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THE EXPONENTIATED GENERALIZED EXTENDED GOMPERTZ DISTRIBUTION

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ABSTRACT

This paper presents a new generalization of the extended Gompertz distribution. We defined the so-called exponentiated generalized extended Gompertz distribution, which has at least three important advantages: (i) Includes the exponential, Gompertz, extended exponential and extended Gompertz distributions as special cases; (ii) adds two parameters to the base distribution, but does not use any complicated functions to that end; and (iii) its hazard function includes inverted bathtub and bathtub shapes, which are particularly important because of its broad applicability in real-life situations. The work derives several mathematical properties for the new model and discusses a maximum likelihood estimation method. For the main formulas related to our model, we present numerical studies that demonstrate the practicality of computational implementation using statistical software. We also present a Monte Carlo simulation study to evaluate the performance of the maximum likelihood estimators for the EGEG model. Three real- world data sets were used for applications in order to illustrate the usefulness of our proposal.

Keywords: Applied results, exponentiated generalized class, Gompertz distribution, probability models with applications, real data sets.

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1 Introduction

From both theoretical and applied perspectives, proposing new probability distributions is crucial to describing natural phenomena. There are several ways to extend well-known distributions, one of the most popular being to consider distribution generators such as Exponentiated (Lehmann, 1953), MarshallOlkin (Marshall and Olkin, 1997), beta (Eugene et al., 2002), gamma (Zografos and Balakrishnan, 2009; Ristic and Balakrishnan, 2011; Nadarajah et al., 2015), Kumaraswamy (Cordeiro and de Castro, 2011), McDonald (Alexander et al., 2012) and exponentiated generalized (Cordeiro et al., 2013) classes of models. These generators are consecrated in the specialized literature and, over the last twenty years, several works considering this approach in different contexts have been published. Notably, in particular, a large number of new continuous probability distributions were proposed in the so-called exp-G, MO-G, beta-G, gamma-G, Kw-G, Mc-G and (Cordeiro et al., 2013)'s-G classes. Here, it is worth mentioning a recent study by Tahir and Nadarajah (2015), who carried out a comprehensive review of the literature and listed numerous continuous univariate distributions, defined on the basis of many of the aforementioned classes.

Recently, El-Gohary et al. (2013) used the approach proposed by Lehmann (1953) to define a generalization of the Gompertz model by adding one parameter. Specifically, those authors employed a Lehmann Type I transformation in the usual Gompertz distribution and defined the model they called generalized Gompertz distribution. It is usual to refer to the distributions obtained through Lehmann Type I transformations as Exp-G models, so that a natural nomenclature for the model proposed by El-Gohary et al. (2013) would be Exp-Gompertz distribution. However, we refer to this model as the extended Gompertz (EG) distribution in order to facil- itate the final nomenclature of the model proposed in this paper. Thus, here and henceforth, the EG distribution is the model by El-Gohary et al. (2013). The cdf G(x) and pdf g(x) of the EG distribution are given by

$$\mathrm{G}(x;\theta,\gamma,\beta)=(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta} \tag{1}$$

and

$$\mathrm{g}(x;\theta,\gamma,\beta)=\theta\beta e^{\gamma x}e^{-\frac{\beta}{\gamma}((e^{\gamma x}-1))}(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta-1} \tag{2}$$

where $\beta>0$, $\gamma\geq 0$, $\theta>0$ and $x\geq 0$.

We believe that adding parameters to the EG model may give rise to new, more flexible models for fitting to real data. Therefore, we defined in this paper an extension of the model above. To that end, we considered the methodology proposed by Cordeiro et al. (2013). For a given continuous baseline cdf G(x), and $x \in R$, those authors defined the exponentiated generalized class of distributions with two extra shape parameters $a > 0$ and $b > 0$ with cdf

F(x) and pdf f(x) given by

$$\mathrm{F(x)} = \{1 - [1 - \mathrm{G(x)}]^{a}\}^{b} \quad (3)$$

and

$$\mathrm{f(x)} = \mathrm{ab}[1 - \mathrm{G(x)}]^{a-1}\{1 - [1 - G(x)]^{a}\}^{b-1} g(x) \quad (4)$$

respectively, in which the dependence on the parameters of G(x), are implicit.

To illustrate the flexibility of the exponentiated generalized model, Cordeiro et al. (2013) applied (3) to extend some well-known distributions such as the Frechet, normal, gamma and Gumbel distributions. Moreover, they presented several properties for the exponentiated generalized class, which motivate the adoption of this generator. Next, we discuss some of these motivations. The first important point to note is the simplicity of equations (3) and (4). They have no complicated functions and will be always tractable when the cdf and pdf of the baseline distribution have simple analytic expressions. It is very easy, for example, to obtain the inverse of the cdf (3). Another important feature is that the model by Cordeiro et al. (2013) contains as especial cases the two classes of Lehmann's alternatives. In fact, for a = 1, (3) reduces to $\mathrm{F(x)} = \mathrm{G(x)}^{b}$ and, for b = 1, we obtain $\mathrm{F(x)} = 1 - [1 - \mathrm{G(x)}]^{a}$, which corresponds to the cdf's of the Lehmann type I and II families Lehmann (1953), respectively. For this reason, the model by Cordeiro et al. (2013) encompasses both Lehmann type I and type II classes. Therefore, the exponentiated generalized family can be derived from a double transformation using these classes. The two extra parameters a and b in the density (4) can control both tail weights, allowing the generation of flexible distributions, with heavier or lighter tails, as appropriate. There is also an attractive physical interpretation of the model (3) when a and b are positive integers. This interpretation is described in Cordeiro and Lemonte (2014): They initially suppose that a certain device is composed of b components in a parallel system. The authors also consider that, for each component b, there exists an independent series of subcomponents a distributed according to G(x). They also assume that each component b fails if some a sub-component fails. Let $X_{j1}, \dots, X_{ja}$ denote the lifetimes of the subcomponents within the jth component, j = 1, . . . , b, with common cdf G(x). Let Xj denote the lifetime of the jth component and let X denote the lifetime of the device. Thus, the cdf of X is

$$\begin{aligned} \mathrm{P(X \le x)} &= \mathrm{P}(X_1 \le \mathrm{x}, \dots, X_b \le \mathrm{x}) = \mathrm{P}(X_1 \le \mathrm{x})^{b} = [1 - P(X_1 > x)]^{b} \\ &= [1 - \mathrm{P}(X_{11} > \mathrm{x}, \dots, X_{11} > \mathrm{x})]^{b} = [1 - P(X_{11} > \mathrm{x})^{a}]^{b} \\ &= [1 - \{1 - \mathrm{P}(X_{11} \le \mathrm{x})\}^{a}]^{b} \end{aligned}$$

Hence, the physical interpretation can be summarized as follows: The lifetime of the device obeys the exponentiated generalized family of distributions.

The above properties and many others have been discussed and explored in recent works for the Cordeiro et al. (2013)'s class. Here, we refer to the papers and baseline distributions: Cordeiro and Lemonte (2014) for the Birnbaum-Saunders distribution, Silva et al. (2015) for the Dagum distribution, De Andrade et al. (2015) and De Andrade et al. (2018) for the Gumbel model, Cordeiro and Lemonte (2016) for the arcsine distribution, De Andrade et al. (2016) for an extended exponential model, Oguntunde et al. (2016) for the exponential distribution, De Andrade and Zea (2018) for the extended Pareto distribution, among.

In this paper, we define the exponentiated generalized extended Gompertz (EGEG) distribution by inserting (1) in equation (3). Our study of the EGEG model has very clear and forceful motivations:

1. The model proposed in this paper generalizes at least four important distributions that are well established in the literature: the exponential, Gompertz, extended exponential and extended Gompertz distributions.
2. Although our EGEG model has five parameters, its does not have any complicated form for the density, cumulative or likelihood functions, among others. This represents a gain since it facilitates obtaining analytical and numerical results.
3. We studied the structural properties of the EGEG model and verified that all formulas associated with the proposed model are simple and manageable using computational resources.
4. The hazard function of the new EGEG model is flexible enough to accommodate all the classic forms, such as increasing, decreasing, inverted bathtub and bathtub shapes, among others. The inverted bathtub and bathtub shapes are particularly important because of their great applicability in practical situations.
5. Three real-world data sets to illustrate the goodness-of-fit of the new EGEG model.

For the reasons listed above, we strongly believe it is important to study in detail the EGEG distribution. We hope that this new distribution will be part of the arsenal of applied researchers and will be used in many practical situations.

Besides this introduction, the paper is organized as follows. In Section 2, the new distributions are detailed. Shapes are discussed in Section 3. The quantile function and its applications are investigated in Section 4. In Section 5, some mathematical properties of the new model are derived and numerical studies are detailed. Estimation and inference are addressed in Section

6. A Monte Carlo simulation study is presented in Section 7. In Section 8, we used three real-world data sets in order to illustrate the usefulness of our proposal. Section 9 is devoted to final considerations.

2 The EGEG distribution

The construction of the probabilistic model of a random variable X with support on the set of positive real numbers and EGEG(a, b, θ, γ, β) distribution, say $X \sim$ EGEG (a, b, θ, γ, β), is defined by inserting (1) in equation (3). Thus, the cdf of X is given by

$$\mathrm{F(x)} = \{1 - [1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}]^{a}\}^{b} \tag{5}$$

where $a > 0, b > 0, \theta > 0, \gamma \geq 0, \beta > 0,$ and $x \geq 0.$

Note that (5) has a simple closed-form. This feature is important because, as we shall see, it is possible to generate EGEG variables in a very simple manner by using the method of inversion (by means of a small simulation study presented in Section 4 we use the method of inversion to generate EGEG random variables). From equation (5), it is easily observed that the EGEG model includes the following distributions as special cases:

- The EG distribution, proposed by El-Gohary et al. (2013), comes from the equation (5) when a = b = 1.
- The Gompertz (Gom) distribution, proposed by Gompertz (1825), appears when a = b = θ = 1 in the equation (5).
- The exponential distribution arise from the equation (5) when γ = 0 and a = b = θ = 1.
- The extended exponential (EE) (common referred in the literature as exponentiated exponential) distribution, previously investigated by Gupta et al. (1998), comes from the equation (5) whem γ = 0 and a = b = 1.
- The exponentiated generalized exponential (EGE) distribution, already investigated by De Andrade et al. (2016), comes from the equation (5) when γ = 0 and θ = 1.
- The new exponentiated generalized Gompertz (EGG) distribution, appears when θ = 1 in the equation (5).

