the vector of the baseline covariates, respectively, for the historical data. Similar to (4), the pdf of y_{0i} is given by

$$f(y_{0i}|\boldsymbol{x}_{0i}, \boldsymbol{\beta}, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y_{0i} - \boldsymbol{x}_{0i}'\boldsymbol{\beta})^2\right\}, \tag{5}$$

where $\boldsymbol{x}_{0i} = (z_{0i}, (\boldsymbol{x}_{0i}^*)')'$. Write $D_0 = \{\boldsymbol{y}_0, \boldsymbol{X}_0, n_0\}$ as the historical data, where $\boldsymbol{y}_0 = (y_{01}, \ldots, y_{0n_0})'$ and $\boldsymbol{X}_0 = (\boldsymbol{x}_{01}, \ldots, \boldsymbol{x}_{0n_0})'$.

The ordinary least squares (OLS) estimates of γ, $\boldsymbol{\beta}$, and σ^2 are reported in Table 4 for the ADNI data. In the ADAS model, MCI is very significant in ADNI–1 (< 0.001) but less significant in ADNI–GO2 (0.077). Male is more significant in ADNI–1 (0.009) than in ADNI–GO2 (0.958). Both RAVLT and APOE4 are slightly more significant in ADNI–GO2 ((0.003, 0.044) in ADNI–1 and (< 0.001, 0.001) in ADNI–GO2). The p-values of the baseline ADAS are quite similar (0.075 in ADNI–1 and 0.080 in ADNI–GO2). Marital status, race, and education are not significant in either dataset. In the MMSE model, MCI is still significant in both ADNI–1 (0.001) and ADNI–GO2 (0.001). The intercept is more significant in ADNI–GO2 (< 0.001) than ADNI–1 (0.034), which is much more significant compared to the ADAS model. Age is highly insignificant in ADNI–1 (0.953) but more significant in ADNI–GO2 (0.067). On the contrary, the male covariate is significant in ADNI–1 (0.043) but not significant in ADNI–GO2 (0.827). RAVLT and APOE4 are significant in both datasets, which is the same situation as in the ADAS model. Marital status and race are not significant in both datasets ((0.616, 0.732) in ADNI–1 and (0.562, 0.494) in ADNI–GO2). Finally, education in ADNI–GO2 (0.012) is much more significant than in ADNI–1 (0.500).

3 The Development of Power Priors

3.1 Basic Setting

Let the data from the current study be denoted by $D = (n, \boldsymbol{y}, \boldsymbol{X})$, where n denotes the sample size, $\boldsymbol{y}$ denotes the $n \times 1$ response vector, and $\boldsymbol{X}$ denotes the $n \times p$ matrix of covariates. Denote the likelihood for the current study by $L(\boldsymbol{\theta}|D)$, where $\boldsymbol{\theta}$ is the vector of model parameters. Thus, $L(\boldsymbol{\theta}|D)$ can be a general likelihood function for an arbitrary regression model, such as a generalized linear model, a random effects model, a nonlinear model, or a survival model with

Table 4: OLS Estimates of Regression Coefficients and Error Variances for the ADNI data.

	ADAS			MMSE		
Variable	Estimate	SE	p-value	Estimate	SE	p-value
ADNI–1						
Intercept	−2.000	3.119	0.522	6.570	3.095	0.034
MCI	2.756	0.547	< 0.001	−1.095	0.327	0.001
Age	0.019	0.035	0.581	−0.001	0.020	0.953
Male	−1.247	0.476	0.009	0.550	0.271	0.043
RAVLT	0.023	0.008	0.003	−0.024	0.004	< 0.001
APOE4	0.715	0.354	0.044	−0.725	0.200	< 0.001
ADAS/MMSE_bl	−0.109	0.061	0.075	−0.185	0.087	0.034
Marital	−0.811	0.571	0.157	−0.163	0.326	0.616
Race	1.648	0.934	0.078	0.184	0.536	0.732
Education	−0.012	0.077	0.879	−0.030	0.044	0.500
$\hat{\sigma}$	4.809			2.743		
ADNI–GO2						
Intercept	−3.455	2.305	0.135	13.685	2.237	< 0.001
MCI	0.732	0.413	0.077	−0.699	0.214	0.001
Age	0.019	0.025	0.459	−0.025	0.013	0.067
Male	0.019	0.364	0.958	0.042	0.193	0.827
RAVLT	0.030	0.006	< 0.001	−0.021	0.003	< 0.001
APOE4	0.886	0.276	0.001	−0.340	0.147	0.022
ADAS/MMSE_bl	−0.090	0.051	0.080	−0.443	0.061	< 0.001
Marital Status	0.584	0.398	0.143	−0.123	0.212	0.562
Race	0.692	0.627	0.270	0.228	0.334	0.494
Education	−0.016	0.067	0.815	0.092	0.036	0.012
$\hat{\sigma}$	3.871			2.061		

right censored data. Denote the historical or external data by $D_0 = (n_0, \boldsymbol{y}_0, \boldsymbol{X}_0)$. Let $\pi_0(\boldsymbol{\theta})$ denote the prior distribution for $\boldsymbol{\theta}$ before the historical data D_0 are observed. Note that $\pi_0(\boldsymbol{\theta})$ is called the initial prior distribution for $\boldsymbol{\theta}$, which is typically taken to be improper. Given the power a_0, the power prior (Ibrahim and Chen, 2000a) of $\boldsymbol{\theta}$ for the current study is defined as

$$\pi(\boldsymbol{\theta}|D_0, a_0) \propto L(\boldsymbol{\theta}|D_0)^{a_0}\pi_0(\boldsymbol{\theta}), \tag{6}$$

where a_0 is a scalar prior parameter that weights the historical data relative to the likelihood of the current study. In (6), a_0 controls the influence of the historical data on $\pi(\boldsymbol{\theta}|D_0, a_0)$ and a_0 can be interpreted as a discounting parameter, a precision parameter, and a parameter which reflects the heterogeneity (compatibility) between current and external data. It is reasonable to restrict the range of a_0 to be between 0 and 1, and thus we take $0 \leqslant a_0 \leqslant 1$ unless otherwise mentioned. Mathematically, a_0 controls the heaviness of the tails of the prior for $\boldsymbol{\theta}$. As a_0 becomes smaller, the tails of $\pi(\boldsymbol{\theta}|D_0, a_0)$ become heavier. Using (6), the resulting posterior distribution is given by

$$\pi(\boldsymbol{\theta}|D, D_0, a_0) \propto L(\boldsymbol{\theta}|D)L(\boldsymbol{\theta}|D_0)^{a_0}\pi_0(\boldsymbol{\theta}).$$

Assuming that a_0 is random, the normalized power prior (Duan et al., 2006; Neuenschwander et al., 2009) refers to a joint prior of $\boldsymbol{\theta}$ and a_0, which is defined as a product of the conditional density of $\boldsymbol{\theta}$ given the historical data D_0 and a_0, and a marginal density of a_0. Specifically, this joint prior is given by

$$\pi(\boldsymbol{\theta}, a_0|D_0) = \pi(\boldsymbol{\theta}|D_0, a_0)\pi_0(a_0) \equiv \frac{L(\boldsymbol{\theta}|D_0)^{a_0}\pi_0(\boldsymbol{\theta})}{\int L(\boldsymbol{\theta}|D_0)^{a_0}\pi_0(\boldsymbol{\theta})d\boldsymbol{\theta}}\pi_0(a_0), \tag{7}$$

where $\pi_0(\boldsymbol{\theta})$ is an initial prior for $\boldsymbol{\theta}$ and $\pi_0(a_0)$ is a marginal prior for a_0. For the normalized power prior, we must have

$$\int L(\boldsymbol{\theta}|D_0)^{a_0}\ \pi_0(\boldsymbol{\theta})d\boldsymbol{\theta} < \infty, \quad 0 < a_0 \leqslant 1.$$

Theoretical properties were examined for the power prior as defined in (6) (Ibrahim et al., 2003), as well as for the normalized power prior (Carvalho and Ibrahim, 2021; Ye et al., 2022; Pawel et al., 2023a). Computational algorithms for the normalized power prior were developed in Carvalho and Ibrahim (2021) and Han et al. (2023b).

