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Flowthrough Centrality: A Stable Node Centrality Measure

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Abstract

This paper introduces flowthrough centrality, a node centrality measure determined from the hierarchical maximum concurrent flow problem (HMCFP). Based upon the extent to which a node is acting as a hub within a network, this centrality measure is defined to be the fraction of the flow passing through the node to the total flow capacity of the node. Flowthrough centrality is compared to the commonly-used centralities of closeness centrality, betweenness centrality, and flow betweenness centrality, as well as to stable betweenness centrality to measure the stability (i.e., accuracy) of the centralities when knowledge of the network topology is incomplete or in transition. Perturbations do not alter the flowthrough centrality values of nodes that are based upon flow as much as they do other types of centrality values that are based upon geodesics. The flowthrough centrality measure overcomes the problem of overstating or understating the roles that significant actors play in social networks. The flowthrough centrality is canonical in that it is determined from a natural, realized flow universally applicable to all networks.

Keywords *incomplete networks; max-min fairness; network centrality measures; network hubs; robustness of network measures*

1 Introduction

Measures of centrality are used to determine the prominence of nodes in a network (Freeman, 1977; Freeman et al., 1991; Segarra and Ribeiro, 2014). Applications of node centrality include finding key infrastructure nodes in telecommunication or urban transportation networks, and "super-spreaders" of disease Wikipedia (2022). Figure 1 is from a highly cited social network complied by Krebs (2002). The data come from publicly available sources linking members of the extended network of the September 11, 2001 terrorists who crashed four airliners into the Twin Towers of the World Trade Center in New York City; the Pentagon in Arlington, Virginia; and a Pennsylvania field. Nineteen of the network members were onboard the airliners, and another 43 members involved in the operation complete the 62 nodes of the network which has 153 links. The names Mohammed Atta and Marwan al-Shehhi are no doubt familiar to most readers; they are represented by nodes 38 and 35, respectively in the figure. These nodes stand out as having relatively large centrality according to several measures from the literature. Node 24, representing Imad Eddin Barakat Yarkas, has two connections to each of the two apparent subnetworks in the figure. As one of the few such nodes in the network, node 24 ought to have

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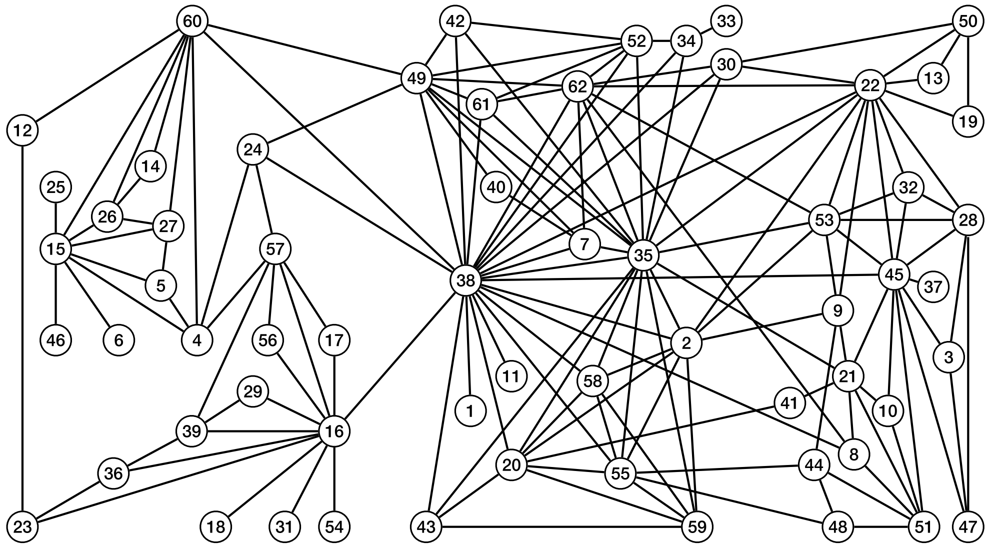

Figure 1: 9-11 conspirators network compiled by Krebs.

a relatively high centrality value, but it is not detected as such by the standard measures. The node centrality measure introduced in this paper not only identifies node 24 as an important node in the Krebs network but is also shown to be a more stable measure than the standard measures from the literature.

Prominence of a node can be measured by its connection to neighbors (degree centrality) (Knoke and Burt, 1983), the distance of a node to its neighbors (closeness centrality) (Sabidussi, 1966), or its influence on the network. An influential node exerts control by restricting or conveying the flow of a commodity, such as information, to and from or through it. Flow through a node between pairs of other nodes is called *passthrough flow*. Hubs are nodes with a relatively higher amount of passthrough flow than other nodes (Mann, 2008). Centrality measures that count geodesics in their calculation, such as betweenness centrality, assign a higher value to those hub-like nodes that lie on the shortest paths between a larger number of node pairs. In models where influence or information flow along shortest paths, nodes on the fewest number of geodesics have the lowest centrality values.

Another way of looking at a node's hub property is through the concept of network flow. Like water flowing in pipes to adjacent nodes, flow can be withheld or transmitted through nodes. The more flow passing through the node, the more it is acting as a hub. The design of the *flowthrough centrality* measure is based on this concept of network flow. Flowthrough centrality was invented to measure the extent to which a node functions as a hub in a network (Mann, 2008). A hub determined from flow tends to be persistent in its role as a conveyor of flow between other pairs of nodes and is little affected by the convolution of the paths connecting those nodes. However, hubs defined by their position in geodesics (e.g., betweenness centrality), are abruptly affected by changes in the length of paths connecting pairs of nodes sharing that hub. The omission of only one link in the geodesic (e.g., deleting the link from node 38 to node 16) could sever their path connection through that hub as alternate geodesics through other nodes are manifested. The omission of that same link when using a model based on flow may have little effect on the amount of flow passing through that hub and therefore little change in its flowthrough centrality value as the flow is rerouted on other non-shortest paths between the

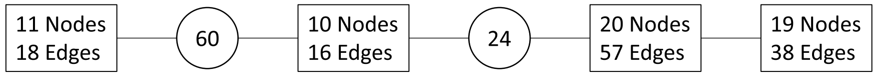

Figure 2: Flowthrough partition of the network in Figure 1 showing 4 major clusters and 2 significant intermediary single nodes.

nodes whose flow pass through the hub. Therefore, flowthrough centrality is more useful than other measures of centrality in situations where we are not certain that our observations capture the full topology of a network.

Another advantage of flowthrough centrality is that its calculation exploits the duality between flows and cuts (node partitions) in networks to determine a hierarchical clustering of the nodes. For example, Figure 2 shows the effective linear arrangement of four substantial clusters in Figure 1 after several levels of flowthrough-determined cuts. The twenty-three leftmost nodes in Figure 1 are seen further partitioned into two clusters of ten and eleven nodes with separating nodes 24 and 60 chaining the network together. Nodes 24 and 60 each have two links to the rightmost nodes and at least one to each cluster on the left. These associations suggest that who you know in different groups is more critical than how many you know.