The EGEG density, for x > 0, reduces to

$$\begin{aligned} \mathrm{f(x)} = ab\beta\theta e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta-1}\{1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}\}^{a-1} \\ \times \{1 - [1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}]^{a}\}^{b-1} \end{aligned} \tag{6}$$

Besides the cdf (5) and pdf (6), other functions can be used to characterize the EGEG model such as the survival function (sf) and hazard rate function (hrf). These are particularly important to analyze survival data that involve the time associated to an event of interest such as the time that a certain component fails, the death of a patient or a disease relapse. Here, it is worth quoting Lee (1992) Chapter 2, page 8:

The distribution of survival times is usually described or characterized by three functions: (i) the survivorship function, (ii) the probability density function, and (iii) the hazard function. These three functions are mathematically equivalent – if one of them is given, the other two can be derived.

The sf and hrf of X are given by

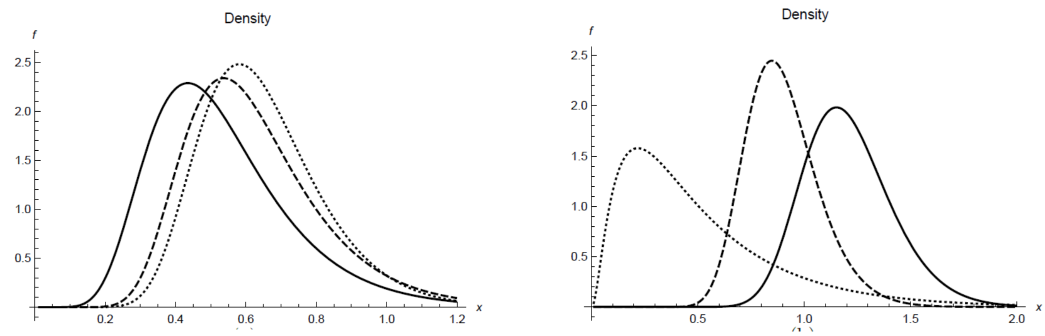
$$S(x) = 1 - \{1 - [1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}]^{a}\}^{b} \tag{7}$$

and

$$\begin{aligned} h(x) = {} & ab\beta\theta e^{\gamma x} e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta-1}\{1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}\}^{a-1} \\ & \times \{1 - [1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}]^{a}\}^{b-1} \\ & \times \{1 - \{1 - [1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}]^{a}\}^{b}\}^{-1} \end{aligned} \tag{8}$$

respectively.

The EGEG density plots for the values of the selected parameters are shown in Figure 1. Figure 2 provides some possible shapes of the EGEG hazard function for appropriate choice of the parameter values, including bathtub, inverted bathtub, increasing and decreasing shape. These plots indicate that the EGEG model is fairly flexible and can be used to fit several types of positive data.



(a) (b)

Figure 1: Plots of the EGEG density function for some parameter values.

As a further characterization of the EGEG distribution, we provide the cumulative hazard rate (chrf) H(x) and reversed hazard rate (rhrf) r(x) functions:

$$H(x) = -\log(1 - \{1 - [1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}]^{a}\}^{b})$$

and

$$r(x) = \frac{ab\beta\theta e^{\gamma x} e^{\gamma x} - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta-1}\{1 - (1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)})^{\theta}\}^{a-1}}{1 - \{1 - [1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}]^{\theta}\}^{a}}$$

respectively. These last two functions are less used in practical situations. However, they have outstanding theoretical importance. For example, we can express it as $f(x) = r(x)\exp\{-H(x)\}$. For more details, see Lee (1992).

3 Shapes

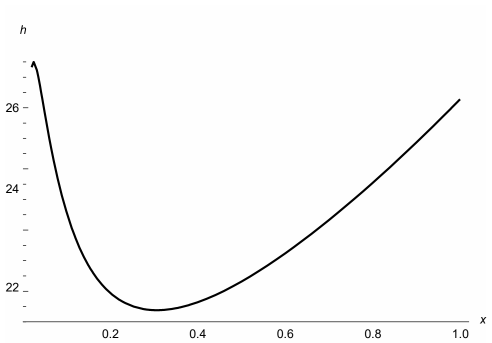
The main features of the density's form of a distribution can be perceived through the study of its first and second derivative. For this reason, many papers that are proposed to study new distributions of probability include a section called "Shapes", which is intended to present conclusions on the characteristics of new models obtained from investigations in its first and second derivatives. Here, we refer to a few of this papers: Nadarajah (2006), Nadarajah et al. (2011), Cordeiro et al. (2017) and De Andrade and Zea (2018), among others.

Regarding EGEG distribution, we have that the first derivative of log{f (x)} is

$$\frac{d\ \log\{f(x)\}}{dx} = \gamma + \beta e^{\gamma x} v_1(x)\{\frac{\theta - 1}{v_2(x)} - \frac{\theta(a-1)v_2(x)}{v_3(x)} + \frac{a\theta(b-1)v_2^{\theta-1}(x)v_3^{a-1}(x)}{v_4(x)}\}$$

where $v_1(x) = e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}, v_2(x) = 1 - v_1(x), v_3(x) = 1 - v_2^{\theta}(x), and\ \ v_4(x) = 1 - v_3^{a}(x)$

Hzazrd



(a) bathtub a = 3, b = 20, θ = 0.2, γ = 0.56, β = 5

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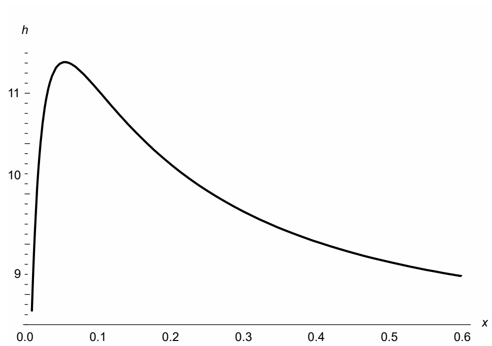
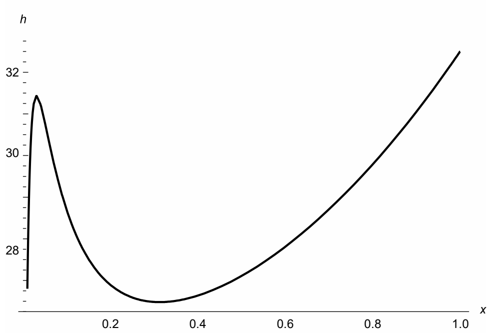


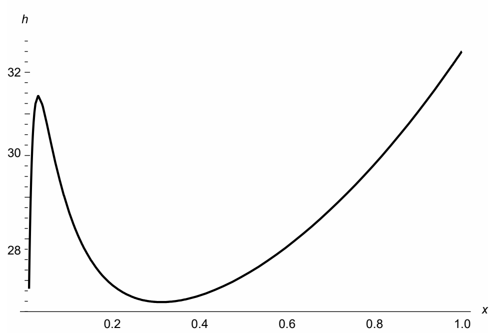
(b) inverted bathtub a = 3, b = 20, θ = 0.2, γ = 0.01, β = 2.14

Hzazrd



(c) inverted bathtub - bathtub a = 6, b = 20, θ = 0.3, γ = 1, β = 2

Hzazrd



(d) increasing a = 2, b = 2, θ = 1.2, γ = 0.64, β = 0.5

Figure 2: Plots of the EGEG hazard function for some parameter values.

Thus, the critical values of f (x) are the roots of the equation

$$\frac{\theta(a-1)v_2(x)}{v_3(x)} - \frac{a\theta(b-1)v_2^{\theta-1}(x)v_3^{a-1}(x)}{v_4(x)} = \frac{\gamma}{\beta e^{\gamma x}v_1(x)} + \frac{\theta-1}{v_2(x)} \tag{9}$$

If the point x=x_0 is a root of (9), then we can classify it as local maximum, local minimum or inflection point when we have, respectively, λ(x_0) < 0, λ(x_0) > 0 and λ(x_0) = 0, where λ(x) =$d^2 \log\{f(x)\}/dx^2$. The second derivative of log{f (x)} comes as

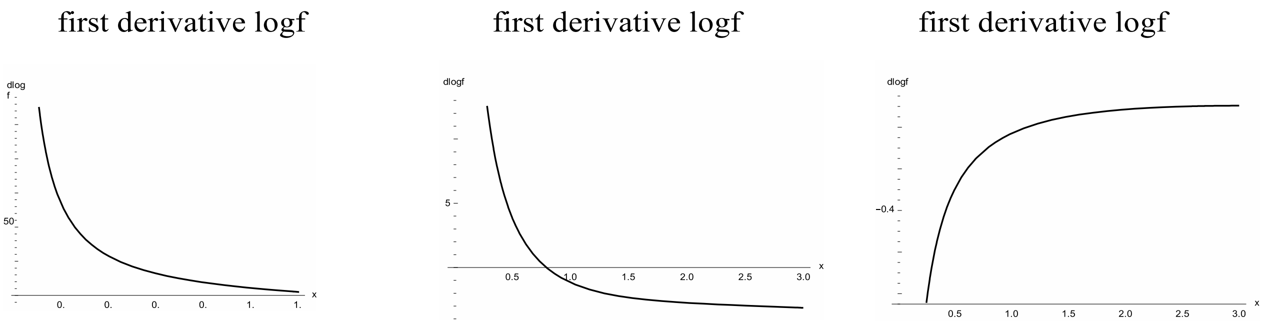
$$\frac{d^2\log\{f(x)\}}{dx^2} = \beta(\theta-1)e^{\gamma x}v_1(x)\frac{-\beta+v_2(x)(\gamma-\beta e^{\gamma x})}{v_2^2(x)} - \beta\theta e^{\gamma x}v_1(x)v_2^{\theta-1}(x)$$

$$\times\{\frac{a-1}{v_3^2(x)}\left[v_3(x)\left(\gamma-\beta e^{\gamma x}+\beta(\theta-1)e^{\gamma x}v_1(x)v_2^{-1}(x)\right)+\beta\theta e^{\gamma x}v_1(x)v_2^{-1}(x)\right]$$

$$+\frac{a(b-1)v_3^{a-1}(x)}{v_4^2(x)}[v_4(x)\left(\gamma-\beta e^{\gamma x}+\beta(\theta-1)e^{\gamma x}v_1(x)v_2^{-1}(x)\right)$$

$$-\beta\theta e^{\gamma x}v_1(x)v_2^{\theta-1}(x)v_3^{-1}(x)(av_3^a(x)+(a-1)v_4(x))]\}$$

In general, it is quite complicated to obtain analytical solutions for the critical points, since expressions such as presented in equation (9). Therefore, it is common to obtain numerical solutions with high accuracy through optimization routines in most mathematical and statistical platforms. Some plots of the first derivative of log{f (x)} for selected parameter values are displayed in Figure 3.