3.2 Extension to Multiple Historical Datasets

The power prior in (6) can also be extended to leverage multiple historical datasets when available. Assume we have K historical datasets, denoted by D_{0k}, $k = 1, \ldots, K$. Write $\boldsymbol{D}_0 = (D_{01}, \ldots, D_{0K})$. Following Ibrahim and Chen (2000a) and Ibrahim et al. (2015a), we have

$$\pi(\boldsymbol{\theta}|\boldsymbol{D}_0, \boldsymbol{a}_0) \propto \prod_{k=1}^{K} L(\boldsymbol{\theta}|D_{0k})^{a_{0k}}\pi_0(\boldsymbol{\theta}), \tag{8}$$

where $\pi_0(\boldsymbol{\theta})$ is the initial prior for $\boldsymbol{\theta}$, $\boldsymbol{a}_0 = (a_{01}, \ldots, a_{0K})$, $0 \leqslant a_{0k} \leqslant 1$ for $k = 1, \ldots, K$, and $\sum_{k=1}^{K} a_{0k} \leqslant 1$. The power prior for multiple historical datasets has been extensively discussed in literature and used in several applications, including those cited in Ibrahim et al. (2015a), Gravestock and Held (2019), and Yuan et al. (2022a).

When $\boldsymbol{a}_0$ is random, the normalized power prior in (7) can be extended to

$$\pi(\boldsymbol{\theta}|\boldsymbol{D}_0, \boldsymbol{a}_0) \propto \frac{\prod_{k=1}^{K} L(\boldsymbol{\theta}|D_{0k})^{a_{0k}}\pi_0(\boldsymbol{\theta})}{\int \prod_{k=1}^{K} L(\boldsymbol{\theta}|D_{0k})^{a_{0k}}\pi_0(\boldsymbol{\theta})d\boldsymbol{\theta}}\pi_0(\boldsymbol{a}_0), \tag{9}$$

where $\pi_0(\boldsymbol{\theta})$ is an initial prior for $\boldsymbol{\theta}$ and $\pi_0(\boldsymbol{a}_0)$ is a marginal prior for $\boldsymbol{a}_0$. A beta prior is typically assumed for $\pi_0(a_0)$ in (7), while a Dirichlet prior is specified for $\pi_0(\boldsymbol{a}_0)$ in (9). The normalized power prior for multiple historical datasets with binary endpoints was considered in Banbeta et al. (2019) and Gravestock and Held (2019).

Again, the theoretical properties of the power prior in (8) and the normalized power prior in (9) were examined in Ibrahim et al. (2003) and Ye et al. (2022), respectively.

3.3 Power Prior for the Binomial Regression Model

For the Kociba and NTP studies, using (2), the likelihood function given the historical data $\boldsymbol{D}_0$ is given by

$$L(\boldsymbol{\beta}|D_0) = \prod_{i=1}^{n_0} \binom{m_{0i}}{y_{0i}} \frac{\exp\{y_{0i}(\beta_0 + \beta_1 x_{0i})\}}{\{1 + \exp(\beta_0 + \beta_1 x_{0i})\}^{m_{0i}}}$$

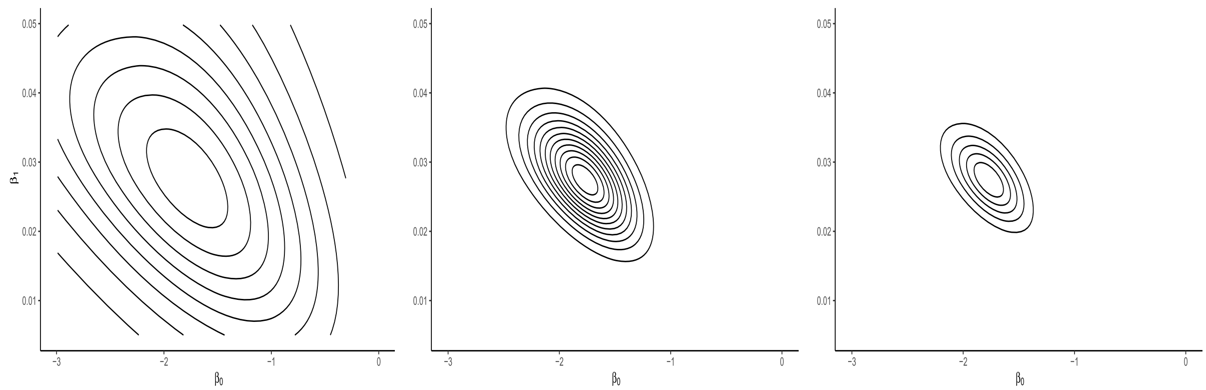

Figure 1: Contours of the Power Prior for $a_0 = 0.05$, 0.5, 0.9.

and the power prior with an improper uniform initial prior, i.e., $\pi_0(\boldsymbol{\beta}) \propto 1$, is thus given by

$$\pi(\boldsymbol{\beta}|D_0, a_0) \propto \prod_{i=1}^{n_0} \Big[\frac{\exp\{y_{0i}(\beta_0 + \beta_1 x_{0i})\}}{\{1 + \exp(\beta_0 + \beta_1 x_{0i})\}^{m_{0i}}} \Big]^{a_0}. \tag{10}$$

Using the data given in Table 1, the contours of the power prior $\pi(\boldsymbol{\beta}|D_0, a_0)$ in (10) for $a_0 = 0.05$, 0.5, and 0.9 are plotted in Figure 1. From this figure we see that (i) the centers, namely the modes, of the power prior remain the same for different a_0 values; (ii) the tails of the power priors become heavier as a_0 becomes larger; and (iii) the prior surfaces become flatter as a_0 becomes smaller.

Under the binomial regression model, the normalized power prior in (7) reduces to

$$\pi(\boldsymbol{\beta}, a_0|D_0) \propto \frac{\pi_0(\beta_0, \beta_1) \prod_{i=1}^{n_0} \left[\frac{\exp\{y_{0i}(\beta_0+\beta_1 x_{0i})\}}{\{1+\exp(\beta_0+\beta_1 x_{0i})\}^{m_{0i}}} \right]^{a_0} \pi_0(a_0)}{\int\int \pi_0(\beta_0^*, \beta_1^*) \left[\frac{\exp\{y_{0i}(\beta_0^*+\beta_1^* x_{0i})\}}{\{1+\exp(\beta_0^*+\beta_1^* x_{0i})\}^{m_{0i}}} \right]^{a_0} d\beta_0^* d\beta_1^*}, \tag{11}$$

where $\pi_0(\beta_0, \beta_1)$ is the initial prior of $\boldsymbol{\beta}$ and $\pi_0(a_0)$ is the marginal prior of a_0.

3.4 Power Prior for the Normal Linear Regression Model

For the ADNI study, using (5), the likelihood function given the historical data D_0 from ADNI–1 is given by

$$L(\boldsymbol{\beta}, \sigma^2|D_0) \propto \frac{1}{(\sigma^2)^{\frac{n_0}{2}}} \exp\Big\{ -\frac{1}{2\sigma^2}(\boldsymbol{y}_0 - \boldsymbol{X}_0\boldsymbol{\beta})'(\boldsymbol{y}_0 - \boldsymbol{X}_0\boldsymbol{\beta}) \Big\}$$

$$\overset{\substack{\text{ANOVA}\\ \text{Decomposition}}}{=} \frac{1}{(\sigma^2)^{\frac{p}{2}}} \exp\Big\{ -\frac{1}{2\sigma^2}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)'\boldsymbol{X}_0'\boldsymbol{X}_0(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)' \Big\} \times \frac{1}{(\sigma^2)^{\frac{n_0-p}{2}}} \exp\Big\{ -\frac{\text{SSE}_0}{2\sigma^2} \Big\}, \tag{12}$$

where $p = q + 1$, $\hat{\boldsymbol{\beta}}_0 = (\boldsymbol{X}_0'\boldsymbol{X}_0)^{-1}\boldsymbol{X}_0'\boldsymbol{y}_0$ is the least squares estimate of $\boldsymbol{\beta}$ based on D_0, and $\text{SSE}_0 = (\boldsymbol{y}_0 - \boldsymbol{X}_0\hat{\boldsymbol{\beta}}_0)'(\boldsymbol{y}_0 - \boldsymbol{X}_0\hat{\boldsymbol{\beta}}_0)$. Assume that the initial prior $\pi_0(\boldsymbol{\beta}, \sigma^2) \propto 1/\sigma^2$. Conditional on σ^2, the power prior for $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta}|\sigma^2, D_0, a_0 \sim N\Big(\hat{\boldsymbol{\beta}}_0, \frac{\sigma^2}{a_0}(\boldsymbol{X}_0'\boldsymbol{X}_0)^{-1}\Big). \tag{13}$$

From (13), we see that the smaller a_0 is, the larger the prior variance becomes. Thus, in this sense, a_0 can also be viewed as a precision parameter.