Centrality measures are used to identify key nodes in a network (Freeman, 1977). But, in social networks, particularly covert networks where some of the connections (i.e., links) may not be known, centrality values in the observed network can differ in important ways from those of the true network (Mann et al., 2008). Through the use of the hierarchical version of the maximum concurrent flow problem, equitable peer-to-peer flows can be determined for every node-demand pair in the network. This information can, in turn, be used to find the amount of flow that passes through a node and define a new measure of centrality, the flowthrough centrality, that not only identifies nodes that act as hubs in a network, but also provides a centrality measurement that is more stable than other types of centrality measurements when networks are perturbed by the removal of links (Mann, 2008).

The flowthrough centrality measurement is shown by statistical analysis of real-world and randomly-generated networks to be better at identifying network hubs and being more stable during perturbations than other major centrality measures. By its basic design, the flowthrough centrality measurement identifies nodes that are acting as hubs within networks and therefore finds true hubs better than other centrality measurements that do not focus on this property nor measure it. Because the flowthrough centrality measurement is calculated from network flows rather than by the geodesics employed by other centrality measurements, its value does not change as abruptly when a link is removed from the network. Therefore, it is a more stable and accurate measurement of node centrality than the other, more-commonly used types of centrality measures calculated from geodesics.

1.1 Overview of Centrality Measures

Centrality measures are based upon various fundamental network properties. One of the most commonly-used measures is degree centrality based upon the number of adjacent-node neighbors. We do not consider degree centrality here. The five centrality measures in this study are based upon either geodesics or flow; and three of them additionally involve the property of betweenness. Betweenness measures the extent to which a node occupies a position in a path between other nodes (Freeman, 1977). In the following sections, we briefly look at the three geodesic-type measures: closeness centrality, betweenness centrality, and stable betweenness centrality (Segarra and Ribeiro, 2014). The latter two of these are also betweenness-type measures as is flow betweenness centrality. Flow betweenness centrality and flowthrough centrality are the two flow-based centrality measures.

Consider a connected and undirected network $G = (N, E)$ where the set N of nodes represents the actors in the network and the set E of links represents the links or connections between the actors. Associated with E is an $n \times n$ binary adjacency matrix A such that $a_{ij} = 1$ if a link is present between i and j and 0 otherwise. The number of nodes in a network is represented as n. We assume that the nodes are numbered $1, 2, \dots, n$ such that $N = \{1, 2, 3, \dots, n\}$ where $n = |N|$. Links are unordered pairs of nodes such that link $(i, j) \in E$ can also be termed link (j, i).

The *source node* from where a path begins is represented as $s \in N$, and the *sink node* where the path ends is represented as $t \in N$. A path in G with origin s and destination t is an alternating sequence of nodes and links where none of the nodes are repeated. The length of a path is the number of links it contains, and the shortest-path distance between nodes i and j, also known as the *geodesic* of i and j, is denoted by $d(i, j)$.

Two of the centrality measures we discuss are based on the concept of network flow. In these cases, flow is permitted in either direction and c_{ij} denotes the maximum number of units of flow that link (i, j) can support (i.e., the capacity of the link). In social network analysis c_{ij} is often used to model the strength of the tie between actors i and j (Bavelas, 1950). In applications where all ties are equally strong, we say that the network is unweighted and treat all links as having unit capacity.

Closeness Centrality The intuitive concept of closeness for a node is derived from the summation of the distances (i.e., geodesics) to all the other vertices in the network. Historically this has also been called *farness*. Bavelas (1950) defined closeness to be the reciprocal of farness to express the closeness centrality so that a larger number means a higher measure of centrality in the same way that other centrality measures indicate greater centrality with higher values. It is further normalized by multiplying this value by the number of geodesics being summed, $n - 1$, where n is the number of nodes in the network. This closeness centrality measure can be described as the reciprocal of the average length geodesic to the node being measured.

The closeness centrality for node i is defined as:

$$C^{CL}(i) = \sum_{j \in N \setminus \{i\}} \frac{n-1}{d(i, j)}.$$

Betweenness Centrality The betweenness centrality measure was introduced to evaluate the strategic importance of an individual related to shortest paths between all other nodes in the network (Freeman, 1977). It measures, for a node i, the proportion of all geodesics between the pairs of other nodes which include i on those paths.

Let $\sigma_{s,t}$ represent the total number of shortest paths from a source node $s \in N$ to a sink node $t \in N$. The number of geodesics between s and t that contain the node $i \in N$ is represented as $\sigma(i)_{s,t}$. The value of the betweenness centrality for node i is defined as:

$$C^{BW}(i) = \sum_{\{s<t \in N \setminus \{i\}\}} \frac{\sigma(i)_{s,t}}{\sigma_{s,t}}.$$

Stable Betweenness Centrality The stable betweenness centrality measure shows how significant a node is in connecting other nodes in a network via paths through the node being measured. This is accomplished by comparing the length of the geodesics passing through the node being measured to the length of the geodesics in a subnetwork that has had that node removed. In a network $G = (N, E)$ a node $i \in N$ is the node to be measured. A subnetwork is defined $G^i = (N^i, E^i)$ where $N^i = N \setminus \{i\}$ and $E^i = \{(k, l) \in E : k \neq i \text{ and } l \neq i\}$. The subnetwork G^i has the node i removed from the set of nodes N^i, and all of the links incident upon that node have been removed from the set of links E^i. The length of the geodesic between nodes s and t in the G network is $d(s, t)$ and between nodes s and t in the G^i subnetwork is $d^i(s, t)$. The equation for the stable betweenness centrality measure defined by Segarra and Ribeiro (2014) is:

$$C^{SB}(i) = \sum_{\{s<t\in N^i\}} (d^i(s, t) - d(s, t)). \tag{1}$$

Because the first term in (1) uses subnetwork G^i that does not have node i and its incident links in it, every geodesic between nodes s and t must be equal to or longer than the geodesics between nodes s and t in the network G that does contain node i and its incident links. This ensures that the difference between the two terms in the equation will never be negative. Furthermore, this form of the stable betweenness centrality is not normalized to restrict the range of its possible values.