(a) a = 0.2, b = 6, θ = 1.2, γ = 0.5, β = 5

(b) a = 1.2, b = 8, θ = 0.9, γ = 0.1, β = 2

(c) a = 1.2, b = 0.8, θ = 0.9, γ =0.1, β = 0.2

Figure 3: Plots of the EGEG d log{f (x)}/dx function for some parameter values.

Similarly, we provide the first and second derivatives of log{h(x)} to the EGEG distribution.The first derivative of log{h(x)} is given by

$$\frac{d\ \log\{f(x)\}}{dx} = \gamma + \beta e^{\gamma x}v_1(x)\{\frac{\theta-1}{v_2(x)} - \frac{\theta(a-1)v_2(x)}{v_3(x)} + \frac{a\theta(b-1)v_2^{\theta-1}(x)v_3^{a-1}(x)}{v_4(x)}$$

$$-\frac{ab\theta(b-1)v_2^{\theta-1}(x)v_3^{a-1}(x)v_4^{b-1}(x)}{v_4(x)v_5(x)}\}$$

where $v_1(x) = e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}, v_2(x) = 1 - v_1(x), v_3(x) = 1 - v_2^{\theta}(x), and\ \ v_4(x) = 1 -$

$v_3^a(x)$ and $v_5(x) = 1 - v_4^b(x)$

Thus, the critical values of h(x) are the roots of the equation

$$-\frac{ab\theta(b-1)v_2^{\theta-1}(x)v_3^{a-1}(x)v_4^{b-1}(x)}{v_4(x)v_5(x)} = -\frac{\gamma}{\beta e^{\gamma x}v_1(x)} - \frac{\theta-1}{v_2(x)} + \frac{\theta(a-1)v_2(x)}{v_3(x)} - \frac{a\theta}{v_4(x)} \quad (10)$$

The modes of the h(x) are the points x=x_0 witch provides d log{h(x)}/dx = 0 or, similarly, satisfy the equality established in (10). Then we can classify it as local maximum, local minimum or inflection point when we have, respectively, δ(x0) < 0, δ(x_0) > 0 and δ(x_0) = 0, where δ(x) =$d^2 \log\{h(x)\}/dx^2$. The second derivative of log{h(x)} is given by

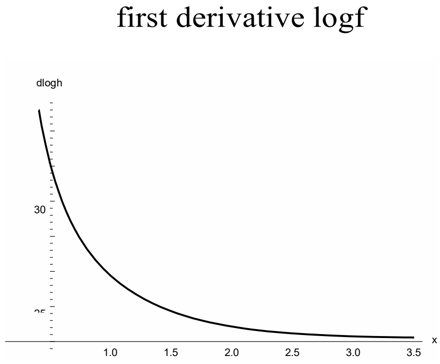
$$\frac{d^2\log\{f(x)\}}{dx^2} = -\beta e^{\gamma x}v_1(x)v_2^{\theta-1}(x)\Big\{(\theta-1)v_2^{-(\theta+1)}(x)[\beta e^{\gamma x}v_1(x) - v_2(x)(\gamma - \beta e^{\gamma x})]\Big\}$$
$$+\theta(a-1)v_3^{-2}(x)\left\{v_3(x)\left[\gamma - \beta e^{\gamma x}\left(1-(\theta-1)v_1(x)v_2^{\theta-1}(x)\right)\right]\right.$$
$$\left.+\beta\theta e^{\gamma x}v_1(x)v_2^{\theta-1}(x)\right\}$$
$$-\frac{a\theta(b-1)v_3^{a-1}(x)}{v_4^2(x)}\{v_4(x)[\gamma$$
$$-\beta e^{\gamma x}\left(1 - v_1(x)v_2^{-1}(x)\times\left[\theta-1-\theta(a-1)v_2^{\theta}(x)v_3^{-a}(x)\right]\right)]$$
$$-a\beta\theta e^{\gamma x}v_1(x)v_2^{\theta-1}(x)v_3^{a-1}(x)\}$$
$$-\frac{ab\theta v_3^{a-1}v_4^{b-1}}{v_5^2(x)}\{v_5(x)[\gamma$$
$$-\beta e^{\gamma x}\left(1-\theta v_1(x)v_2^{\theta-1}(x)v_3(x)\{a(b-1)v_4^{-1}(x)-(a-1)v_3^{-2}(x)\}\right)]$$
$$+ab\beta\theta e^{\gamma x}v_1(x)v_2^{\theta-1}(x)v_3^{a-1}(x)v_4^{b-1}(x)\}$$

where $v_5(x) = 1 - v_4^b(x)$

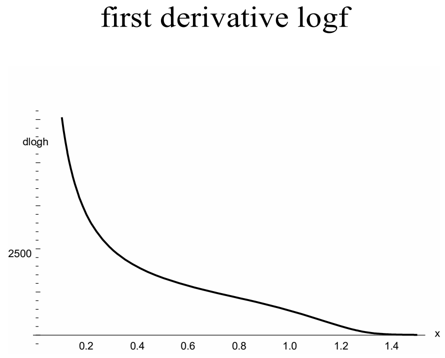
Of course, the degree of difficulty in obtaining analytical solutions for the equation (9) is similar to that presented for obtaining such solutions for (10). Fortunately, again in this situation, we can use solutions obtained numerically, through optimization routines in most mathematical and statistical platforms. Some plots of the first derivative of log{h(x)} for selected parameter values are displayed in Figure 4.

4 Quantile function

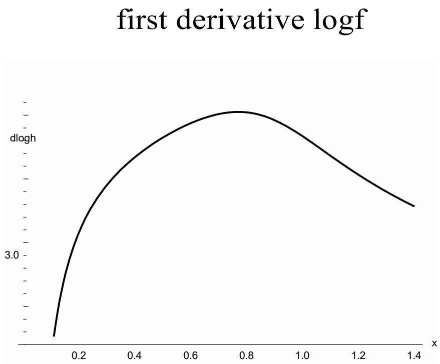
For many applications it is important to determine the quantile function (qf) of X. Based on this function, we can, for example, generate occurrences of X ~ EGEG (a, b, θ, γ, β), obtain



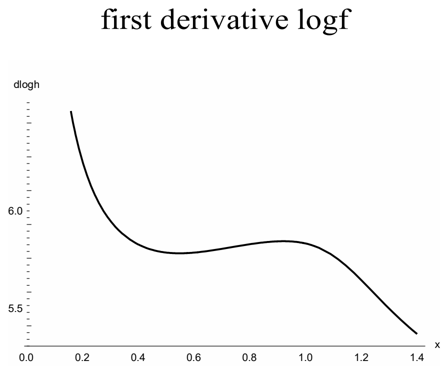
(a) a = 0.2, b = 10, θ = 2, γ = 0.5, β = 3



(b) a = 6, b = 40, θ = 6, γ = 2, β = 0.3



(c) a = 12, b = 0.4, θ = 2, γ = 2, β = 0.3



(d) a = 12, b = 0.4, θ = 4, γ = 2, β = 0.3

Figure 4: Plots of the EGEG d log{h(x)}/dx function for some parameter values

the median of the EGEG distribution and compute measures of skewness and kurtosis, among others. The qf of the EGEG distribution is obtained in an explicit form by inverting (5)

$$\mathrm{Q}(\mathrm{a},\mathrm{b},\theta,\gamma,\beta;\mu)=\frac{1}{\gamma}\log\{1-\frac{\gamma}{\beta}\log\{1-[1-(1-\mu^{\frac{1}{b}})^{\frac{1}{a}}]^{\frac{1}{\theta}}\}\} \qquad (11)$$

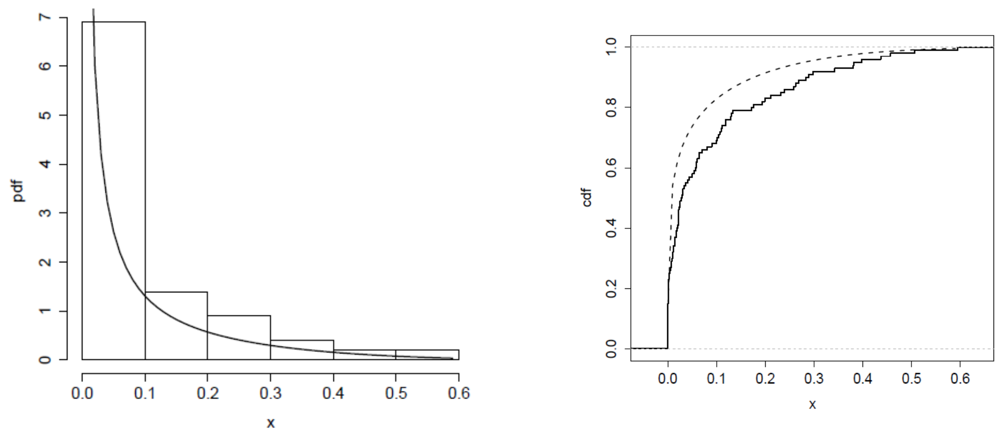
for $0<\mathrm{u}<1$

It is important to note that the qf has a closed form and is easily implemented in any programming language. Occurrences of the EGEG distribution, for example, are obtained by taking $\mathrm{X}=\mathrm{Q}(a_0,b_0,\theta_0,\gamma_0,\beta_0;\mathrm{U})$ for fixed $a_0,b_0,\theta_0,\gamma_0$ and β_0 parameters values and adopting uniform outcomes as inputs in (11). An algorithm for a random generator of the EGEG distribution can be write as follows:

- Step 1: Fix $a=a_0,\mathrm{b}=b_0,\theta=\theta_0,\gamma=\gamma_0,\beta=\beta_0$ and n = N where $a_0,b_0,\theta_0,\gamma_0$ and β_0 are arbitrary values chosen within the parametric space and n = N is the desired number of realizations of the random variable;
- Step 2: Fix i = 1;
- Step 3: Generate an occurrence of uniform distribution in the interval (0, 1), say ui;
- Step 4: Use (11) to compute Q($a_0,b_0,\theta_0,\gamma_0,\beta_0$; ui);

- Step 5: Record the value of the first realization of the random variable as xi = Q($a_0, b_0, \theta_0, \gamma_0, \beta_0$; ui);
- Step 6: While i < N , do i = i + 1 and restart the process from Step 3.