4 Variations of Power Priors

4.1 Partial Borrowing Power Priors

Ibrahim et al. (2012b), Chen et al. (2014b), and Ibrahim et al. (2015a) introduced the partial borrowing power prior. The idea of the partial borrowing power prior is to borrow information from the historical data only for parameters shared by both the current and historical data.

To illustrate this prior, we first consider the binomial regression models in (1) and (2). For the Kociba and NTP study, as discussed in Section 2.1, The $\hat{\beta}_0$'s (intercept) are quite different while the $\hat{\beta}_1$'s (slope) are quite similar. Thus, when analyzing the NTP data, it is reasonable to limit our borrowing of the Kociba data to information regarding the slope. To this end, we extend the power prior in (10) to define the partial borrowing power prior as

$$\pi(\boldsymbol{\beta}|D_0, a_0) \propto \int \prod_{i=1}^{n_0} \Big[\frac{\exp\{y_{0i}(\beta_{0h} + \beta_1 x_{0i})\}}{\{1+\exp(\beta_{0h} + \beta_1 x_{0i})\}^{m_{0i}}} \Big]^{a_0} \pi_0(\beta_{0h}) d\beta_{0h} \pi_0(\beta_0, \beta_1), \tag{14}$$

where $\pi_0(\beta_{0h})$ is an initial prior for β_{0h} (intercept) for fitting the Kociba data only and $\pi_0(\beta_0, \beta_1)$ is an initial prior for β_0 and β_1. Under this partial borrowing power prior, we essentially assume that the historical and current data share a common slope (β_1) but have different intercepts (β_{0h} and β_0). Therefore, we integrate out β_{0h}, which is the intercept for the historical data in (14). Shi et al. (2021) and Han et al. (2023b) introduced the partial borrowing normalized power prior. Specifically, when a_0 is random, the partial borrowing power prior in (14) can be extended to the partial borrowing normalized power prior given by

$$\pi(\boldsymbol{\beta}, a_0|D_0) \propto \frac{\int \pi_0(\beta_{0h}) \pi_0(\beta_0, \beta_1) \prod_{i=1}^{n_0} \left[\frac{\exp\{y_{0i}(\beta_{0h}+\beta_1 x_{0i})\}}{\{1+\exp(\beta_{0h}+\beta_1 x_{0i})\}^{m_{0i}}} \right]^{a_0} d\beta_{0h} \pi_0(a_0)}{\iiint \pi_0(\beta_{0h}) \pi_0(\beta_0^*, \beta_1^*) \prod_{i=1}^{n_0} \left[\frac{\exp\{y_{0i}(\beta_{0h}+\beta_1^* x_{0i})\}}{\{1+\exp(\beta_{0h}+\beta_1^* x_{0i})\}^{m_{0i}}} \right]^{a_0} d\beta_{0h} d\beta_0^* d\beta_1^*}, \tag{15}$$

where $\pi_0(\beta_{0h})$, $\pi_0(\beta_0, \beta_1)$, and $\pi_0(a_0)$ are the initial priors. Note that the prior in (15) is properly defined if $\int \pi_0(\beta_0, \beta_1) d\beta_0 < \infty$. We further note that the denominator in (15) involves a three-dimensional integral instead of a two-dimensional integral in (11) since an additional parameter β_{0h} needs to be integrated out due to the partial borrowing.

For the ADNI data, we see from Table 4 that for the response variable ADAS, $\hat{\sigma} = 4.809$ for ADNI–1 and $\hat{\sigma} = 3.871$ for ADNI–GO2. Thus, we may not borrow the information about the variance from the ADNI–1 data while analyzing the ADNI–GO2 data. Thus, if we assume that the current data $\boldsymbol{D}$ and the historical data $\boldsymbol{D}_0$ share the common regression coefficients $\boldsymbol{\beta}$ only while the variances are different, which are denoted by σ^2 and σ_h^2. Then, the power prior in (13) can be extended as the partial borrowing power prior given by

$$\begin{aligned} \pi(\boldsymbol{\beta}, \sigma^2|D_0, a_0) \propto \int \Bigg[\frac{1}{(\sigma_h^2)^{\frac{p}{2}}} \exp\Bigg\{ -\frac{1}{2\sigma_h^2} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)' \boldsymbol{X}_0' \boldsymbol{X}_0 (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0) \Bigg\} \\ \times \frac{1}{(\sigma_h^2)^{\frac{n_0-p}{2}}} \exp\Bigg\{ -\frac{\mathrm{SSE}_0}{2\sigma_h^2} \Bigg\} \Bigg]^{a_0} \pi_0(\sigma_h^2) d\sigma_h^2 \pi_0(\boldsymbol{\beta}, \sigma^2), \end{aligned} \tag{16}$$

where $\pi_0(\sigma_h^2)$ and $\pi_0(\boldsymbol{\beta}, \sigma^2)$ are the initial priors. When $\pi_0(\sigma_h^2) \propto 1/\sigma_h^2$, (16) reduces to

$$\pi(\boldsymbol{\beta}, \sigma^2 | D_0, a_0) \propto \left\{ \mathrm{SSE}_0 + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)' \boldsymbol{X}_0' \boldsymbol{X}_0 (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0) \right\}^{-\frac{a_0 n_0}{2}} \pi_0(\boldsymbol{\beta}, \sigma^2). \tag{17}$$

When $\pi_0(\boldsymbol{\beta}, \sigma^2) \propto \pi_0(\sigma^2)$, the marginal partial borrowing power prior for $\boldsymbol{\beta}$ is a p-dimensional multivariate t-distribution with $a_0 n_0 - p$ degrees of freedom. In this special case, when $\pi_0(\sigma^2)$ is a proper density and a_0 is random, the partial borrowing normalized power prior is given by

$$\pi(\boldsymbol{\beta}, \sigma^2, a_0 | D_0) = \frac{\Gamma(a_0 n_0/2) \det(\boldsymbol{X}_0' \boldsymbol{X}_0)^{1/2}}{\Gamma[(a_0 n_0 - p)/2] \pi^{p/2} \mathrm{SSE}_0^{p/2}} \times \left\{ 1 + \frac{1}{a_0 n_0 - p} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)' \left\{ \frac{(a_0 n_0 - p)}{\mathrm{SSE}_0} \boldsymbol{X}_0' \boldsymbol{X}_0 \right\} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0) \right\}^{-\frac{a_0 n_0}{2}} \pi_0(\sigma^2) \pi_0(a_0), \tag{18}$$

where $\pi_0(a_0)$ is a proper initial prior. Note that the prior in (17) is proper if $a_0 n_0 - p > 0$ when $\pi_0(\boldsymbol{\beta}, \sigma^2) \propto \pi_0(\sigma^2)$ and furthermore, the prior in (18) is properly defined if $\pi_0(a_0)$ has a support $\{a_0 : a_0 > p/n_0\}$. Thus, a_0 cannot be too small. A similar phenomenon was observed in Han et al. (2023a).

4.2 Borrowing-by-Parts Power Priors

Yuan et al. (2022b) first introduced this variation of the power prior. The main idea is to borrow information for different parts of the parameters separately, with each part having its own discounting parameter.

The borrowing-by-parts power prior harmonizes well with conditional inference. The exact conditional logistic regression was originally proposed by Cox (1970) and the computational methods were discussed in Hirji et al. (1987), Hirji (1992), Mehta et al. (1992), and Corcoran et al. (2005). Using the asymptotic normality of large samples, the approximate conditional logistic regression was discussed in Breslow and Day (1980), Lachin (2000), and Stokes et al. (2000). For the binomial logistic regression model (10), let $T_0 = \sum_{i=1}^{n_0} Y_{0i}$ and $T_1 = \sum_{i=1}^{n_0} x_{0i} Y_{0i}$, where Y_{0i} is the underlying random variable with realized value y_{0i}. Also, let $t_0 = \sum_{i=1}^{n_0} y_{0i}$ and $t_1 = \sum_{i=1}^{n_0} x_{0i} y_{0i}$ denote the observed values of T_0 and T_1. It is clear that (T_0, T_1) are the (minimal) sufficient statistics for (β_0, β_1), and T_0 is a Poisson-binomial random variable. Given an observed value $t_0 \in \{0, 1, \ldots, \sum_{i=1}^{n_0} m_{0i}\}$, we have

$$P(T_0 = t_0 | \beta_0, \beta_1) = \sum_{\boldsymbol{y}_0^* \in \mathcal{S}_{t_0}} \left[\prod_{i=1}^{n_0} \frac{\binom{m_{0i}}{y_{0i}^*}}{\{1 + \exp(\beta_0 + \beta_1 x_{0i})\}^{m_{0i}}} \right] \exp\left(\beta_0 t_0 + \beta_1 \sum_{i=1}^{n_0} x_{0i} y_{0i}^* \right),$$

where

$$\mathcal{S}_{t_0} = \left\{ \boldsymbol{y}_0^* = (y_{01}^*, \ldots, y_{0n_0}^*) : \sum_{i=1}^{n_0} y_{0i}^* = t_0 \text{ and } y_{0i}^* \in \{0, 1, \ldots, m_{0i}\}, i = 1, \ldots, n_0 \right\}.$$