Flow Betweenness Centrality The flow betweenness centrality (C^{FB}) measure is based on the maximum flow model of Ford and Fulkerson (1956, 1962) and was introduced by Freeman et al. (1991). The maximum flow (not the maximum concurrent flow) through the network $G = (N, E)$ from node $s \in N$ to node $t \in N$ where $s \neq t$ is defined to be the largest number of units of flow that can be sent simultaneously from s to t, and is denoted by $MF(G, s, t)$. In the maximum flow model, flow from s to t is not limited to geodesics; instead, flow is allowed on any collection of paths between the end points as long as the total amount of flow on any particular link, (k, l), does not exceed the link capacity c_{kl}. Intuitively, flow betweenness centrality may be seen as the flow analog of betweenness centrality. That is, C^{FB} of node i represents the extent to which node i contributes to the sum of the maximum flow values between all pairs of nodes in $N \setminus (i)$. Formally, C^{FB} of node i is defined as:

$$C^{FB}(i) = 1 - \frac{\sum_{\{s<t\in N\setminus\{i\}\}} MF(G^i, s, t)}{\sum_{\{s<t\in N\setminus\{i\}\}} MF(G, s, t)}, \tag{2}$$

where G^i is the subnetwork of G induced by removing i and all links incident on i as in the Stable Betweeness Centrality measure.

2 Flowthrough Centrality Measurement

The amount of flow passing through the nodes serves as the basis for the new centrality measure that provides an indication of the extent to which nodes are serving as hubs. Defining it to be the fraction of the non-terminating flow passing through the node to the total flow (capacity) of the node, this new centrality measurement is named flowthrough centrality (Mann, 2008). Its value can range between zero and one. A node of degree one (i.e., it is adjacent to only one other node) can not serve as an intermediary node in an efficient routing between pairs of other nodes. It can only have peer-to-peer terminal flow; therefore, the flowthrough centrality value for a degree-one node is zero. However, a multiple-degreed node that happens to have very little terminal flow would have a large amount of flow passing through it and therefore a relatively large flowthrough centrality value. In a network of flow, there are certain nodes known as hubs through which other, non-adjacent nodes exchange flow. Flowthrough centrality is designed to exploit that property and identify the extent to which nodes function as hubs in a network based upon the idealized, realizable network flow. As in a network of aircraft flights representing the links between airports that represent the nodes, passenger traffic flows from airports of departure to airports of destination. These are the terminal flows. The airports, where the passengers make an intermediate stop perhaps to change aircraft, are the hubs. The fraction of these "layover" passengers to the total passengers traveling through an airport is its flowthrough centrality value.

Flowthrough centrality is a measure of the extent to which a node is a hub by considering the flow that passes through the node. Flow that terminates at the particular node is not included in the passthrough flow for that node. The flow terminating at a particular node is determined by the hierarchical maximum concurrent flow algorithm. The difference between the capacity of a node and its terminating flow is the amount of flow that passes through it. The fraction of the node's capacity that is occupied by the passthrough flow is the flowthrough centrality measure. Before formally introducing this measure, we explain the concepts of network density, sparsest cuts, and the maximum concurrent flow problem.

2.1 Network Density and Sparsest Cuts

The density of an (S, T)-cut in a network is $\frac{|(S,T)|}{(|S||T|)}$ where $|(S, T)|$ is the actual number of links between node sets S and T, and $|S||T|$ is the maximum number of links possible. A minimum density cut in the network is a *sparsest cut*. When the links are weighted, the density is the average weight of the $|(S, T)|$-cut links, with absent links treated as links of zero weight. The density of a network with n nodes and m links is defined analogously to cut density as $\frac{m}{\binom{n}{2}}$. Mann, Matula, and Olinick (2008) propose the divisive *MCF Cut algorithm*, which uses sparse cuts to recursively partition clusters, and establish theoretical conditions assuring that the densities increase monotonically with each cut (Mann et al., 2008). They apply MCF Cut to well-studied, real-world networks to evaluate how effectively the algorithm agrees with the accepted hierarchical community structures. In particular, the algorithm perfectly identifies the correct structure of NCAA football conferences and divisions demonstrating its robust nature in determining a complex hierarchy of variable-size communities.

For a more detailed example, consider the network representation of the frequently analyzed Florentine Families data set shown in Fig. 3 (DuBois, 2008; Breiger and Pattison, 1986; Padgett and Ansell, 1993). The vertices in the network represent prominent 15th century Florentine families; the edges represent marriages and business ties between the families. The sparsest cut in the network is the single link (9, 13) with a density of $\frac{1}{26} \approx 0.0385$. The next cut in the partition is $\{(3, 9), (4, 7), (12, 14)\}$, which has density $\frac{3}{50} = 0.06$. The network as a whole has a density of $\frac{20}{105} \approx 0.19$ and the red (five-node), green (two-node), and blue (eight-node) sub-networks have densities of 1, $\frac{6}{10}$, and $\frac{9}{28} \approx 0.32$, respectively. Thus, the three cuts identify three sub-communities of the social network that are each considerably more tight knit than the community as a whole.

The sparsest-cut problem is NP-hard (Matula and Shahrokhi, 1990); it is unlikely that it can be solved in all cases with a polynomial-time algorithm. However, it can be shown that the value of the maximum concurrent flow (MCF) in a network is less than or equal to the density of

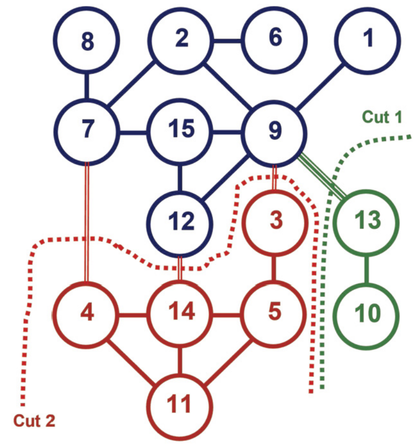

Figure 3: Sparse cuts in the Florentine Families network.

a sparsest cut (Matula, 1985). The *maximum concurrent flow problem* (MCFP) is a peer-to-peer network flow problem defined on a link-capacitated network where the objective is to maximize the ratio of the flow delivered between each peer-to-peer pair in comparison to the corresponding demand for that pair (Shahrokhi and Matula, 1990). This *throughput* ratio must be the same for all pairs (Matula and Shahrokhi, 1990). Every MCFP instance has a set of *critical links* that are saturated with flow by every optimal solution, and when the MCF solution identifies a partition into between two and four parts, a sparsest cut can be identified (Matula, 1985). The MCFP can be solved efficiently via linear programming (LP) (Dong et al., 2015; Bauguion et al., 2015), and a subset of critical links that form a throughput-constraining cut can be identified via LP duality theory. Thus, the method exploits a duality between flow and density, which generalizes the max-flow/min-cut duality Bavelas (1950); Freeman (1977) employed by Zachary in his seminal analysis of a university karate club (Zachary, 1977).

The LP formulation of MCFP is given by

$$\max \quad z, \tag{3}$$

$$\text{s.t.} \quad \sum_{p \in P(i,j)} f_p = z r_{ij} \quad \forall i < j \in N, \tag{4}$$

$$\sum_{p \in Q(i,j)} f_p \leqslant c_{ij} \quad \forall (i,j) \in E, \tag{5}$$

$$f_p \geqslant 0 \qquad \forall p \in P, \tag{6}$$

where z is the throughput, P is the set of paths in the graph, $P(i, j)$ is the set of all paths between nodes i and j, f_p is the flow on path p, $Q(i, j)$ is the set of paths traversing edge (i, j), and r_{ij} is the requested flow (demand) between nodes i and j.