In order to provide a simple numerical example, we use (11) and the R language (https://www.r-project.org/) to generate 100 EGEG(1.5, 1.2, 0.2, 2, 2) random variables. Figure 5 shows the histogram and empirical cdf for the simulated data and also the exact pdf and cdf of X. These plots reinforce the adequacy model for practical applications.



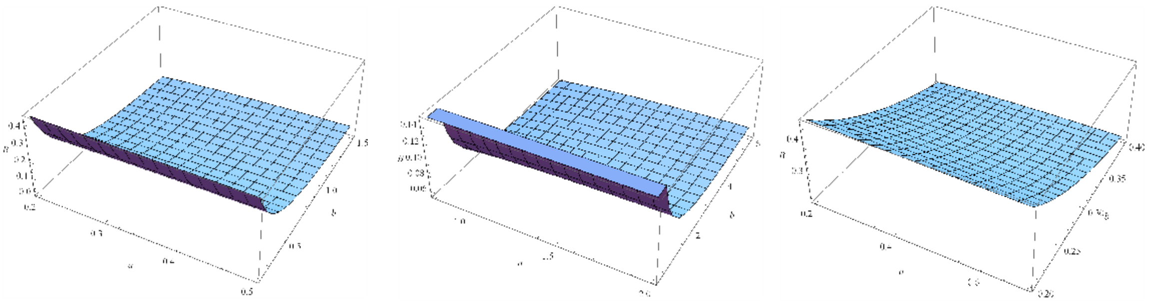
(a) Histogram and exact pdf (b) Empirical and exact cdfs

Figure 5: Plots of the EGEG(1.5, 1.2, 0.2, 2, 2) pdf, histogram, exact and empirical cdfs for simulated data with n = 100.

As a second application, we use the qf of X to determine the Bowley's skewness Kenney and Keeping (1962) (B) and Moors's kurtosis Moors (1988) (M). These measures are given by

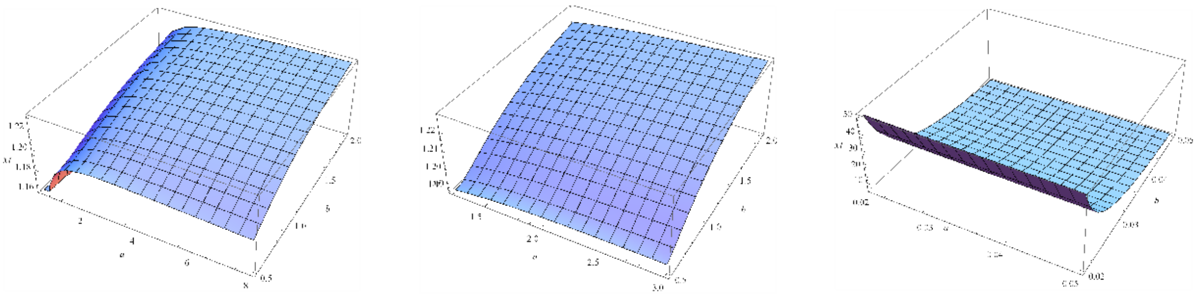
$$\mathrm{B} = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \quad and \quad M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

In Figures 6 and 7, we present 3D plots of the B and M measures for selected baseline parameter values. These plots are obtained using the Wolfram Mathematica software (http://www.wolfram.com/mathematica/). Based on these plots, it is possible to conclude that, for fixed baseline parameter values, changes in the additional parameters a and b have a considerable impact on the skewness and kurtosis of the EGEG model, thus corroborating for its greater flexibility. So, theses plots reinforce the importance of the additional parameters.



(a) θ = 2, γ = 0.5, β = 1.2 (b) θ = 2, γ = 0.5, β = 1.2 (c) θ = 2, γ = 0.5, β = 1.2

Figure 6: Plots of the Bowley's skewness for the EGEG model.



(a) θ = 2, γ = 0.5, β = 1.2 (b) θ = 2, γ = 0.5, β = 1.2 (c) θ = 2, γ = 0.5, β = 1.2

Figure 7: Plots of the Moors's kurtosis for the EGEG model.

5 Properties

5.1 useful representation

It is hardly necessary to emphasize the importance of the exponentiated distributions for the study of new lifetime models. To give an idea of the importance of this class, a recent paper by Tahir and Nadarajah (2015) lists over seventy works related to exponentiated distributions. Here, we refer to a few of these papers: Nadarajah and Gupta (2007) for exponentiated Gamma, Carrasco et al. (2008) for exponentiated modified Weibull and Cordeiro et al. (2011) for exponentiated generalized Gamma.

For an arbitrary continuous baseline cdf G(x), a random variable Y is said to have the exponentiated-G ("exp-G" for short) distribution with power parameter a > 0, say $Y \sim \exp - G(a)$, if its cdf and pdf are $H_a(\mathrm{x}) = \mathrm{G(x)}^a$ and $h_a(x) = ag(x)\mathrm{G(x)}^{a-1}$, respectively. Thus, "exp-G" denotes the Lehmann type I transformation of G(x). Based on some results in Cordeiro and Lemonte (2014), we can express the Cordeiro et al. (2013)'s cdf (3) as

$$\mathrm{F(x)} = \sum_{j=0}^{\infty} w_{j+1} H_{j+1}(x) \qquad (12)$$

where $w_{j+1} = \sum_{m=1}^{\infty} (-1)^{j+m+1} \binom{b}{m} \binom{ma}{j+1}$ and $H_{j+1}(x) = G(x)^{j+1}$ is the exp-G cdf with power parameter $j+1$. By differentiating (12), we obtain a similar mixture representation for f(x) as

$$\mathrm{f(x)} = \sum_{j=0}^{\infty} w_{j+1} h_{j+1}(x) \tag{13}$$

where $h_{j+1}(x) = \frac{dH_{j+1}(x)}{dx}$.

By using (12) and (13) for the EG distribution (1), $h_{j+1}(x)$ becomes the exp-Gompertz pdf with power parameter $\theta(j+1)$ (for j ≥ 0) or, using our notation, we obtain the extended Gompertz pdf with power parameter $\theta(j+1)$ (for j ≥ 0) given by

$$h_{\theta(j+1)}(x) = \theta(j+1)\beta e^{\gamma x} e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{\theta(j+1)-1} \tag{14}$$

Combining equations (13) and (14) we have an important result: the EGEG density function is a mixture of extended Gompertz densities. This result can be used to derive some mathematical properties of X from those of the EG distribution proposed by El-Gohary et al. (2013).

5.2 Ordinary moments

The ordinary moments of a distribution play an important role in applications. The nth moment of X can be determined based on (13) as

$$\mathrm{E}(X^r) = \sum_{j=0}^{\infty} w_{j+1}\, E\left(Y_{\theta(j+1)}^r\right) \tag{15}$$

where $Y_{\theta(j+1)}$ denotes a random variable that follows the EG distribution with power parameter $\theta(j+1)$. The rth moment of $Y_{\theta(j+1)}$, say $\mathrm{E}(Y_\theta^r)$, can be obtained in recent papers by El-Gohary et al. (2013) and Jafari et al. (2014), adopting a convenient manipulations of indexes.

5.3 Moment generating function

The moment generating function (mgf) of X can be obtained from (13) as

$$M_x(t) = \sum_{j=0}^{\infty} w_{j+1} M_{Y_{\theta(j+1)}(t)} \tag{16}$$

where $Y_{\theta(j+1)}$ denotes a random variable that follows the EG distribution with power parameter $\theta(j+1)$ and $M_{Y_{\theta(j+1)}}(t)$ is the mgf of $Y_{\theta(j+1)}$, that can be obtained from Jafari et al. (2014).

5.4 Dual generalized Order statistics

The concept of generalized order statistics, which seems to have been introduced first by Kamps (1995), is extremely useful, especially in applied works. Order statistics, upper record values and sequential order statistics are some of the important results obtained from generalized order statistics studies. In a recent work, Burkschat et al. (2003) made an important contribution in this line of research and introduced the concept of dual generalized order statistics (dgos) as a model for descending ordered random variables and admits the special cases reversed ordered statistics, lower k-records and lower Pfeifer records.

In this Section, we broke the frontier of knowledge in this line of research and present general expressions for dgos for the exponentiated generalized class. Thus, we derive an explicit expression for the density of the rth ($1 \leq r \leq n$) dual generalized order statistic from a random sample of size n following a distribution in the Cordeiro et al. (2013)'s class. Besides that, we present particularized results for the EGEG distribution, obtaining dgos to this model. These two outcomes are the main contributions of this Section.