By conditioning on $T_0 = t_0$, the conditional probability of $T_1 = t_1$ is free from β_0, giving the conditional likelihood of β_1:

$$P(T_1 = t_1 | T_0 = t_0, \beta_1) \propto \frac{\exp(\beta_1 t_1)}{\sum_{\boldsymbol{y}_0^* \in \mathcal{S}_{t_0}} \left[\prod_{i=1}^{n_0} \binom{m_{0i}}{y_{0i}^*} \right] \exp\left(\beta_1 \sum_{i=1}^{n_0} x_{0i} y_{0i}^* \right)}. \tag{19}$$

The borrowing-by-parts power prior focusing on β_1 is then given by

$$\begin{aligned}\pi(\boldsymbol{\beta}|D_0, a_{01}, a_{02}) &\propto P(T_1 = t_1|T_0 = t_0, \beta_1)^{a_{01}} P(T_0 = t_0|\beta_0, \beta_1)^{a_{02}} \pi_0(\boldsymbol{\beta}) \\ &\propto \left[\frac{\exp(\beta_1 t_1)}{\sum_{y_0^* \in \mathcal{S}_{t_0}} \left[\prod_{i=1}^{n_0} \binom{m_{0i}}{y_{0i}^*} \right] \exp\left(\beta_1 \sum_{i=1}^{n_0} x_{0i} y_{0i}^* \right)} \right]^{a_{01}} \\ &\times \left[\sum_{y_0^* \in \mathcal{S}_{t_0}} \left[\prod_{i=1}^{n_0} \frac{\binom{m_{0i}}{y_{0i}^*}}{\{1 + \exp(\beta_0 + \beta_1 x_{0i})\}^{m_{0i}}} \right] \exp\left(\beta_0 t_0 + \beta_1 \sum_{i=1}^{n_0} x_{0i} y_{0i}^* \right) \right]^{a_{02}} \\ &\times \pi_0(\boldsymbol{\beta}), \end{aligned} \tag{20}$$

where $0 \leqslant a_{01} \leqslant 1$ is a discounting parameter for β_1, $0 \leqslant a_{02} \leqslant 1$ is another discounting parameter for both β_0 and β_1, and $\pi_0(\boldsymbol{\beta})$ is an initial prior. In (20), when $a_{02} = 0$, we borrow the historical information for β_1 alone. If we wish to borrow the historical information focusing on β_0, we need to use an alternative formulation via

$$\pi(\boldsymbol{\beta}|D_0, a_{01}, a_{02}) \propto P(T_0 = t_0|T_1 = t_1, \beta_0)^{a_{01}} P(T_1 = t_1|\beta_0, \beta_1)^{a_{02}} \pi_0(\boldsymbol{\beta}).$$

For the normal linear regression model, using (12), the borrowing-by-parts power prior is given by

$$\begin{aligned}\pi(\boldsymbol{\beta}, \sigma^2|D_0, a_{01}, a_{02}) \propto &\left[\frac{1}{(\sigma^2)^{\frac{p}{2}}} \exp\{ -\frac{1}{2\sigma^2} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)' \boldsymbol{X}_0' \boldsymbol{X}_0 (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0) \} \right]^{a_{01}} \\ &\times \left[\frac{1}{(\sigma^2)^{\frac{n_0 - p}{2}}} \exp\left\{ -\frac{\mathrm{SSE}_0}{2\sigma^2} \right\} \right]^{a_{02}} \pi_0(\boldsymbol{\beta}, \sigma^2),\end{aligned} \tag{21}$$

where $0 \leqslant a_{01} \leqslant 1$ and $0 \leqslant a_{02} \leqslant 1$ are the discounting parameters for "mean" $\hat{\boldsymbol{\beta}}_0$ and "sample variance" SSE_0, and $\pi_0(\boldsymbol{\beta}, \sigma^2)$ is an initial prior. Yuan et al. (2022b) further developed the conditional borrowing-by-parts power prior approach to leverage multiple historical datasets in Bayesian design of superiority trials. In their recent study, Baron et al. (2024) tailored a stratified borrowing-by-parts power prior approach for incorporating data from an external control arm into a current randomized controlled trial. They proposed minimal plausibility indexes to determine the discounting parameters.

4.3 Partial Borrowing-by-Parts Power Priors

Sheikh et al. (2022) first introduced this variation of the power prior. For the ADNI data, in (3), suppose that we assume that ADNI–1 and ADNI–GO2 share the common coefficient γ for the exposure effect but not the regression coefficients $\boldsymbol{\beta}_1$ for adjusting covariates. In this regard, we assume $\boldsymbol{\beta}_{1h}$ for ADNI–1 and $\boldsymbol{\beta}_1$ for ADNI–GO2. A partial borrowing-by-parts power prior is a generic integration of the partial borrowing power prior and the borrowing-by-parts power prior so that it allows us to borrow the historical information for different parts of the parameters separately as well as to borrow the historical information only for the parameters shared by both the current and historical data.

For the ADNI data, let $\boldsymbol{z}_0 = (z_{01}, \ldots, z_{0n_0})'$ and $\boldsymbol{X}_0^* = (\boldsymbol{x}_{01}^*, \ldots, \boldsymbol{x}_{0n_0}^*)'$. Then, we have $\boldsymbol{X}_0 = (\boldsymbol{z}_0, \boldsymbol{X}_0^*)$. Note that $\hat{\boldsymbol{\beta}}_0 = (\boldsymbol{X}_0'\boldsymbol{X}_0)^{-1}\boldsymbol{X}_0'\boldsymbol{y}_0$. After some algebra, we can show that

$$\hat{\boldsymbol{\beta}}_0 = (\hat{\gamma}_0, (\hat{\boldsymbol{\beta}}_{10})')',$$

where

$$\hat{\gamma}_0 = \frac{z_0'(\boldsymbol{I}_{n_0} - \boldsymbol{H}_0^*)\boldsymbol{y}_0}{z_0'(\boldsymbol{I}_{n_0} - \boldsymbol{H}_0^*)z_0},$$
$$\hat{\boldsymbol{\beta}}_{10} = (\boldsymbol{X}_0^{*\prime}\boldsymbol{X}_0^*)^{-1}\boldsymbol{X}_0^{*\prime}\left\{\boldsymbol{I}_{n_0} - \frac{z_0 z_0'(\boldsymbol{I}_{n_0} - \boldsymbol{H}_0^*)}{z_0'(\boldsymbol{I}_{n_0} - \boldsymbol{H}_0^*)z_0}\right\}\boldsymbol{y}_0,$$

$\boldsymbol{I}_{n_0}$ is the $n_0 \times n_0$ identity matrix, and $\boldsymbol{H}_0^* = \boldsymbol{X}_0^*(\boldsymbol{X}_0^{*\prime}\boldsymbol{X}_0^*)^{-1}\boldsymbol{X}_0^{*\prime}$ is the orthogonal projection matrix onto the column space of $\boldsymbol{X}_0^*$. Then, the partial borrowing-by-parts power prior is given by

$$\begin{aligned}\pi(\boldsymbol{\beta}, \sigma^2 | D_0, a_{01}, a_{02})& \\ \propto \int \Bigg[\frac{1}{(\sigma^2)^{\frac{p}{2}}} \exp\Bigg\{ &- \frac{1}{2\sigma^2}\begin{pmatrix}\gamma - \hat{\gamma}_0 \\ \boldsymbol{\beta}_{1h} - \hat{\boldsymbol{\beta}}_{10}\end{pmatrix}' \begin{pmatrix} z_0' z_0 & z_0' \boldsymbol{X}_0^* \\ \boldsymbol{X}_0^{*\prime} z_0 & \boldsymbol{X}_0^{*\prime}\boldsymbol{X}_0^*\end{pmatrix}\begin{pmatrix}\gamma - \hat{\gamma}_0 \\ \boldsymbol{\beta}_{1h} - \hat{\boldsymbol{\beta}}_{10}\end{pmatrix}\Bigg\}\Bigg]^{a_{01}} \pi_0(\boldsymbol{\beta}_{1h}) d\boldsymbol{\beta}_{1h} \\ \times \Bigg[\frac{1}{(\sigma^2)^{\frac{n_0-p}{2}}} \exp\Bigg\{&-\frac{\text{SSE}_0}{2\sigma^2}\Bigg\}\Bigg]^{a_{02}} \pi_0(\gamma, \boldsymbol{\beta}_1, \sigma^2), \end{aligned} \tag{22}$$