Constraint (4) ensures that all node pairs receive the same throughput z, while constraints (5) and (6) ensure that flow values assigned to paths are non-negative and do not exceed the link capacities (weights). To find the sparsest cut in the network in Fig. 3, the MCFP LP with unit link weights and demands is solved (e.g., each pair of cities on a long-distance network requests a unit's worth of bandwidth for a communication circuit and each link can carry at most one unit of bandwidth at any given time). In Fig. 3, link (9, 13) must support all 26 z units of flow between node set {10, 13} and the rest of the nodes. Thus, it limits the throughput to 1/26.

A "fair" bandwidth allocation supports 1/26 (3.8%) of all requests simultaneously; link (9, 13) is saturated but residual bandwidth on the other links allows node pairs on either side of the cut to receive a larger fraction of their demand.

2.2 Hierarchical Maximum Concurrent Flow Problem

Although the cut identified by the MCFP is not always a sparsest cut, the LP approach has proven very effective in our experiments with the MCF Cut algorithm and suggests a novel divisive clustering algorithm that retains information from the entire data set at each level of the hierarchy (i.e., cut links remain in the network). The critical links partition the network into clusters. Then, the *hierarchical MCFP* (HMCFP), also termed max-min fairness (Nace and Pioro, 2006), further maximizes the common throughput in the links that connect pairs of nodes within clusters and determines a second throughput level and subpartition(s) following from a second set of critical links. Iterating further, a series of throughput levels is determined until all links are critical. The extension is straightforward by fixing the throughput for demand pairs that would otherwise be cut off by removing the critical links and re-solving iteratively over the whole network until all links are critical at some level (i.e., throughput is fixed for all demand pairs). Furthermore, the HMCFP determines uniquely the amount of flow between each node pair (i,k), $F(i,k)$. The flowthrough centrality of node i is defined as

$$C^{FT}(i) = \frac{\sum_{\{j\in N:(i,j)\in E\}} c_{ij} - \sum_{k\in N} |F(i,k)|}{\sum_{\{j\in N:(i,j)\in E\}} c_{ij}}, \tag{7}$$

where $F(k,k) = 0$. The numerator in the fraction above is referred to as the amount of flowthrough at node i, the denominator is a measure of the capacity of the node to transmit flow, and flowthrough centrality indicates the extent to which nodes are important hubs for flow between other pairs of nodes in the network (Mann, 2008; Matula and Olinick, 2016).

Equation (7) is the general formulation for the flowthrough centrality of a node. It is applicable not only for undirected networks but also for *digraphs* whose links have direction. The second term of the numerator takes the absolute value of the flows of each link when summing because the flows are consuming capacity of the node no matter which direction the flow is directed.

The numerator of the equation calculates double the value of the flow passing through the node. The capacity of the node minus the terminating peer-to-peer flows results in the passthrough flow. But, the passthrough flow is comprised of equal amounts of the flow coming into the node and flow going out of the node as it passes through. Therefore, the quantity calculated in the numerator of the flowthrough centrality equation is twice the amount of the actual passthrough flow. However, this has no effect on the relative flowthrough centrality values for ranking the nodes in a network, so it is unnecessary to perform the additional calculation of dividing by two.

In Fig. 3 node 9 representing the prominent Medici family has the largest flowthrough centrality (0.81), while the singleton nodes (1, 6, 8, 10) have flowthrough centralities of zero. The flow identified within the network is either of two types—either peer-to-peer or passthrough. The solution to the HMCFP identifies the peer-to-peer type flow for every node. In a network $G = (N, E)$ a node $i \in N$ serves as a terminal node to all other nodes (peer-to-peer) where $k \neq i \in N$ represents another node. Then $F(i,k)$ is the peer-to-peer type flow between nodes k and i. The difference between the flow capacity of the node and the sum of the peer-to-peer type flows at that node is the passthrough type flow between all other nodes with node i in their path.

Knowing these ideal peer-to-peer flows that terminate at the nodes, the amount of flow at each node that passes through the node to terminate between other pairs of nodes can be determined. This is the flow that passes through the node as opposed to terminating at it. The passthrough flow is determined by subtracting the terminal flow from the capacity of the node. For example, in a network where every link has a capacity weighting of one, a node incident to four links has a capacity of four. If the HMCFP solution tells us that this node has 1.5 units of terminal flow to it, then there are 2.5 units ($4-1.5=2.5$) of flow passing through the node. The node is acting as a hub in the network for 2.5 units of traffic that is routed through this node to terminate at other pairs of non-adjacent nodes. The flowthrough centrality for this node is 0.625. It is the passthrough flow of 2.5 divided by the flow capacity of 4 giving $\frac{2.5}{4.0}=0.625$.

The hierarchical maximum concurrent flow (HMCF), which leads Equation (7), was developed as an approximate method for partitioning a network by sparse cuts into dense subnetworks. In a social network, this partitioning provides a hierarchy of distinct communities ranging from the extremes of a single community of n individuals to n one-person communities. Partitioning a network by HMCF is an effective approach to community detection in social science applications and hierarchical clustering in general. In Section 3 we explore the stability of each of the centrality measures via simulation, and in Section 4 we apply the centrality measures to the network of 9-11 hijackers. We show that flowthrough centrality is the most stable centrality measure when links between nodes are missing. Furthermore, flowthrough centrality provides insights into the structure of the 9-11 hijacker network that other measures do not.

3 Experimental Confirmation

In order to explore the stability of the various measures of centrality, baseline networks of 50, 100, and 150 nodes were randomly generated with average node degrees of 6, 8, and 10. These networks were perturbed by randomly removing from 3% to 12% of the original links. The five types of centrality measures of this study (defined in Section 1.1) were calculated for each of the nodes before and after the network perturbations and compared for stability. The results show that the flowthrough centrality measure overcomes the problem of overstating or understating the prominence of hubs in networks, particularly those networks that are in transition changing over time. Flowthrough centrality was shown to be more stable, and therefore more reliable, than geodesic-based measures for the uncertain topology of networks for which it may be difficult to collect accurate information. A stable and reliable measure of centrality is particularly important when it is necessary to determine the prominent actors in networks for which full information is not available.

A random network-generator algorithm was used to create the baseline networks necessary for the analysis and comparison of the other centrality measures to the flowthrough centrality measure in order to determine stability. The other centrality measures examined in this study were closeness, betweenness, flow betweenness, and stable betweenness. The series of experiments conducted for this research has shown the flowthrough centrality measure to be more stable than the commonly-used betweenness centrality and other centrality measures based upon geodesics.