Suppose a random sample of size n, $X_1, \ldots, X_n$, from Cordeiro et al. (2013)'s class. We derive an explicit expression for the density of the rth ($1 \leq r \leq n$) dual generalized order statistic $X^*(r,n,m,k)$, say $f_{X^*(r,n,m,k)}(x)$, from this sample. Using the definition of the dgos, the $f_{X^*(r,n,m,k)}(x)$ can be expressed as

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} F^{\gamma r-1}(x) g_m^{r-1}(F(x)) f(x) \qquad (17)$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i, \gamma_r = k + (n-r)(m+1) \geq 1, g_m(\mu) = h_m(\mu) - h_m(1), \mu \epsilon [0,1]$ and

$$h_m(\mu) = \begin{cases} -\dfrac{1}{m+1}\mu^{m+1} & , if \ \ m \neq -1 \\ -log\mu, if \ \ m = -1 \end{cases}$$

with $k \geq 1, m \epsilon \mathcal{R}$.

Next, according to the equation (17), we can rewrite the $f_{X^*(r,n,m,k)}(x)$ considering two cases, as shown below

$$f_{X^*(r,n,m,k)}(x) = \begin{cases} \dfrac{C_{r-1}}{(r-1)!} F^{\gamma r-1}(x) \{\dfrac{1}{m+1}[1 - F^{m+1}(x)]\}^{r-1} f(x), if \ \ m \neq -1 \\ \dfrac{C_{r-1}}{(r-1)!} F^{\gamma r-1}(x) \{-\log[F(x)]\}^{r-1} f(x), if \ \ m = -1 \end{cases} \qquad (18)$$

Then, we will investigate the equation (18), considering f (x) and F (x) to be, respectively, the density and the distribution functions of theCordeiro et al. (2013)'s class.

Case I: $m \neq 1$

Using the binomial expansion in the first sentence of (18) and inserting cdf (3) and pdf (4), we readily obtain

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}(m+1)^{1-r}}{(r-1)!}\sum_{p=0}^{r-1}(-1)^p\binom{r-1}{p}[1-G(x)]^{a-1}\times\{1 - [1-G(x)]^a\}^{b[\gamma_r+p(m+1)]-1}g(x) \quad (19)$$

Now, we consider the generalized binomial expansion

$$(1-\mathrm{z})^\beta = \sum_{k=0}^{\infty}(-1)^k\binom{\beta}{k}z^k \quad (20)$$

which holds for any real non-integer β and |z| < 1.

Using (20) twice in equation in (19) and after simple algebric manipulation, we write $f_{X^*(r,n,m,k)}(x)$ as

$$f_{X^*(r,n,m,k)}(x) = \sum_{s=0}^{\infty}\varepsilon_{p,q}(m)\,h_{s+1}(x) \quad (21)$$

where

$$\varepsilon_{p,q}(m) = \sum_{p=0}^{r-1}\sum_{q=0}^{\infty}\frac{(-1)^{p+q+s}abC_{r-1}(m+1)^{1-r}\Gamma(a[1+q])\Gamma(b[\gamma_r+p(m+1)])}{(s+1)q!\,s!\,(r-1)!\,\Gamma(a[1+q]-s)\Gamma(b[\gamma_r+p(m+1)-q])}\binom{r-1}{p}$$

Case II: $\mathrm{m} = 1$

By expanding the logarithm function in power series and then using an equation for a power series raised to a positive integer given in Gradshteyn and Ryzhik (2007) (Section 0.314), we have

$$\{-\log[-\log[\mathrm{F(x)}]]\}^{r-1} = \{-\log[-1-\bar{\mathrm{F}}(X)]\}^{r-1}$$
$$= \sum_{p=0}^{\infty}\sum_{q=0}^{p+r-1}(-1)^q\,c_{r-1,p}\binom{p+r-1}{q}F^q(x) \quad (22)$$

where $\bar{F}(x) = 1 - F(x)$ and the coefficients $c_{r-1,p}(\mathrm{for.p} = 1,2..)$ are determined from the recurrence expression

$$c_{r-1,p} = (\mathrm{p}a_0)^{-1}\sum_{v=1}^{p}[v\gamma - p]a_v\,c_{r-1,p-v}$$

with $c_{r-1,0} = a_0^{r-1}$

From the equation (22), the second sentence of (18) reduces to

$$f_{X^*(r,n,m,k)} = \frac{C_{r-1}}{(r-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{p+r-1}(-1)^q\,c_{r-1,p}\binom{p+r-1}{q}F^{\gamma_r+q-1}(x)f(x). \quad (23)$$

Inserting (3) and (4) in the previous equation and applying expression (20) twice the pdf $f_{X^*(r,n,m,k)}$ can be expressed as

$$f_{X^*(r,n,m,k)} = \sum_{j=0}^{\infty} \varepsilon'_{p,q,s}\,(\mathrm{m}) h_{j+1}(x) \tag{24}$$

where

$$\varepsilon_{p,q,s}(m) = \sum_{s,p=0}^{\infty} \sum_{q=0}^{p+r-1} \frac{(-1)^{q+s+j} abC_{r-1}c_{r-1,p}\Gamma(a[s+1])\Gamma(b[\gamma_r+q])}{(j+1)!\,j!\,s!\,(r-1)!\,\Gamma(a[s+1]-j)\Gamma(b[\gamma_r+q]-s)} \binom{p+r-1}{p}$$

Our work on obtaining an explicit expression for the density of the rth ($1'r \leq n$) dual generalized order statistic from a random sample of size n following a distribution in the Cordeiro et al. (2013) 's class is complete and can be summarized as follows

$$f_{X^*(r,n,m,k)}(x) = \begin{cases} \sum_{s=0}^{\infty} \varepsilon_{p,q}(m)\, h_{\theta(s+1)}(x), if \;\; m \;\; \neq -1 \\ \sum_{j=0}^{\infty} \varepsilon'_{p,q,s}(m)\, h_{\theta(j+1)}(x),, if \;\; m \;\; = -1 \end{cases} \tag{25}$$

The equation (25) above is the first important result of this Section and probably one of the most important contributions of this paper. This equation provides a general expression for the density of the dual generalized order statistic from a random sample that following a distribution in the exponentiated generalized class. It is, therefore, a new result that had not been considered in the paper presented by Cordeiro et al. (2013).

Next, as a second important result, the density of the rth ($1'r \leq n$)dual generalized order statistic from a random sample of size n following a EGEG distribution arise immediately from the equation (25), when we consider $h_{\theta(s+1)}(x)$ and $h_{\theta(j+1)}(x)$, respectively, as the extended Gompertz pdf with power parameter θ(s+1) and θ(j+1). Several mathematical quantities of the EGEG dgos, as the incomplete and factorial moments, mgf, mean deviations, among others, can be easily obtained from those quantities of EG distribution. For example the tth moment dgos of the EGEG distribution can be expressed from (25) as

$$\mathrm{E}\big(X^{*t}(\mathrm{r,n,m,k})\big) = \begin{cases} \sum_{s=0}^{\infty} \varepsilon_{p,q}(m)\, E[Y^t_{\theta(s+1)}], if \;\; m \;\; \neq -1 \\ \sum_{j=0}^{\infty} \varepsilon'_{p,q,s}(m)\, E[Y^t_{\theta(j+1)}],, if \;\; m \;\; = -1 \end{cases}$$

where

$$\mathrm{E}\big[Y^t_\eta\big] = \eta\beta\Gamma(\mathrm{t}+1) \sum_{l,d=0}^{\infty} \binom{\eta-1}{l} \frac{(-1)^{l+d}}{\Gamma(\mathrm{d}+1)} \frac{e^{\frac{\beta}{\gamma}(l+1)}}{[\frac{\beta}{\gamma}(l+1)]^{-d}} (\frac{-1}{\gamma(d+1)})^{t+1}$$

and $Y_\eta \sim EG$.

It is well known that if $\mathrm{m}=0, \mathrm{k}=1$ then $X^*(\mathrm{r,n,m,k})$reduces to the $\mathrm{n}-\mathrm{r}+1$ th order statistics, $X_{n-r+1:n}$ from the sample $X_1,..,X_n$ and when, $\mathrm{m}=-1$ then $X^*(\mathrm{r,n,m,k})$ reduces to the rth lower k-record value.

5.5 Order statistics

The density function $f_{i:n}(x)$ of the ith order statistic, say $X_{i:n}$ for $\mathrm{i}=1,\dots,\mathrm{n}$, from a random sample $X_1,..,X_n$ having the Cordeiro et al. (2013)'s class can be expressed as

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}F(x)^{i-1}[1-F(x)]^{n-i}$$

where f(x) is the pdf (4) and F(x) is the cdf (3).

Applying the binomial expansion in the last equation, we have

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}f(x)F(x)^{i+j-1} \quad (26)$$

Substituting (3) and (4) in equation (26) and applying the generalized binomial expansion (20), we can write

$$f_{i:n}(x)=\frac{ab}{B(i,n-i+1)}\sum_{l=0}^{\infty}\sum_{j=0}^{n-i}\sum_{k=0}^{\infty}(-1)^{j+k+l}\binom{n-i}{j}\binom{b(i+j)-1}{k}\binom{a(k+1)-1}{l} \times g(x)G(x)^l$$

Applying the binomial expansion in the last equation, we have

$$f_{i:n}(x)=\sum_{l=0}^{\infty}q_l\,h_{l+1}(x) \quad (27)$$

where q_l is given by

$$q_l=\frac{ab}{B(i,n-i+1)}\sum_{j=0}^{n-i}\sum_{k=0}^{\infty}\frac{(-1)^{j+k+l}}{l+1}\binom{n-i}{j}\binom{b(i+j)-1}{k}\binom{a(k+1)-1}{l}$$

and $h_{l+1}(x)$ denotes the exp-G density function with power parameter l+1 (for $l\geq 0$).

Equation (27) reveals that the density function of the Cordeiro et al. (2013)'s order statistic is a linear mixture of exp-G densities. We emphasize that this result is not new and has already been presented by Cordeiro et al. (2013). However, we now give an alternative way of expressing the weights that compose this linear combination.

In order to specify the previous expression, in order to obtain the density function fi:n(x) of the ith order statistic, say $X_{i:n}$ for i = 1, . . . , n, from a random sample $X_1,\dots,X_n$ having the EGEG distribution, just consider the $h_{l+1}(x)$ as the extended Gompertz pdf with power parameter $l+1$ (for $l\geq 0$) in the equation (27).