where $\boldsymbol{\beta}_{1h}$ is a q-dimensional vector of the regression coefficients, and $\pi_0(\boldsymbol{\beta}_{1h})$ and $\pi_0(\gamma, \boldsymbol{\beta}_1, \sigma^2)$ are the initial priors. The partial borrowing-by-parts power prior facilitates borrowing information about the common regression coefficient γ primarily from $\hat{\gamma}_0$. When $a_{01} > 0$ and $\pi_0(\boldsymbol{\beta}_{1h}) \propto 1$, (22) reduces to

$$\begin{aligned}\pi(\boldsymbol{\beta}, \sigma^2 | D_0, a_{01}, a_{02}) \propto & \frac{1}{(\sigma^2)^{\frac{(a_{01}-1)p+1}{2}}} \exp\left\{-\frac{a_{01} z_0'(\boldsymbol{I}_{n_0} - \boldsymbol{H}_0^*)z_0}{2\sigma^2}(\gamma - \hat{\gamma}_0)^2\right\} \\ & \times \left[\frac{1}{(\sigma^2)^{\frac{n_0-p}{2}}} \exp\left\{-\frac{\text{SSE}_0}{2\sigma^2}\right\}\right]^{a_{02}} \pi_0(\gamma, \boldsymbol{\beta}_1, \sigma^2).\end{aligned} \tag{23}$$

4.4 Propensity Score based Power Priors

The propensity score (PS) is a cornerstone in observational studies, especially when randomization is limited or unattainable (Rosenbaum and Rubin, 1983). Yue (2007) was among the first to introduce the propensity score from a regulatory perspective. Building on this foundation, Wang et al. (2019) developed a PS-integrated power prior that leverages external information for single-arm clinical studies. Their method involves stratifying subjects into K strata based on quantiles of estimated propensity scores, with each stratum assigned a corresponding power prior. Within each stratum, the discounting parameter is tailored according to the similarity in propensity score distributions between the current and external subjects. More recently, Wang et al. (2022) introduced the PS-based subject-specific power prior, which we discuss in further detail below.

Suppose for each historical subject i there exists a subject-specific power parameter $a_{0i} \geqslant 0$. For the binomial regression model (2), this prior is given by

$$\pi(\boldsymbol{\beta} | D_0, a_{01}, \ldots, a_{0n_0}) \propto \prod_{i=1}^{n_0}\left[\frac{\exp\{y_{0i}(\beta_0 + \beta_1 x_{0i})\}}{\{1 + \exp(\beta_0 + \beta_1 x_{0i})\}^{m_{0i}}}\right]^{a_{0i}} \pi_0(\boldsymbol{\beta}, \sigma^2), \tag{24}$$

where $\pi_0(\boldsymbol{\beta}, \sigma^2)$ is an initial prior. For the normal regression model, we have

$$\pi(\boldsymbol{\beta}, \sigma^2 | D_0, D_0, a_{01}, \ldots, a_{0n_0}) \propto \prod_{i=1}^{n_0}\left[(1/\sigma^2)\exp\left\{-\frac{1}{2\sigma^2}(y_{0i} - \boldsymbol{x}_{0i}'\boldsymbol{\beta})^2\right\}\right]^{a_{0i}} \pi_0(\boldsymbol{\beta}, \sigma^2). \tag{25}$$

The propensity score in this context is defined as $P(i \in \text{current group}|\boldsymbol{x}_{0i})$, the conditional probability of a historical subject i being in the current group given the baseline covariates. Let $e(\boldsymbol{x}_{0i})$ denote the estimated propensity score, which is usually obtained by a logistic regression (Wang et al., 2019, 2022). The subject-specific discounting parameter a_{0i} in (24) or (25) is then determined by a function of estimated propensity scores. Wang et al. (2022) considered the inverse probability of treatment weighting (IPTW) with stablized weights and suggested

$$a_{0i} = e(\boldsymbol{x}_{0i}) \cdot w(\boldsymbol{x}_{0i}) = e(\boldsymbol{x}_{0i}) \cdot \frac{e(\boldsymbol{x}_{0i})/\{1 - e(\boldsymbol{x}_{0i})\}}{n_0^{-1} \sum_{j=1}^{n_0} e(\boldsymbol{x}_{0j})/\{1 - e(\boldsymbol{x}_{0j})\}}, \tag{26}$$

where $w(\boldsymbol{x}_{0i})$ is called the stabilized weight in the context, and $\sum_{i=1}^{n_0} w(\boldsymbol{x}_{0i}) = n_0$. When $e(\boldsymbol{x}_{0i})$ approaches 1, the particular subject is highly likely to be in the current group.

Along the PS-stratification path, Baron et al. (2024) developed a PS-integrated borrowing-by-parts power prior, allowing more flexibility for leveraging external information.

5 Software Development of Power Priors

As the power priors gain popularity in vast areas of application, increasingly more statistical software packages have become available to aid the development and use of power priors. The `R` package `BayesPPD` (Bayesian Power Prior Design) (Shen et al., 2023) supports Bayesian power and type I error calculation and model fitting with incorporation of historical data using the power prior and the normalized power prior for generalized linear models. The normalized power prior is also implemented in the `R` package `NPP` (Normalized Power Prior Bayesian Analysis) (Ye et al., 2022), where Markov chain Monte Carlo (MCMC) sampling is used for different distributions with the normalized power priors. The two-arm Bayesian design is built into `R` package `BayesCTDesign` (Bayesian Clinical Trial Design) (Eggleston et al., 2021), in which the power and sample size can be calculated under multiple historical datasets. Packages `ppRep` (Analysis of Replication Studies using Power Priors) (Pawel et al., 2023b) and `BayesPPDSurv` (Bayesian Power Prior Design for Survival Data) (Shen et al., 2024) have recently been published on CRAN (Comprehensive `R` Archive Network).

`SAS` permits model fitting to incorporate historical data with the power priors via the `PROC MCMC` procedure. Either the combined approach or the conventional approach can be used to construct the power prior in `PROC MCMC`. Each method has its respective advantages and disadvantages. The combined approach forms a larger dataset by combining the historical and current data and putting a weight (a_0) on each observation. This allows for easy implementation, but cannot be extended to the normalized power prior or any comparisons of a_0 through DIC. An example of the power prior built in `PROC MCMC` can be found in Chen (2009). The conventional approach, on the other hand, specifies the power prior in its original form (6) by using the historical data to construct the power prior, and the current data for the likelihood function. This approach is more compatible in `SAS`.

A more efficient and model-specific procedure, which supports Bayesian inference, `PROC BGLIMM` (Bayesian Generalized Linear Mixed Model), is available from `SAS/STAT 15.1` onwards. `PROC BGLIMM` fits a narrower range of models but is easier to use. A binomial example of Kociba and NTP data to illustrate `PROC BGLIMM` can be found in SAS Institute Inc (2023), however, only the newest versions of `SAS` may be able to run the codes successfully. The different features of `PROC MCMC` and `PROC BGLIMM` discussed in SAS Institute Inc (2023) and summarized in Chen (2019) are shown below (RE stands for random effects):

	Model	RE Distribution	Linear Predictor	Hierarchy
MIXED	Normal	Normal	$X\beta + Z\gamma$	Nested & Non-Nested
GLIMMIX	GLM	Normal	$X\beta + Z\gamma$	Nested & Non-Nested
NLMIXED	General	Normal	General	Nested
MCMC	General	General	General	Nested & Non-Nested
BGLIMM	GLM	Normal	$X\beta + Z\gamma$	Nested & Non-Nested

Besides the `SAS` built-in statements, `SAS` macros have also been developed to implement the power prior in Bayesian design. For example, `BSMED` (Bayesian Survival Meta-experimental Design) (Ibrahim et al., 2015b) is built for meta-experimental design with historical data, where an exponential regression model and a log-linear fixed-effects model are used for the meta-regression survival model.

Other statistical software, such as `Stan` or `Stata`, currently do not have built-in modules or packages to specifically handle power priors, but users can manually code to sample from the posterior distribution under the power prior. Where analytical forms do not exist, `R` users may find packages `rjags` (Plummer, 2024) and `nimble` (de Valpine et al., 2017, 2024a,b) helpful to accelerate MCMC computation via `JAGS` (Just Another Gibbs Sampler) and `NIMBLE`, separately.