The networks used in the study included networks with 50, 100, and 150 nodes. The average node degrees of the networks were 6, 8, and 10. The degree of the nodes directly affects the number of links in the networks. The intent of our method to perturb the "true" baseline networks was to produce modified networks that would closely resemble the characteristics of networks observed in the real world. The actors in a non-theoretical network are identified through their interactions (i.e., links) with each other. It is more likely that outside observers may fail to see some links than they are to inadvertently fabricate nonexistent links. This is especially true for covert networks where the actors may be deliberately trying to conceal their connections. Therefore, in our experiments, we perturbed the designated true networks by randomly omitting links. Additional links were not added, and networks were required to remain connected so that before and after centrality measures could be compared for every node in the network.

As an example, consider a 100-node network of average degree 8 that is perturbed by 6%. The degree of a node is the number of links incident upon that node. A complete 100-node network where every node is connected to each of the other nodes directly by a link would have 4,950 links with an average degree of 99. Since each link directly connects two nodes, each link contributes two degrees to the overall degree total in a network. A one hundred node network with an average degree of eight would have a degree total of 800 produced by 400 links. A 6% perturbation of such a network would remove 6% of the 400 links meaning that 24 links are randomly removed.

Networks with 50, 100, and 150 nodes were generated and then perturbed by a program that randomly removed a given percentage (3%, 6%, 9%, and 12%) of the links from the initial network. The centrality for the intact network (henceforth called the "true" centrality) was calculated prior to perturbation. Each network was perturbed 100 times in order to obtain an estimate of the accuracy and precision for stable betweenness centrality (SB), flowthrough centrality (FT), flow between centrality (FB), betweenness centrality (BW) and closeness centrality (CL). Relative percentage difference was defined as

$$RPD_i = \frac{2(x_i - y)}{(|x_i| + |y|)}, \tag{8}$$

where y is the true centrality calculated prior to any perturbation and x_i is the observed centrality calculated for all $i = 1, \ldots, 100$ perturbations. Note that the percentage error can be negative because its numerator is the difference between the observed value and true value, rather than the absolute value of this difference. This was done to estimate bias for each measure. Measures that tend to have a negative *RPD* are those for which the observed value after perturbation tends to be less than the true centrality value. Measures which are positive tend to have estimates that overestimate the true centrality value.

Because the relative percentage differences are not normally distributed, the nonparametric Nashimoto-Wright NPM test (Nashimoto and Wright, 2005) for ordered means was computed on the percentage errors for each network. The null hypothesis for this test is that the percentage errors for all five measures are equivalent. We set up the alternative hypothesis to test whether the *PE* for FT $<$ FB $<$ CL $<$ BW $<$ SB. The test produces a p-value for each pair of ordered alternatives; therefore p-values were adjusted for multiple comparisons using the Holm p-value adjustment (Holm, 1979). All calculations were done using the PMCMRplus package (Pohlert, 2018) from R statistical software (R Foundation for Statistical Computing, 2018).

The results of our study were remarkably consistent across nodes, degrees, and perturbations. Therefore, to avoid repetition, the detailed results of only the smallest and the largest amount of perturbation to the average 6-degree networks are discussed here. Table 1 shows the p-values for the pairwise Nashimoto-Wright test for networks with 50 nodes and 6 degrees and 3% and 12% perturbations in the networks. Perturbations are given in the table heading. The

Table 1: P-values for the pairwise Nashimoto-Wright test for networks with 50 nodes and 6 average degrees and 3% and 12% perturbations in the networks.

	Perturbation percentages							
	3%				12%			
Type	FT	FB	CL	BW	FT	FB	CL	BW
FB	0.17				1.0			
CL	1.0	1.0			1.0	1.0		
BW	< 0.0001	< 0.01	< 0.001		< 0.001	< 0.001	< 0.001	
SB	< 0.001	< 0.001	< 0.001	0.23	< 0.001	< 0.001	< 0.001	0.0007

different methods of measuring centrality are given in the far left column and the column headings. FT = flowthrough, FB = flow between, CL = closeness, BW = betweenness and SB = stable betweenness.

The number in the intersection between two methods is the p-value for the test that the relative percentage difference for the two methods is equivalent versus the alternative hypothesis that the percentage error for measure on the column is less than that of the measure on the row. For example, the p-value for the test $RPD(\text{FT}) = RPD(\text{BW})$ vs. $RPD(\text{FT}) < RPD(\text{BW})$ is < 0.0001 (in bold face type Table 1). This p-value is small (the actual p-value is 1.3×10^{-9}), indicating strong evidence that the population mean percentage error for flowthrough centrality is less than the population mean percentage error for betweenness centrality. Other p-values can be interpreted similarly. P-values larger than 0.05 indicate that the null hypothesis is retained.

In Table 2, as in the previous table, we see that there is no evidence of a difference in the population mean percentage error for FT, FB, and CL measures. However, the BW and SB measures differ significantly from the other three measures and from each other. Therefore, BW and SB are not as stable, in terms of percentage error, as are FB, FT, and CL.

The centrality values for the nodes of a network are not evenly distributed throughout its range. In order to compare the high, middle, and low scoring nodes of each centrality type more easily, a tier ranking was established. This tier level is a ranking of the nodes of a network by centrality value from the highest value assigned to tier level 1, the second-highest value assigned to tier level 2, down to the lowest value assigned to the largest-numbered tier. If more than

Table 2: P-values for the pairwise Nashimoto-Wright test for networks with 50 nodes and 10 average degrees and 3% and 12% perturbations in the networks.

	Perturbation percentages							
	3%				12%			
Type	FT	FB	CL	BW	FT	FB	CL	BW
FB	0.079				0.92			
CL	1.0	1.0			1.0	1.0		
BW	< 0.0001	< 0.01	< 0.001		< 0.001	< 0.001	< 0.001	
SB	< 0.001	< 0.001	< 0.001	0.23	< 0.001	< 0.001	< 0.001	0.0007