5.6 Reliability

Type $\mathrm{P}(X_1 > X_2)$ probabilities frequently arise in applied studies, especially in life time works. Here, we derive the reliability, say R, when $X_1 \sim \mathrm{EGEG}(a_1, b_1, \theta, \gamma, \beta)$ and $X_2 \sim \mathrm{EGEG}(a_2, b_2, \theta, \gamma, \beta)$ are two independent random variables. Let $f_1(x)$ denote the pdf of X_1 and $F_2(x)$ denote the cdf of X_2. The reliability can be expressed as $R = \mathrm{P}(X_1 > X_2) = \int_0^\infty f_1\, F_2(x) dx$ and using equations (12) and (13) gives

$$R = \sum_{j,k=0}^{\infty} I_{j,k} \int_0^\infty h_{j+1}(x) H_{k+1}(x) dx,$$

where $\mathcal{I}_{j,k} = \sum_{m,n=1}^{\infty} (-1)^{j+k+m+n+2} \binom{b_1}{m} \binom{m\ a_1}{j+1} \binom{b_2}{n} \binom{na_2}{k+1}$.

Thus, the reliability of X reduces to

$$R = \sum_{j,k=0}^{\infty} \frac{(j+1) I_{j,k}}{(j+k+2)}. \tag{28}$$

Table 1 gives some values of R for different parameter values. Clearly, for $a_1 = a_2$ and $b_1 = b_2$, we obtain $R = \mathrm{P}(X_1 > X_2) = 1/2$. All computations are done using *Wolfram Mathematica* software by taking the upper limits equal to 30 in (28).

Table 1: The reliability of X for $(a_1 = 2, a_2 = 2)$ and some values of b_1 and b_2.

b_2 / b_1	2	3	4	5	6
2	0.50000	0.40000	0.33333	0.28571	0.25000
3	0.60000	0.50000	0.42857	0.37500	0.33333
4	0.66667	0.57143	0.50000	0.44444	0.40000
5	0.71429	0.62500	0.55556	0.50000	0.45455
6	0.75000	0.66667	0.60000	0.54545	0.50000

6 Estimation and inference

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters $a, b, \theta, \gamma,$ and β of the EGEG distribution from complete samples only by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample of size n from the EGEG distribution. The log-likelihood function for the vector of parameters $\theta = (a, b, \theta, \gamma, \beta)^T$, say $\ell(\boldsymbol{\theta})$, can be expressed as

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= n\log(ab\beta\theta) + \gamma\sum_{i=1}^{n} x_i - \frac{\beta}{\gamma}\sum_{i=1}^{n}(e^{\gamma x_i} - 1)\\ &+(\theta-1)\sum_{i=1}^{n}\log\left[1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_i}-1)}\right]\\ &+(a+1)\sum_{i=1}^{n}\log\left\{1-\left[1-e^{-\frac{\beta}{\gamma}(e^{\gamma x_i}-1)}\right]^{\theta}\right\}\\ &+(b-1)\sum_{i=1}^{n}\log\left(1-\left\{1-\left[1-e^{-\frac{\beta}{\gamma}(e^{\gamma x_i}-1)}\right]^{\theta}\right\}^{a}\right).\end{aligned} \tag{29}$$

The components of the score function are:

$$\frac{\partial\ell(\theta)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log\left[v_3(x_i)\right] - (b-1)\sum_{i=1}^{n}\frac{v_3^a(x_i)\log[v_3(x_i)]}{v_4(x_i)},$$

$$\frac{\partial\ell(\theta)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left[v_4(x_i)\right],$$

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial \beta} &= \frac{n}{\beta} - \frac{1}{\gamma}\sum_{i=1}^{n} v_2(x_i) + \frac{\theta-1}{\gamma}\sum_{i=1}^{n}\frac{v_1(x_i)(e^{\gamma x_i}-1)}{v_2(x_i)}\\ &-\frac{\theta(a-1)}{\gamma}\sum_{i=1}^{n}\frac{v_1(x_i)v_2^{\theta-1}(e^{\gamma x_i}-1)}{v_3(x_i)}\\ &+\frac{a\theta(b-1)}{\gamma}\sum_{i=1}^{n}\frac{v_1(x_i)v_2^{\theta-1}v_3^{a-1}(e^{\gamma x_i}-1)}{v_4(x_i)},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial \gamma} &= \sum_{i=1}^{n} x_i - \frac{\beta}{\gamma}\sum_{i=1}^{n} x_i\, e^{\gamma x_i} + \frac{\beta}{\gamma^2}\sum_{i=1}^{n}(e^{\gamma x_i}-1) - (\theta-1)\sum_{i=1}^{n}\frac{v_1(x_i)\eta(x_i)}{v_2(x_i)}\\ &+\theta(a+1)\sum_{i=1}^{n}\frac{v_1(x_i)v_2^{\theta-1}(x_i)\eta(x_i)}{v_3(x_i)} - a\theta(b-1)\sum_{i=1}^{n}\frac{v_1(x_i)v_2^{\theta-1}(x_i)v_3^{a-1}\eta(x_i)}{v_4(x_i)},\end{aligned}$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log\left[v_2(x_i)\right] - (a-1)\sum_{i=1}^{n} \frac{v_2^{\theta}(x_i)\log[v_2(x_i)]}{v_3(x_i)}$$

$$+a(b-1)\sum_{i=1}^{n} \frac{v_2^{\theta}(x_i)v_3^{a-1}(x_i)\log[v_2(x_i)]}{v_4(x_i)}$$

where $v_1(x_i) = e^{-\frac{\beta}{\gamma}(e^{\gamma x_i}-1)}, v_2(x_i) = 1 - v_1(x_i), v_3(x_i) = 1 - v_2^{\theta}(x_i), v_4(x_i) = 1 - v_3^{a}(x_i)$ and $\eta(x_i) = \frac{\beta}{\gamma^2}[-1 + e^{\gamma x_i}(1 - \gamma x_i)]$.

We determine the 4×4 observed information matrix given by $J(\boldsymbol{\theta}) = \{-U_{rs}\}$, whose elements $U_{rs} = \partial^2 \ell(\theta)/(\partial r \partial s)$ for $r, s \in \{a, b, \theta, \gamma, \beta\}$ can be obtained from the authors upon request.

7 Simulation study

In this Section, we verify if the parameter estimates are obtained with precision since the inferences and the decision processes will depend directly on the quality of the estimates. In this context, one of the most used simulation methods to evaluate the performance of estimators is by Monte Carlo simulation, see, for example, the following works: Lemonte (2013), Cordeiro and Lemonte (2014) and De Andrade et al. (2015).

We investigate the behavior of the MLEs for the parameters of the EGEG model by generating from (11) samples sizes n = 100, 300, 500, 1000 with selected values for a, b, θ, γ and β. We consider 5, 000 Monte Carlo replications. The simulation process is performed in the R software using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) maximization method in the optim script. To ensure the reproducibility of the experiment, we use the seed for the random number generator: set.seed (103).

The results of the simulations are presented in Table 2, including the means, biases, variances and the mean square error (MSE). These results reveal that the EGEG estimates have desirable properties even for small to moderate sample sizes. In general, the MSE decrease as the sample size increases, as expected.

Table 2: Means, biases, variances and MSE of $\hat{a}, \hat{b}, \hat{\beta}, \hat{\theta}$ and $\hat{\gamma}$ for the EGEG model ($a = 1.5, b = 1.5, \beta = 2.0, \theta = 1.5$ and $\gamma = 2$ as a true parameter values).

n	Parameter	Mean	Biases	Variances	MSE
100	a	1.3826	-0.1174	1.8724	3.5197
	b	1.2323	-0.2677	1.1904	1.4887
	β	3.0508	1.0508	1.8110	4.3837
	θ	2.8886	1.3886	1.7632	5.0371
	γ	3.3968	1.3968	1.8013	5.1958
300	a	1.8775	0.3775	1.9095	3.7889
	b	1.5345	0.0345	1.0279	1.0578
	β	2.4971	0.4971	1.2164	1.7268
	θ	2.1120	0.6120	1.3799	2.2786
	γ	2.3997	0.3997	1.1904	1.5768
500	a	1.7987	0.2987	1.6972	2.9695
	b	1.5836	0.0836	0.8583	0.7437
	β	2.5248	0.5248	1.2018	1.7196
	θ	1.9703	0.4703	1.3064	1.9279
	γ	2.2054	0.2054	0.9129	0.8756
1000	a	1.9915	0.4915	1.5825	2.7459
	b	1.6945	0.1945	0.7878	0.6585
	β	2.2327	0.2327	0.8377	0.7559
	θ	1.7188	0.2188	1.0643	1.1806
	γ	1.9448	-0.0552	0.6001	0.3632

8 Real data set applications: the power of adjustment of the EGEG model

The estimation and inference process is valuable for statisticians and applied researchers. Several estimation methods are available in the literature, and the maximum likelihood method is probably one of the most used. In this Section, we consider three real life data sets to show that our proposed EGEG model can provide better fit than its corresponding sub models. For this reason, we fit EGEG$(a, b, \theta, \gamma, \beta)$ extended Gompertz $[\mathrm{EG}(\theta, \gamma, \beta)]$ and Gompertz $[\mathrm{Gom}(\gamma, \beta)]$ models and compare the results for the three data sets:

Data set I: The data consist in 346 nicotine measurements made from several brands of cigarettes in 1998. These data have been collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection [http: //www.ftc.gov/reports/tobacco or http://pw1.netcom.com/rdavis2/smoke.html.]:

1.3, 1.0, 1.2, 0.9, 1.1, 0.8, 0.5, 1.0, 0.7, 0.5, 1.7, 1.1, 0.8, 0.5, 1.2, 0.8, 1.1, 0.9, 1.2, 0.9, 0.8, 0.6, 0.3, 0.8, 0.6, 0.4, 1.1, 1.1, 0.2, 0.8, 0.5, 1.1, 0.1, 0.8, 1.7, 1.0, 0.8, 1.0, 0.8, 1.0, 0.2, 0.8, 0.4, 1.0, 0.2, 0.8, 1.4, 0.8, 0.5, 1.1, 0.9, 1.3, 0.9, 0.4, 1.4, 0.9, 0.5, 1.7, 0.9, 0.8, 0.8, 1.2, 0.9, 0.8, 0.5, 1.0, 0.6, 0.1, 0.2, 0.5, 0.1, 0.1, 0.9, 0.6, 0.9, 0.6, 1.2, 1.5, 1.1, 1.4, 1.2, 1.7, 1.4, 1.0, 0.7, 0.4, 0.9, 0.7, 0.8, 0.7, 0.4, 0.9, 0.6, 0.4, 1.2, 2.0, 0.7, 0.5, 0.9, 0.5, 0.9, 0.7, 0.9, 0.7, 0.4, 1.0, 0.7, 0.9, 0.7, 0.5, 1.3, 0.9, 0.8, 1.0, 0.7, 0.7, 0.6, 0.8, 1.1, 0.9, 0.9, 0.8, 0.8, 0.7, 0.7, 0.4, 0.5, 0.4, 0.9, 0.9, 0.7, 1.0, 1.0, 0.7, 1.3, 1.0, 1.1, 1.1, 0.9, 1.1, 0.8, 1.0, 0.7, 1.6, 0.8, 0.6, 0.8, 0.6, 1.2, 0.9, 0.6, 0.8, 1.0, 0.5, 0.8, 1.0, 1.1, 0.8, 0.8, 0.5, 1.1, 0.8, 0.9, 1.1, 0.8, 1.2, 1.1, 1.2, 1.1, 1.2, 0.2, 0.5, 0.7, 0.2, 0.5, 0.6, 0.1, 0.4, 0.6, 0.2, 0.5, 1.1, 0.8, 0.6, 1.1, 0.9, 0.6, 0.3, 0.9, 0.8, 0.8, 0.6, 0.4, 1.2, 1.3, 1.0, 0.6, 1.2, 0.9, 1.2, 0.9, 0.5, 0.8, 1.0, 0.7, 0.9, 1.0, 0.1, 0.2, 0.1, 0.1, 1.1, 1.0, 1.1, 0.7, 1.1, 0.7, 1.8, 1.2, 0.9, 1.7, 1.2, 1.3, 1.2, 0.9, 0.7, 0.7, 1.2, 1.0, 0.9, 1.6, 0.8, 0.8, 1.1, 1.1, 0.8, 0.6, 1.0, 0.8, 1.1, 0.8, 0.5, 1.5, 1.1, 0.8, 0.6, 1.1, 0.8, 1.1, 0.8, 1.5, 1.1, 0.8, 0.4, 1.0, 0.8, 1.4, 0.9, 0.9, 1.0, 0.9, 1.3, 0.8, 1.0, 0.5, 1.0, 0.7, 0.5, 1.4, 1.2, 0.9, 1.1, 0.9, 1.1, 1.0, 0.9, 1.2, 0.9, 1.2, 0.9, 0.5, 0.9, 0.7, 0.3, 1.0, 0.6, 1.0, 0.9, 1.0, 1.1, 0.8, 0.5, 1.1, 0.8, 1.2, 0.8, 0.5, 1.5, 1.5, 1.0, 0.8, 1.0, 0.5, 1.7, 0.3, 0.6, 0.6, 0.4, 0.5, 0.5, 0.7, 0.4, 0.5, 0.8, 0.5, 1.3, 0.9, 1.3, 0.9, 0.5, 1.2, 0.9, 1.1, 0.9, 0.5, 0.7, 0.5, 1.1, 1.1, 0.5, 0.8, 0.6, 1.2, 0.8, 0.4, 1.3, 0.8, 0.5, 1.2, 0.7, 0.5, 0.9, 1.3, 0.8, 1.2, 0.9

Data set II: The data represents the times of failures and running times for samples of devices from an eld-tracking study of a larger system. The data has been previously studied by Meeker and Escobar (1988). The data has 30 observations:

2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66

Data set III: The data set was first obtained by Smith and Naylor (1987). These data consist in 63 observations the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England:

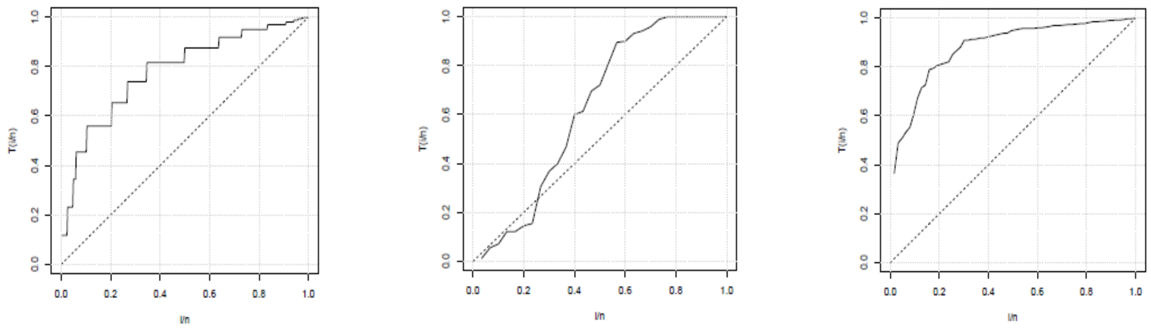
0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24

Some descriptive statistics for the three data sets considered are presented in Table 3, including mean, median, variance, skewness, among others. The graphs of total test time (TTT curves) to these data are presented in Figure 8.

For all fitted models, we compute the MLEs of the model parameters (with the correspondning standard errors in parentheses) and also the values of the Akaike information criterion

Table 3: Descriptive statistics for the data sets.

Statistic	Data set I	Data set II	Data set II
n	346	30	63
Mean	0:85260	1:77033	1:50683
Median	0:90000	1:96500	1:59000
Variance	0:11201	1:32232	0:10506
Skewness	0:17219	-0:28405	-0:89993
Kurtosis	0:31555	-1:54634	0:92376
Minimum	0:10000	0:02000	0:55000
Maximum	2:00000	3:00000	2:24000



(a) TTT curve to data set I (b) TTT curve to data set II (c) TTT curve to data set III

Figure 8: The graphs of total test time (TTT curves).

(AIC), Hannan-Quinn information criterion (HQIC) and consistent Akaike information crite -rion (CAIC) as a methods of comparing fits of distributions to data. In general, it is considered that lower values of these statistics indicate the better fit to the data. Besides that, since the EGEG distribution reduces to $\mathrm{EG}(\theta,\gamma,\beta)$ when $a=b=1$, to $Gom(\gamma,\beta)$ for $a=b=\theta=1$, we consider the likelihood ratio (LR) test for nested models to check the following hypotheses:

- $\mathrm{H}_0: a=b=1$ that is the sample is from $\mathrm{EG}(\theta,\gamma,\beta)\times \mathrm{H}_1: a\neq 1, b\neq 1$, that is, the sample is from $\mathrm{EGEG}(a,b,\theta,\gamma,\beta)$.
- $\mathrm{H}_0: a=b=\theta=1$ that is the sample is from $\mathrm{Gom}(\gamma,\beta)\times \mathrm{H}_1: a\neq 1, b\neq 1, \theta\neq 1$, that is, the sample is from $\mathrm{EGEG}(a,b,\theta,\gamma,\beta)$.

To the test of the above hypotheses, the LR test statistic is given by $\mathrm{LR}=-2\log\left[L(\hat{\theta}^*;x)/L(\hat{\theta};x)\right]$, where $\theta,\hat{\theta}^*$ is the restricted MLEs under the null hypothesis H_0 and $\hat{\theta}$ is the unrestricted MLEs under the alternative hypothesis H_1. Under the null hypothesis, the LR criterion follows chi-square distribution. The null hypothesis can not be accepted for p-value less than 0.05.

Table 4 and Table 5 present the results related to the first data set. Table 4 lists the MLEs of the model parameters (with the corresponding standard errors and confidence intervals in parentheses) for all fitted models. Table 5 present the values of the values of the AIC, HQIC, CAIC and the LR test statistics. These figures in this Tables reveals that the EGEG model has the lowest AIC, HQIC, CAIC and the LR values among all fitted models. Thus, the proposed distribution is the best model to explain these data. To a visual comparison, Figure 9 displays the histogram of the data and the estimated pdf and cdf for all fitted models. These plots reveal that the proposed model is quite suitable for these data.

Table 4: MLEs, standard errors, confidence interval (in parentheses) for the data set I.

Models	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\beta}$
$\mathrm{Gom}(\gamma,\beta)$	-	-	-	0.237	2.422
				(0.028)	(0.127)
				(0.18,0.29)	(2.17,2.67)
$\mathrm{EG}(\theta,\gamma,\beta)$	-	-	2.188	1.484	0.771
			(0.308)	(0.188)	(0.152)
			(1.58, 2.79)	(1.12, 1.85)	(0.47, 1.07)
$\mathrm{EGEG}(a,b,\theta,\gamma,\beta)$	2.727	0.881	2.525	1.165	0.585
	(3.965)	(0.827)	(2.535)	(0.547)	(0.472)
	(0, 10.49)	(0, 2.50)	(0, 7.49)	(0.09,2.24)	(0, 1.51)

Table 5: AIC, CAIC, HQIC and LR (P-value) values for the data set I.