6 Empirical Studies

6.1 Analysis of the Kociba and NTP Data

As discussed in Section 2.1, we consider the NTP data as the current data and the Kociba data as the historical data. The Kociba and NTP data were analyzed in Shi et al. (2021), Shi et al. (2022), and Han et al. (2023b). We fit the binomial regression models in (1) and (2) to the NTP and Kociba data, respectively. In our analysis, we consider and compare various priors, including the power prior (PP) in (6) with an improper uniform prior for $\boldsymbol{\beta}$, i.e., $\pi_0(\boldsymbol{\beta}) \propto 1$; the normalized power prior (nPP) in (11) with an improper uniform prior for $\boldsymbol{\beta}$ and a proper uniform prior for a_0, i.e., $\pi_0(a_0) = 1$ for $0 < a_0 < 1$; and the partial borrowing power prior for β_1 (pPP_1) in (14) with improper uniform priors for both $\pi_0(\beta_{0h})$ and $\pi_0(\beta_0, \beta_1)$. We also specify the partial borrowing power prior for β_0 (pPP_0) given by

$$\pi(\boldsymbol{\beta}|D_0, a_0) \propto \int \prod_{i=1}^{n_0} \Big[\frac{\exp\{y_{0i}(\beta_0 + \beta_{1h}x_{0i})\}}{\{1+\exp(\beta_0 + \beta_{1h}x_{0i})\}^{m_{0i}}} \Big]^{a_0} \pi_0(\beta_{1h}) d\beta_{1h} \pi_0(\beta_0, \beta_1),$$

where $\pi_0(\beta_{1h}) \propto 1$ and $\pi_0(\beta_0, \beta_1) \propto 1$ are the improper initial priors. Furthermore, we implement the propensity score based power prior (iptwPP) in (24) with the IPTW power a_{0i} defined by (26). The marginal posterior densities of β_1 under no borrowing, PP with $a_0 = 0.1$, nPP, and pPP_1 with $a_0 = 1$ are shown in Figure 2. The posterior estimates (means), the standard deviations (SDs), and the 95% highest posterior density (HPD) intervals of β_0 and β_1 under these priors are reported in Table 5. MCMC sample size of 50,000 is used in all calculations and convergence checks can be found in the Supplemental Material.

From Figure 2, we see that (i) the posterior density under no borrowing is flatter than the other three densities; (ii) the posterior density under pPP_1 with $a_0 = 1$ has the sharpest peak among the four curves; and (iii) the posterior densities under PP with $a_0 = 0.1$ are nPP are very

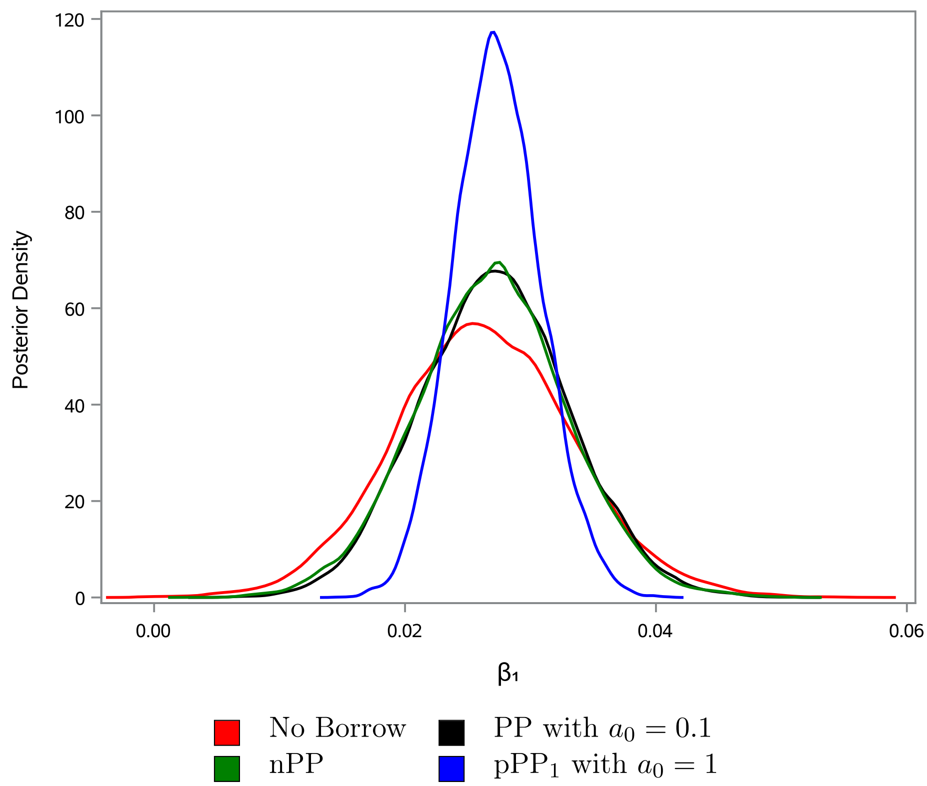

Figure 2: Posterior Densities of $\boldsymbol{\beta}_1$ under no borrowing, PP with $a_0 = 0.1$. nPP, and pPP_1 with $a_0 = 1$.

close to each other, which is expected since the posterior mean a_0 under nPP is 0.121, which is close to 0.1.

Under no borrowing circumstances, included in Table 5, the posterior estimates and SDs are -3.031 and 0.366 for β_0 and 0.026 and 0.007 for β_1, respectively, which are close to the corresponding MLEs from Table 2. We note that when we fit the Kociba data alone using an improper uniform prior for $\boldsymbol{\beta}$, the posterior estimates, SDs, and 95% HPD intervals are -1.787, 0.213, and (-2.221, -1.384) for β_0 and 0.028, 0.004, and (0.020, 0.036) for β_1, respectively. Those posterior estimates are also close to the MLEs of $\boldsymbol{\beta}$ for the Kociba data shown in Table 2. From Table 5, the posterior means of β_1 under no borrowing, PP with $a_0 = 0.1$, nPP, and pPP_1 with $a_0 = 1$ are very close to each other. These results show that pPP allows for borrowing the full Kociba data without inducing bias in estimating $\boldsymbol{\beta}_1$. These findings are consistent with those in Shi et al. (2021) and Han et al. (2023b). On the other hand, if we carry out a full-borrowing conditional inference for β_1, where the current conditional likelihood has a similar form as (19) and the prior is proportional to (19), then the posterior estimate, SD, and 95% HPD interval are 0.027, 0.003, and (0.020, 0.034), respectively. The results are also consistent with PP and pPP_1 when $a_0 = 1$.

From Table 5, we also see that (i) when a_0 increases, the posterior estimates of β_0 under PP or pPP_0 move away from the one under no borrowing and get closer to the one by fitting the Kociba data alone, while the SDs are getting smaller; and (ii) the posterior estimates of β_0 and β_1 under iptwPP are similar to those under PP with $a_0 = 0.5$, which induces substantial

Table 5: Posterior Estimates of $\boldsymbol{\beta}$ for the NTP Data Using the Historical Kociba Data.

		β_0			β_1		
Prior	a_0	Estimate	SD	95% HPD	Estimate	SD	95% HPD
No Borrow	0	−3.031	0.366	(−3.779, −2.335)	0.026	0.007	(0.012, 0.041)
PP	0.1	−2.887	0.321	(−3.508, −2.262)	0.027	0.006	(0.015, 0.039)
	0.5	−2.517	0.231	(−2.972, −2.072)	0.028	0.004	(0.020, 0.036)
	1.0	−2.298	0.181	(−2.660, −1.944)	0.027	0.003	(0.021, 0.034)
nPP	0.121*	−2.877	0.344	(−3.528, −2.192)	0.027	0.006	(0.016, 0.039)
pPP_0	0.1	−2.815	0.316	(−3.461, −2.232)	0.023	0.007	(0.011 0.037)
	0.5	−2.419	0.222	(−2.851, −1.987)	0.017	0.006	(0.005, 0.028)
	1.0	−2.211	0.179	(−2.570, −1.868)	0.014	0.006	(0.004, 0.026)
pPP_1	0.1	−3.054	0.344	(−3.733, −2.400)	0.027	0.006	(0.015, 0.039)
	0.5	−3.056	0.297	(−3.646, −2.489)	0.027	0.004	(0.019, 0.036)
	1.0	−3.052	0.276	(−3.574, −2.514)	0.027	0.003	(0.021, 0.034)
iptwPP	(26)	−2.465	0.222	(−2.917, −2.050)	0.025	0.004	(0.016, 0.034)

Note: 0.121* is the posterior mean of a_0 under nPP.

bias in estimating β_0. Finally, as reported in Han et al. (2023b), the posterior estimate and 95% HPD interval of $\boldsymbol{\beta}_1$ are 0.027 and (0.019, 0.036) for the partial borrowing normalized power prior (pnPP) in (15), which are almost the same as those obtained under pPP_1 with $a_0 = 0.5$. This is not surprising as the posterior mean of a_0 under pnPP is 0.554 as reported in Han et al. (2023b).