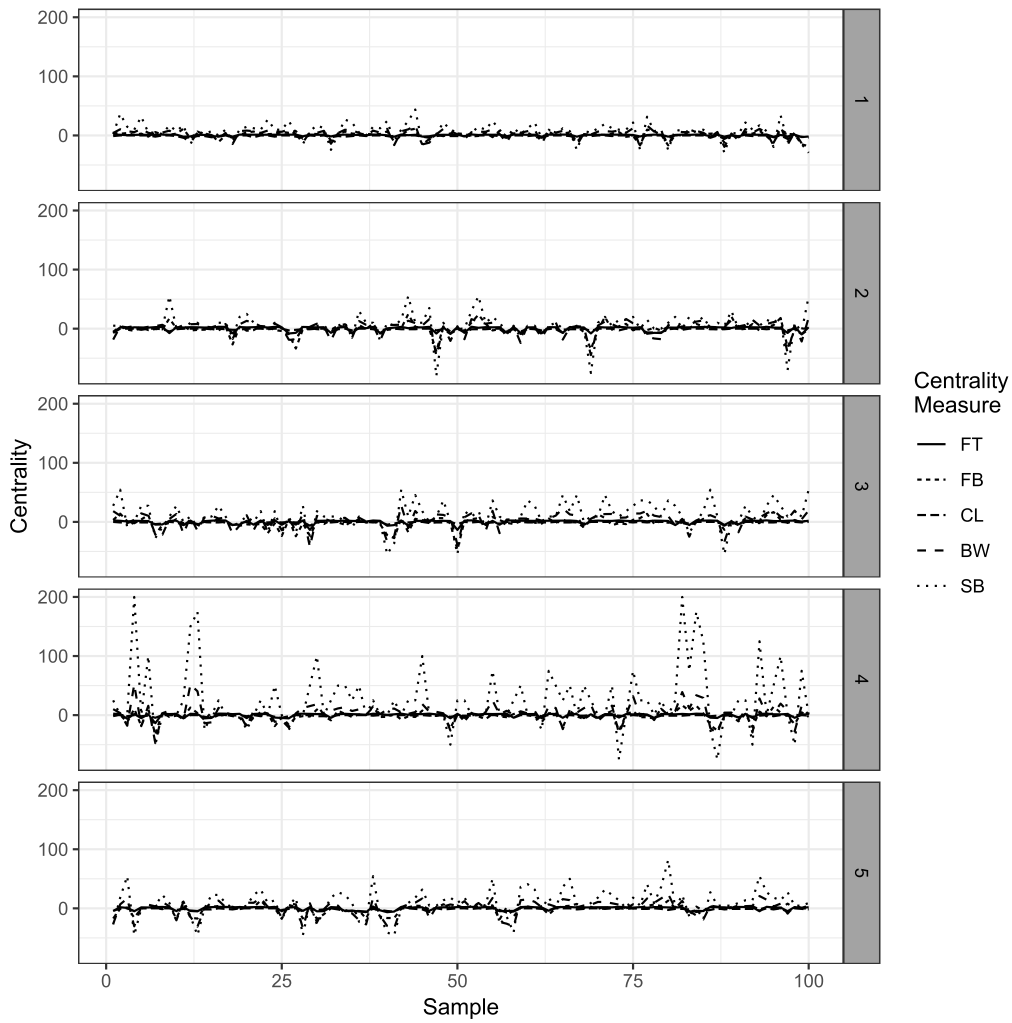

Figure 4: Stability of each centrality measure across all samples. Each measure is represented by a different line type. The percentage error is on the left vertical axis and the (arbitrary) sample number is on the horizontal axis. Each panel represents a different tier level, with the tier level indicated in the gray strip to the right of each panel.

one node received the same centrality value, those nodes were assigned to the same tier level to indicate they had an equal rank. Figures 4 and 5 show the relative percentage error for each of the centrality measures represented as lines or different types. Figure 4 shows results 50 nodes with 6 degrees and 3% perturbation and Figure 5 is for 50 nodes with 6 degrees and 12% perturbation. For both Figure 4 and Figure 5, the x-axis is the arbitrary sample number (1 to 100) and the y-axis is the value of the centrality measure. Each panel is a different tier level of node. For example, the top panel represents the centrality values for tier level 1 nodes.

One can see that, as the tier level increases, regardless of perturbation percentage, all measures become less stable. This makes sense as nodes which are less important when removed have varying effects on the network flow. However, SB is most unstable for all tier levels, as shown by the jagged nature of the dotted line corresponding to SB. FT, FB, and CL are indistinguishable in the first and second tier levels, and these three measures are very stable across

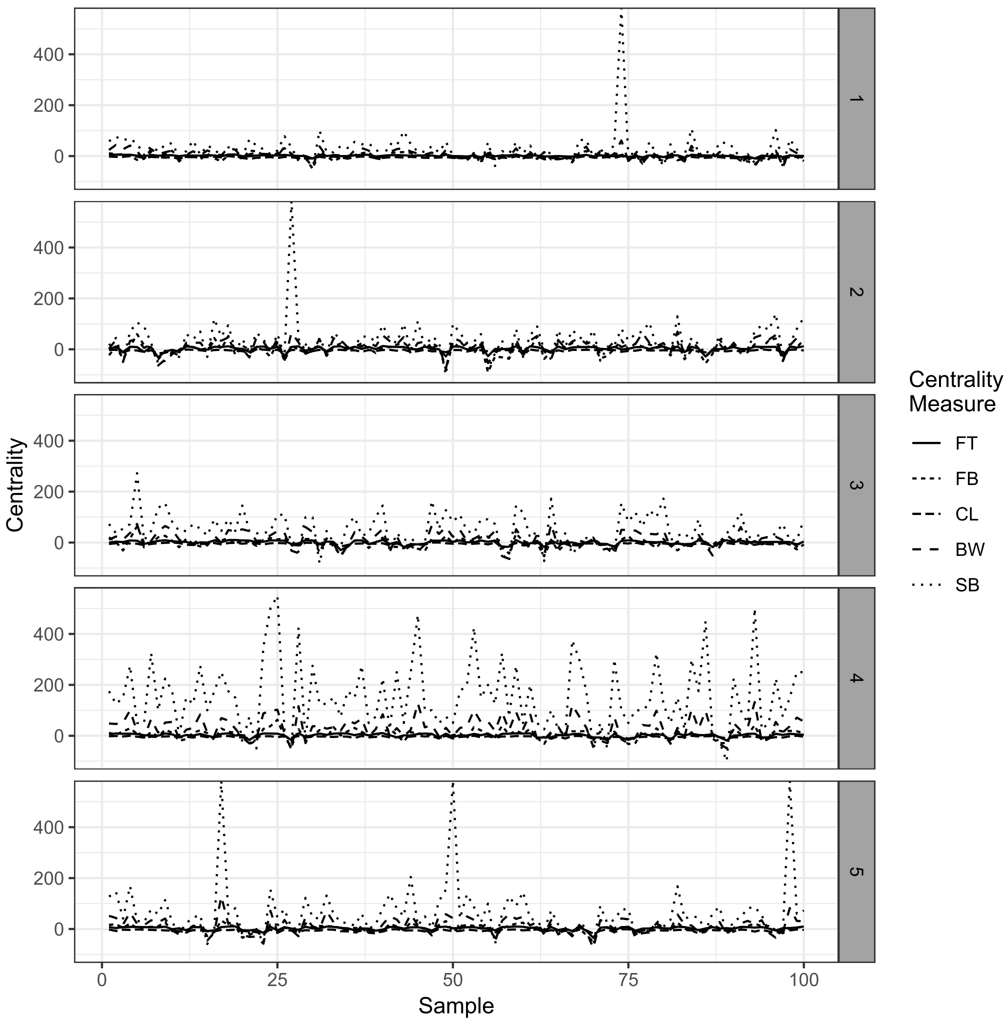

Figure 5: Stability of each centrality measure across all samples. Each measure is represented by a different line type. The percentage error is on the left vertical axis and the (arbitrary) sample number is on the horizontal axis. Each panel represents a different tier level, with the tier level indicated in the gray strip to the right of each panel.

all samples. This plot fortifies what was indicated by the p-values from the Nashimoto-Wright test in Table 1.