Models	$-l_{max}$	*AIC*	*CAIC*	*HQIC*	*LR (p-value)*
$\mathrm{Gom}(\gamma,\beta)$	128.35	260.70	260.73	263.78	34.68 (0.001)
$\mathrm{EG}(\theta,\gamma,\beta)$	115.04	236.08	236.15	240.70	8.06 (0.02)
$\mathrm{EGEG}(a,b,\theta,\gamma,\beta)$	111.01	232.02	232.19	239.72	-

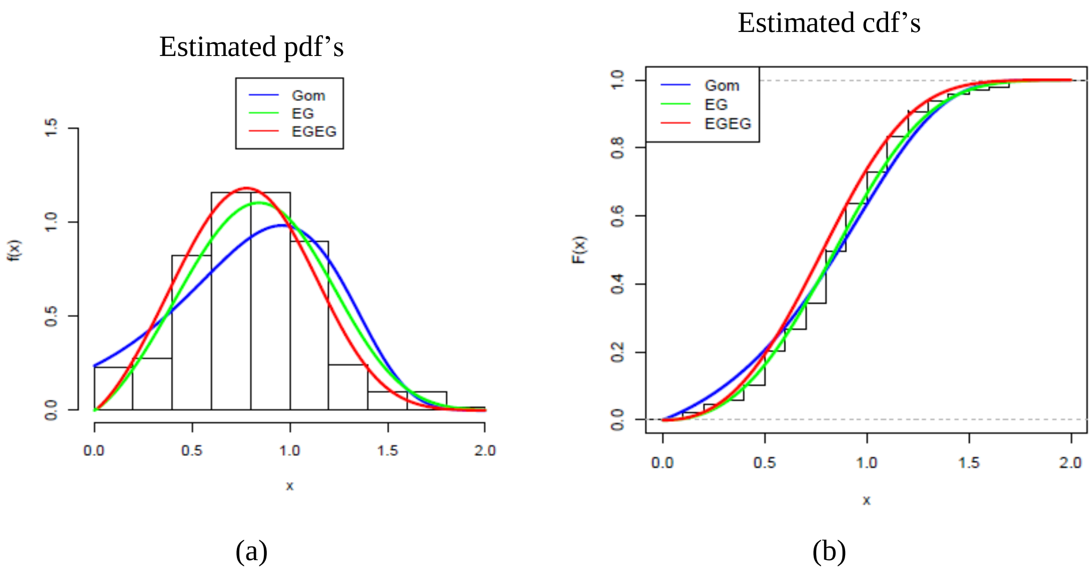


Figure 9: Plots of the estimated pdf and cdf of the models for the data set I.

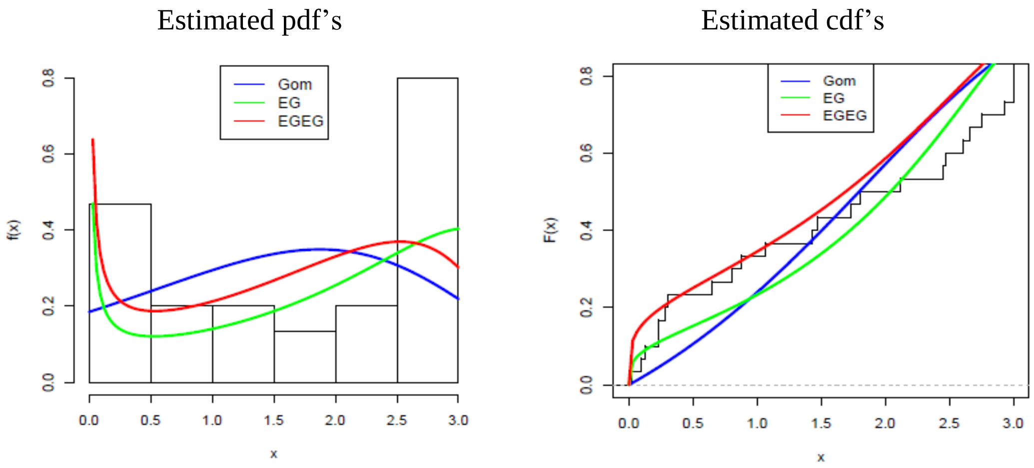
Table 6 and Table 7 present the results related to the second data set. Table 6 lists the MLEs of the model parameters (with the corresponding standard errors and confidence intervals in parentheses) for all fitted models. Table 7 present the values of the values of the AIC, HQIC, CAIC and the LR test statistics. These figures in this Tables reveals that the EGEG model has the lowest AIC, HQIC, CAIC and the LR values among all fitted models. Thus, the proposed distribution is the best model to explain these data. To a visual comparison, Figure 10 displays the histogram of the data and the estimated pdf and cdf for all fitted models. These plots reveal that the proposed model is quite suitable for these data.

Table 6: MLEs, standard errors, con_dence interval (in parentheses) for the data set II.

Models	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\beta}$
$\text{Gom}(\gamma,\beta)$	-	-	-	0.185	0.739
				(0.080)	(0.223)
				(0.03, 0.34)	(0.30,1.18)
$\text{EG}(\theta,\gamma,\beta)$	-	-	0.270	2.487	0.0005
			(0.116)	(3.201)	(0.0001)
					(0.0003,0.0007)
$\text{EGEG}(a,b,\theta,\gamma,\beta)$	0.043	0.291	1.191	2.107	0.098
	(0.012)	(0.066)	(0.015)	(0.009)	(0.007)
	(0.02, 0.07)	(0.16,0.42)	(1.16,1.22)	(2.09, 2.12)	(0.08,0.11)

Table 7: AIC, CAIC, HQIC and LR (P-value) values for the data set II.

Models	$-l_{max}$	*AIC*	*CAIC*	*HQIC*	*LR(p-value)*
$\text{Gom}(\gamma,\beta)$	42.35	88.70	89.14	89.58	16.58 (0.0008)
$\text{EG}(\theta,\gamma,\beta)$	39.95	85.90	86.82	87.22	11.78 (0.003)
$\text{EGEG}(a,b,\theta,\gamma,\beta)$	34.06	78.12	80.62	80.30	-



(a) Estimated pdf's (b) Estimated cdf's

Figure 10: Plots of the estimated pdf and cdf of the models for the data set II.

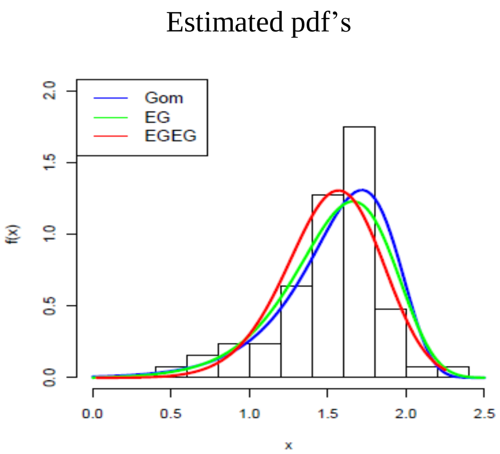
Table 8 and Table 9 present the results related to the third data set. Table 8 lists the MLEs of the model parameters (with the corresponding standard errors and confidence intervals in parentheses) for all fitted models. Table 9 present the values of the values of the AIC, HQIC, CAIC and the LR test statistics. These figures in this Tables reveals that the EGEG model has the lowest AIC, HQIC, CAIC and the LR values among all fitted models. Thus, the proposed distribution is the best model to explain these data. To a visual comparison, Figure 11 displays the histogram of the data and the estimated pdf and cdf for all fitted models. These plots reveal that the proposed model is quite suitable for these data.

Table 8: MLEs, standard errors, con_dence interval (in parentheses) for the data set III.

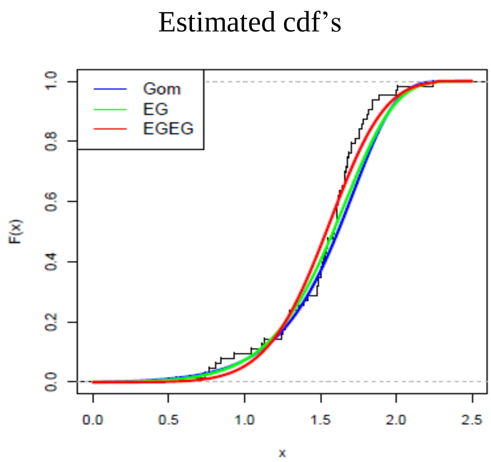
Models	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\beta}$
$\mathrm{Gom}(\gamma, \beta)$	-	-	-	0.008 (0.007) (0, 0.02)	3.547 (0.287) (2.98, 4.11)
$\mathrm{EG}(\theta, \gamma, \beta)$	-	-	1.606 (0.636) (0.36,2.85)	2.583 (0.627) (1.35, 3.81)	0.046 (0.038) (0, 0.12)
$\mathrm{EGEG}(a, b, \theta, \gamma, \beta)$	0.185 (0.026) (0.13,0.24)	3.995 (1.847) (0.37,7.62)	0.749 (0.086) (0.58,0.92)	1.804 (0.003) (1.79, 1.81)	(1.156) (0.004) (1.15, 0.16)

Table 9: AIC, CAIC, HQIC and LR (P-value) values for the data set III.

Models	$-l_{max}$	*AIC*	*CAIC*	*HQIC*	*LR (p-value)*
$Gom(\gamma,\beta)$	14.81	33.62	33.82	35.30	8.90 (0.03)
$EG(\theta,\gamma,\beta)$	14.14	34.28	34.69	36.80	7.56 (0.02)
$EGEG(a,b,\theta,\gamma,\beta)$	10.36	30.72	31.77	34.92	-



(a) Estimated pdf's



(b) Estimated cdf's

Figure 11: Plots of the estimated pdf and cdf of the models for the data set III.

From the findings presented in the Tables 5, 7 and 9 on the basis of the different criteria such as AIC, CAIC and HQIC, the $\mathrm{EGEG}(a,b,\theta,\gamma,\beta)$ is found to be a better model than its sub models $\mathrm{EG}(\theta,\gamma,\beta)$ and $\mathrm{Gom}(\gamma,\beta)$ for all three data sets considered here. A visual comparison of the closeness of the fitted densities with the observed histogram and fitted cdfs with the observed ogive of the data sets I, II and III are present in the Figures 9, 10 and 11 respectively. These plots also indicate that the proposed distributions provide comparatively closer fit to these data sets.

9 Conclusions

In this paper, we studied a five-parameter model named *exponentiated generalized extended Gompertz* (EGEG) distribution, which consists of a major extension of the *extended Gompertz* distribution. The paper also provided several mathematical properties of the EGEG model including explicit expressions for the density and quantile functions, ordinary moments and order statistics. We discussed the maximum likelihood method to estimate the model parameters and presented a Monte Carlo simulation study to evaluate the performance of the maximum likelihood estimators for the EGEG model. Finally, three applications illustrate the potential of the EGEG distribution fitting survival data.

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