6.2 Analysis of the ADNI Data

As mentioned in Section 2.2, we consider the change in ADAS or MMSE at 24 months from the baseline as our response variables. The current study consists of ADNI–GO and ADNI–2 datasets, whereas the historical study is formed by the ADNI–1 dataset. We consider a reference-type initial prior $\pi_0(\boldsymbol{\beta}, \sigma^2) \propto 1/\sigma^2$ for the power prior (PP) in (12), the partial borrowing power prior (pPP) in (17), the borrowing-by-parts power prior ($\bar{\text{p}}$PP) in (21), and the propensity score based power prior (iptwPP) in (25). We also specify the initial prior $\pi_0(\gamma, \boldsymbol{\beta}_1, \sigma^2) \propto 1/\sigma^2$ for the partial borrowing-by-parts power prior (p$\bar{\text{p}}$PP) in (23). The posterior distributions have closed forms, except for the partial borrowing power prior, for which we employed a Gibbs sampler of size 50,000 after a burn-in phase of 10,000. The trace plots and the autocorrelation function plots are provided in the Supplementary Material.

We are interested in the exposure effect γ in model (3). Posterior estimates of γ under various priors are summarized in Table 6 for both ADAS and MMSE. In general, the estimates of the exposure effect are more sensitive to the discounting parameters for ADAS than those for MMSE, as the OLS estimates of the current and historical studies are closer for MMSE. Under no borrowing, the posterior means of the exposure effect, 0.732 for ADAS and −0.699 for MMSE in Table 6, are exactly the same as the corresponding OLS estimates $\hat{\gamma}$ within the current study as shown in Table 4, while the posterior standard deviations (SD), 0.414 for ADAS and 0.215 for MMSE, are slightly larger than the frequentist standard errors (SEs), 0.413 and 0.214, reflecting the additional uncertainty introduced by the initial prior within the Bayesian framework. On the other hand, for ADAS, the more we borrow (either larger a_0 in general or larger a_{01} under borrowing-by-parts), the greater the estimated exposure effect, since the OLS estimate $\hat{\gamma} = 2.765$ in ADNI–1 is greater than the one in ADNI–GO2. A similar monotonicity is observed for MMSE. Under $\bar{\text{p}}$PP and p$\bar{\text{p}}$PP, we notice that a_{02} does not affect the posterior mean of γ, but the larger a_{02} is, the greater the posterior SD. This is because the historical $\hat{\sigma} = 4.809$ for ADAS (2.743 for MMSE) is greater than the current $\hat{\sigma} = 3.871$ (2.061); see Table 4.

The posterior estimates of other regression coefficients in model (3) behave similarly under these priors. For the ADAS response, the OLS estimate for the effect of Male in the current study is 0.019 with insignificant p-value$= 0.958$. After borrowing from the historical study, the posterior estimate gradually moves from positive to negative towards the historical estimate. However, most 95% HPD intervals still contain 0, except when $(a_{01}, a_{02}) = (1.0, 0.0)$ under $\bar{\text{p}}$PP, for which the posterior estimate is -0.567 with 95% HPD $(-1.105, -0.030)$. On the other hand, the posterior estimate for the effect of 'Education' to the MMSE response under no borrowing is positive and significant, with mean 0.092, SD 0.036, and HPD interval (0.021, 0.163). Borrowing too much from the historical estimate, however, would cause the effect to be insignificant, because the estimate is dragged towards zero by the historical data source. For instance, $(a_{01} = 1.0, a_{02} = 0.0)$ under $\bar{\text{p}}$PP gives posterior estimate 0.031, SD 0.025, and HPD interval $(-0.019, 0.080)$.

When we do not borrow, the posterior estimate $\hat{\sigma} = 3.878$ is close to but slightly greater than the OLS estimate $\hat{\sigma} = 3.871$ in Table 4. Since the OLS estimate $\hat{\sigma} = 4.809$ for ADNI–1 is greater than that in the current study, it makes sense to expect that the more we borrow from the historical data, the larger the posterior estimate $\hat{\sigma}$ would be. This monotonicity is consistently observed.

Finally, we exchange the role of Age and MCI in (3) so that γ corresponds to the regression coefficient for the covariate Age. In Table 4, the OLS estimates for the coefficient of Age are both 0.019 within the current and historical studies, and they are not significant from zero. Now, the posterior estimate, SD, and 95% HPD interval of the regression coefficient corresponding to Age are 0.019, 0.026, $(-0.031, 0.069)$ under PP with $a_0 = 0$; 0.026, 0.025, $(-0.023, 0.075)$ under PP with $a_0 = 0.1$; 0.040, 0.023, $(-0.005, 0.085)$ under PP with $a_0 = 0.5$; and 0.046, 0.021, (0.005, 0.087) under PP with $a_0 = 1.0$, respectively, for ADAS. Under iptwPP, the posterior estimate, SD, and 95% HPD interval of the regression coefficient corresponding to Age are -0.005, 0.019, $(-0.042, 0.033)$ for ADAS, respectively, where the estimate is changed from positive to negative after borrowing. In contrast, the posterior estimate, SD, and 95% HPD interval of the regression coefficient corresponding to Age are 0.019, 0.025, $(-0.030, 0.067)$ under p$\bar{\text{p}}$PP with $(a_{01}, a_{02}) = (0.1, 0)$; 0.019, 0.022, $(-0.023, 0.061)$ under p$\bar{\text{p}}$PP with $(a_{01}, a_{02}) = (0.5, 0)$; and 0.019, 0.019, $(-0.018, 0.056)$ under p$\bar{\text{p}}$PP with $(a_{01}, a_{02}) = (1.0, 0)$, respectively, for ADAS. The results under p$\bar{\text{p}}$PP are desirable, since the estimates under p$\bar{\text{p}}$PP remain constant with decreasing SDs when a_{01} is increasing.

7 Concluding Remarks

In this paper, we provide a brief overview of the development of the power priors in Section 3 and elaborate several recently developed variations of the power priors in Section 4. The available software on the power priors is reviewed in Section 5. The binomial regression models and the

Table 6: Posterior Estimates of γ in (3) for the ADNI Data.

		ADAS			MMSE		
Prior	a_0	Estimate	SD	95% HPD	Estimate	SD	95% HPD
No Borrow	0.0	0.732	0.414	(−0.080, 1.544)	−0.699	0.215	(−1.119, −0.278)
PP	0.1	0.817	0.406	(0.020, 1.614)	−0.737	0.213	(−1.155, −0.318)
	0.5	1.109	0.373	(0.378, 1.840)	−0.821	0.202	(−1.217, −0.426)
	1.0	1.372	0.337	(0.711, 2.033)	−0.872	0.186	(−1.237, −0.507)
pPP	0.1	0.783	0.400	(−0.025, 1.547)	−0.719	0.209	(−1.127, −0.307)
	0.5	0.977	0.364	(0.271, 1.695)	−0.781	0.192	(−1.159, −0.404)
	1.0	1.186	0.329	(0.539, 1.829)	−0.829	0.178	(−1.174, −0.478)
iptwPP	(26)	1.086	0.328	(0.444, 1.729)	−0.622	0.179	(−0.974, −0.270)
p̄PP	(0.1, 0.0)	0.817	0.398	(0.037, 1.597)	−0.737	0.207	(−1.142, −0.331)
	(0.1, 0.5)	0.817	0.429	(−0.025, 1.659)	−0.737	0.230	(−1.188, −0.285)
	(0.1, 1.0)	0.817	0.445	(−0.055, 1.689)	−0.737	0.241	(−1.209, −0.264)
	(0.5, 0.0)	1.109	0.348	(0.426, 1.792)	−0.821	0.183	(−1.180, −0.463)
	(0.5, 1.0)	1.109	0.385	(0.354, 1.864)	−0.821	0.211	(−1.235, −0.408)
	(1.0, 0.0)	1.372	0.307	(0.770, 1.975)	−0.872	0.162	(−1.190, −0.554)
	(1.0, 0.5)	1.372	0.327	(0.731, 2.014)	−0.872	0.178	(−1.222, −0.522)
pp̄PP	(0.1, 0.0)	0.896	0.401	(0.110, 1.682)	−0.727	0.209	(−1.136, −0.318)
	(0.1, 0.5)	0.896	0.433	(0.047, 1.744)	−0.727	0.232	(−1.182, −0.271)
	(0.1, 1.0)	0.896	0.448	(0.017, 1.774)	−0.727	0.243	(−1.204, −0.249)
	(0.5, 0.0)	1.351	0.349	(0.666, 2.035)	−0.808	0.184	(−1.168, −0.448)
	(0.5, 0.5)	1.351	0.376	(0.613, 2.088)	−0.808	0.205	(−1.209, −0.407)
	(0.5, 1.0)	1.351	0.390	(0.587, 2.114)	−0.808	0.215	(−1.228, −0.387)
	(1.0, 0.0)	1.679	0.305	(1.081, 2.277)	−0.870	0.162	(−1.187, −0.553)
	(1.0, 0.5)	1.679	0.329	(1.035, 2.324)	−0.870	0.181	(−1.224, −0.516)
	(1.0, 1.0)	1.679	0.341	(1.012, 2.347)	−0.870	0.190	(−1.241, −0.498)