Figures 4 and 5 show variability for each measure across samples and tier levels. Figures 6 and 7 display the mean percentage errors for the first five node tier levels for a network of 50 nodes with 6 degrees and 3% perturbation (Figure 6) and 12% perturbation (Figure 7) as a way to summarize the results in Figures 4 and 5. Although 50 node tier levels were calculated, only the first five are shown in order to cut down on graph clutter.

The tier levels are across the bottom and the mean is on the vertical axis. Each stability measure is represented by a different symbol. The SB measure has the largest mean for all tier levels, and this mean fluctuates the most across the tier levels. The mean for FT hovers just above 0 for all tier levels, which is ideal in this scenario. FB and CL (represented by a triangle and a square, respectively) hover just below zero, indicating that these measures tend to give

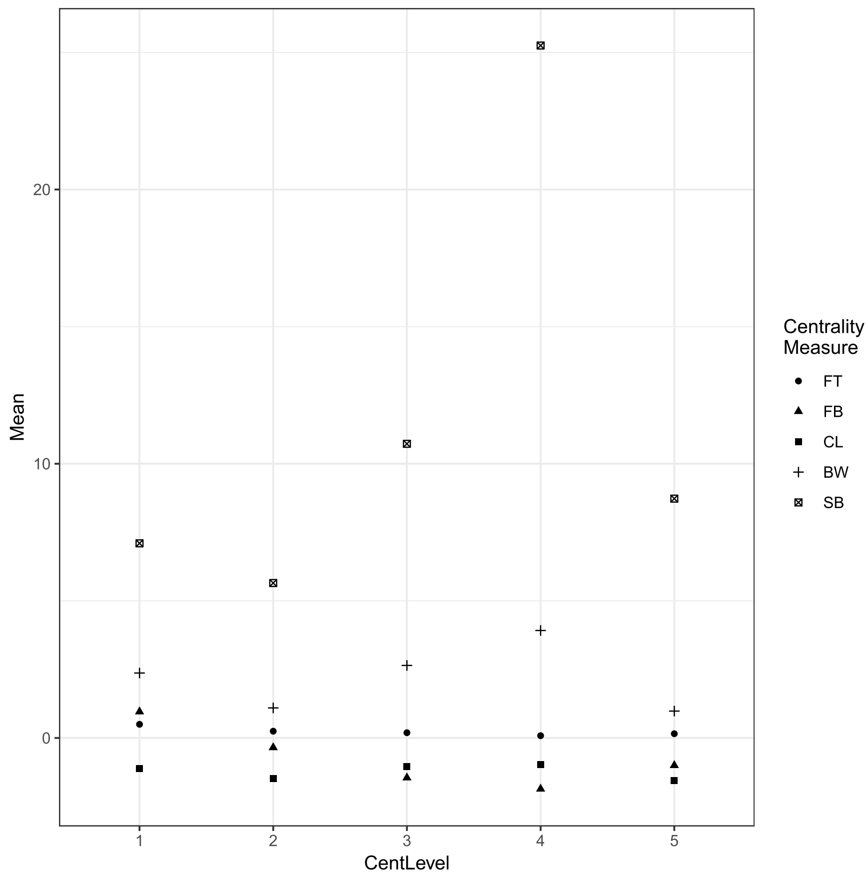

Figure 6: Mean percentage error over all samples for each centrality measure. Each measure is represented by a different shape. The vertical axis shows the mean percentage error and the horizontal axis is the tier level of centrality. The networks have 50 nodes, an average of 6 degree, and 3% perturbation of links between nodes.

perturbed values that are less than the true centrality, on average.

Results for 100 nodes and 6 degrees are much the same as shown in Tables 1 and 2, with one exception. There is no evidence of a difference between the population mean percentage error for FB and FT measure for the 3% perturbation ($p = 0.22$), and there is evidence of a difference for the 12% perturbation ($p = 0.016$). According to the alternative hypothesis, the indication is that FB is more accurate than is FT for this particular instance.

When there are 100 nodes and 10 average degrees, the Nashimoto-Wright test results are the same as the results for 50 nodes and 6 degrees, displayed in Tables 1 and 2. The p-values for the tests between BW and FB, BW and FT, BW and CL are equal to four decimal places due to the fact that there are many ties in these data, and the Nashimoto-Wright NPM test uses ranks of observations rather than actual observations in the calculation of its test statistic. In the Nashimoto-Wright distribution, these ranked values lead to similar test statistics with the same p-values. For that reason, these results are not displayed.

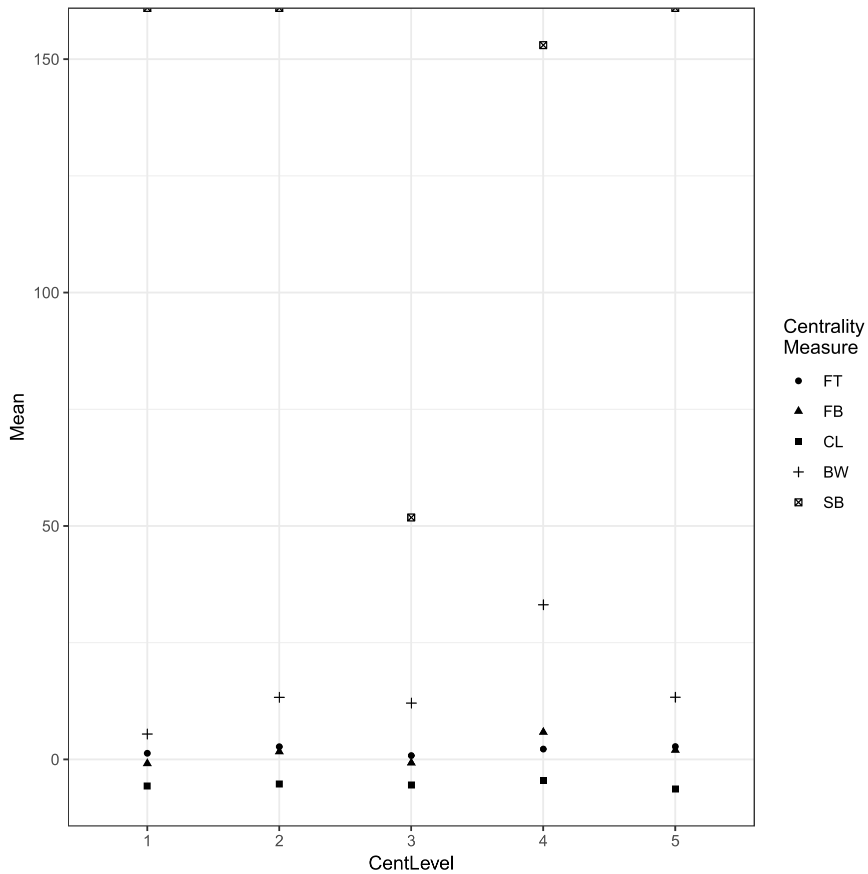

Figure 7: Mean percentage error over all samples for each centrality measure. Each measure is represented by a different shape. The vertical axis shows the mean percentage error and the horizontal axis is the tier level of centrality. The networks have 50 nodes, an average of 6 degree, and 12% perturbation of links between nodes.