normal linear regression models for fitting the Kociba and NTP data and the ADNI data are used to demonstrate the formulations of the power priors, the normalized power priors, and the several variations of the power priors. As discussed in Sections 2.1 and 6.1, the Kociba and NTP data share a similar slope but have very different intercepts. As shown in Section 6.1, in the analysis of the NTP data, PP and nPP may yield a large bias in the posterior estimate of β_0 (the intercept) if we borrow the Kociba data too much, or may not lead to substantial reduction in the posterior SD (uncertainty) if we borrow too little. On the contrary, pPP_1 and the borrowing-by-parts power prior in (20) with $a_{02} = 0$ work remarkably well since the posterior estimates under these two priors remain almost the same as those under no borrowing while the corresponding posterior SDs are substantially reduced. Similarly, in the analysis of the ADNI data in Sections 2.2 and 6.2, we observe that iptwPP, p̄PP, and pp̄PP behave much better than PP, especially in the analysis of MMSE for estimating the exposure effects of MCI. However, upon treating Age as the exposure effect, the effects of borrowing with pp̄PP are remarkable for ADAS, introducing very little bias to the estimate while significantly reducing posterior SD. These empirical analyses clearly demonstrate that the choices among different variations are a challenging task. Ibrahim et al. (2015a) proposed using model selection criteria to guide the choices of the priors. Another option is to adopt a dynamic borrowing power prior, which is another variation of the power prior and is expanded on in more detail below. We further discuss additional various important issues with the power priors and future directions of research on the power priors as follows.

Fixed or random a_0 One important aspect of power prior application is the determination of the discounting parameter a_0. The full Bayesian approach, the normalized power prior, is conceptually simple but computationally intensive; see (15). Algorithms are developed (Carvalho and Ibrahim, 2021; Han et al., 2023b) to alleviate the computational burden. Alternatively, one may adopt criterion-based approaches such as the penalized likelihood-type criterion (Ibrahim et al., 2003, 2015a), the marginal likelihood criterion (Ibrahim et al., 2015a; Gravestock and Held, 2017; Wang et al., 2018), the deviance information criterion (Ibrahim et al., 2015a), and the logarithm of the pseudo-marginal likelihood criterion (Ibrahim et al., 2015a) to elicit a fixed value for a_0. Similarity measures that compare the historical data source with the current data are also popular in eliciting a_0, including the p-values based measures (Liu, 2018; Nikolakopoulos et al., 2018), the information gain measure (Shi et al., 2021), the dissonance measure (Shi et al., 2022), and the minimal plausibility index (Baron et al., 2024). The choice of fixed or random a_0 remains an open question.

Normalized power priors The normalized power prior for a single historical dataset is given in Section 3.1 and for multiple historical datasets in Section 3.2. The partial borrowing normalized power prior was introduced by Shi et al. (2021) and Han et al. (2023b). This variation of the power prior is further discussed in Section 4.1. The normalized version of the borrowing-by-parts power prior discussed in Section 4.2 has yet to be fully developed. Furthermore, the normalized version of the partial borrowing-by-parts power prior discussed in Section 4.3 is still at its pre-birth stage. These variations of the normalized power priors and their corresponding computational developments are well-deserved future research projects.

Outcome-dependent borrowing or covariates-based borrowing When a_0 is fixed, the power prior with the value of a_0 determined by certain Bayesian model comparison criterion such as the deviance information criterion (Ibrahim et al., 2015a) or the marginal likelihood criterion. Gravestock and Held (2017) and Wang et al. (2018) use outcome-dependent borrowing. The propensity score (PS) based inverse probability of treatment weighting (IPTW) power prior (iptwPP) discussed in Section 4.4 uses covariate-based borrowing. The recent literature on covariate-based borrowing includes Wang et al. (2019), Li et al. (2020), Li et al. (2022a), Li et al. (2022b), and Wang et al. (2022). Based on our preliminary empirical investigations given in Section 6, the iptwPP based approach does yield over-borrowing, which leads to biased estimates of the model parameters when the current data and the historical data are not similar. On the contrary, the variations of the normalized power priors may be attractive since the amount of borrowing is automatically determined by the similarity of the current and historical data, although these approaches are outcome-dependent borrowing. This type of borrowing opens up another new research direction of so-called dynamic borrowing. Another potential future research direction is to develop an approach based on hybrid outcome-dependent borrowing and covariate-based borrowing.

Other recent developments on leveraging historical or external data There is a growing literature on leveraging historical or external data other than using the power priors. Examples include the latent exchangeability (LEAP) prior (Alt et al., 2024), which borrows from historical data based on their most relevant subjects; the horseshoe prior (Ohigashi et al., 2022), which assumes potential bias of historical data from current data; Bayesian additive regression trees (BART) (Zhou and Ji, 2021), which consider differences in patient characteristics and other factors that may confound historic data; frequentist-based (Chu and Yi, 2021) and Bayesian (Kaplan et al., 2023) dynamic borrowing approaches which use the current-historical weighted average and joint prior, respectively, to dynamically determine borrowing levels; dynamic borrowing approaches via elastic and self-adapting mixture priors (Yang et al., 2023; Jiang et al., 2023); the similarity-weighted informative prior (König, 2021), which focuses on developing a novel measure of similarity; and various innovations on the meta-analytic-predictive (MAP) priors (Hupf et al., 2021; Zhang et al., 2021; Liu et al., 2021; Zhang et al., 2023). However, these approaches have yet to be directly compared to the variations of the power prior discussed in Section 4. Further empirical or theoretical investigations are needed in future research.

Posterior computation with power priors The power priors are informative priors. Generally speaking, the power priors bring additional information into the posterior, yielding better parameter estimation and reduced model parameter uncertainty *a posteriori*. Thus, the power priors should help accelerate convergence and improve mixing of MCMC sampling from the resulting posterior distribution. As discussed in Section 5, current available software is limited to standard models. For more complex models, more software needs to be developed. These are very important future projects. Such software development will be highly beneficial to researchers and practitioners.

Other future directions of research on the power priors Power priors have been used in various Bayesian designs of clinical trials (Chen et al., 2011, 2014b; Jiang et al., 2015; Li et al., 2015, 2018; van Rosmalen et al., 2018; Yuan et al., 2022b; Baumann et al., 2024). See Chen et al. (2024) for an overview of recent developments on this topic. However, much more research on the performance of the variations elaborated in Section 4 in Bayesian design of clinical trials needs to be carried out. The power priors have also been used in co-data analysis (Neuenschwander et al., 2016; Sheikh et al., 2022); however, the literature on co-data analysis remains sparse and necessitates much more investigation.

Supplementary Material

The posterior densities in Figure 2 and the posterior estimates reported in Table 5 are computed using `SAS` while the posterior estimates in Table 6 are obtained either analytically or using `R`. Additional tables and figures for MCMC convergence checks, which show good convergence and mixing of MCMC samples, are provided in Supplementary Material Sections S.1. Section S.2 contains additional tables and figures for MCMC convergence checks for the ADNI data, which again show good convergence and mixing. Unfortunately, the ADNI data is proprietary. `SAS` code (`Kociba_NTP_example.sas`) for the Kociba and NTP data, `R` code (`analysis_simulated_data.qmd`) for ADNI results, and a simulated dataset (`sim_data.csv`) mimicking the ADNI data can be found at https://github.com/MinLinSTAT/PPreview. The posterior estimates of γ for the simulated dataset are given in Section S.3.

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