4 Perturbations on Actual Network via Link Removal

As a test of the applicability of the stability of the flowthrough centrality measure to actual data, the perturbation testing was conducted on the 9-11 conspirators network provided by Krebs (2002) and illustrated in Figure 1. The true centrality values for each node was calculated for each of the five centrality measures in our study. Then the network was perturbed by a program that randomly removed 3%, 6%, 9%, and 12% of the links to create 10 perturbed networks for each of the four percentages of links removed. The same statistics and hypothesis tests were conducted on this real-world network as were conducted on the randomly-generated networks described earlier.

Three subcommunities can be seen in the 9-ll hijacker network of Figure 1. The 19 hijackers onboard the crashed airliners are included within the two subcommunities on the right. The 23-node subcommunity on the left is tied to the remainder of the network through five cut nodes sharing five links in the cut that separate the two subcommunities of the network. Naturally these cut nodes would be expected to play prominent roles in holding together the network which would be reflected in their centrality values.

Table 3: Centrality values and tier levels for cut nodes in Krebs' 9/11 hijacker network. Tier levels for each node are in parentheses after each centrality value.

Node ID	Flowthrough	Flow betweenness	Betweenness	Stable betweenness	Closeness
38	0.830 (3)	0.200 (1)	0.588 (1)	Infinity (1)	0.587 (1)
49	0.673 (10)	0.088 (4)	0.048 (9)	32 (14)	0.436 (5)
60	0.745 (8)	0.120 (2)	0.232 (3)	6 (6)	0.436 (5)
24	0.891 (1)	0.058 (10)	0.029 (13)	4 (19)	0.409 (8)
16	0.812 (4)	0.113 (3)	0.252 (2)	Infinity (1)	0.433 (6)

All five centrality measures agree on the relative ranking of the two cut nodes, node 38 and node 49, on the right-hand side of the cut as indicated in Table 3. Node 38 has a higher value than node 49 in each of the centrality measures indicating that Node 38 plays a more significant role than node 49. Therefore, node 38 is ranked as a tier level 1 node by four of the five measures, as shown by the value of 1 in parentheses after each centrality value for node 38. However, flowthrough centrality ranks node 38 in tier level 3. Instead, flowthrough centrality identifies Node 24 with its highest value (tier level 1) indicating that node 24 is the most significant hub in this network.

Flow arriving across the cut to node 24 is distributed to a small subcommunity in the upper left via the link to node 4 and to a second small subcommunity in the lower right via the link to node 57. Only flowthrough centrality identifies this significance of node 24's role. Flow betweenness and closeness both give node 60 their highest values for nodes on the left-hand side of the cut. The flow to node 60 is fed into the upper left subcommunity. The remaining centralities of betweenness and stable betweenness give node 16 their highest values for the nodes on the left-hand side of the cut. The flow to node 16 is fed into the lower right subcommunity. Only flowthrough centrality ranks node 24 as the most prominent of the three cut nodes on the left-hand side of the cut. This is the correct position for node 24, as without node 24, none of the subcommunities would be connected. In other words, without this node, the network actors cannot communicate with one another.

Table 4 shows the p-values for the pairwise Nashimoto–Wright test for the 9–11 hijacker network with 3% perturbations (on the left) and 12% perturbations (on the right). The number in the intersection between two methods is the p-value for the test that the percentage error for

Table 4: P-values for the pairwise Nashimoto-Wright test for the 9–11 Hijacker network data for 3% and 12% perturbations in the network. Each network was perturbed 10 times.

	Perturbation percentages							
	3%				12%			
Type	FT	FB	CL	BW	FT	FB	CL	BW
FB	1.0				1.0			
CL	1.0	1.0			1.0	1.0		
BW	**0.036**	0.036	0.036		0.00063	0.00063	0.00063	
SB	0.036	0.036	0.036	1.0	0.00063	0.00063	0.00063	1

the two methods is equivalent versus the alternative hypothesis that the percentage error for measure on the column is less than that of the measure on the row. For example, the p-value 0.036 is in bold face type, and it is at the intersection of BW and FT. Therefore, this is the Holm-adjusted p-value for the test $H_0 : FT = BW$ versus the alternative $H_A : FT < BW$. Here, we find in favor of the alternative, indicating evidence that the relative percentage error for flowthrough centrality (FT) is less than that of the relative percentage error for betweenness centrality (BW) when 3% of the links are randomly removed. The results become stronger when the network is perturbed by 12%. In general, stable betweenness centrality has the largest relative percentage error of all of the centrality measures. FT, FB, and CL give similar results in terms of error when the network is perturbed. These results are the same as were seen for the simulated data.

5 Conclusion

In both the randomly-generated networks as well as the real-world 9-11 conspirators social network, we found that the centrality types that are calculated based upon geodesics are not as accurate as the centrality types based on flow. Betweenness centrality and stable betweenness centrality measures are determined by geodesics. Their measures were the most inaccurate of the five centrality types studied. Closeness centrality, which is also based on geodesics, gave mixed results. The closeness centrality measurements in this study either matched the true values or were far off the mark.

The flowthrough centrality and the flow betweenness centrality were the two centrality types based on flow. Their measurements proved to be the most consistent and the most accurate of the study. These experiments show that the flow-based centrality measures are generally more stable than the geodesic-based centrality measures when links are removed from networks. A centrality based upon geodesics appears to be more disrupted by the removal of links in those paths that create much longer geodesics between certain pairs of nodes. Flow-based centralities simply alter flow between pairs of nodes using multiple, simultaneous paths that are not necessarily the shortest paths making link removals less abrupt in their results. The field of social network analysis addresses the problems of finding the key actors in social networks and measuring the extent of their influence on others and on their interactions such as communication, persuasion, availability, and even the spread of disease. Many of these real-world networks are dynamic. A temporal element is involved with data sampling that causes knowledge of the networks to be limited or lagging. Thus, purposeful responses or interventions into developing situations may be misguided. The flowthrough centrality measure overcomes the problem of overstating or understating the roles that influential actors play in these social networks. Flowthrough centrality is more stable than other centrality measures when the network topology is in transition. It is a more reliable tool than geodesic-based measures for the study of networks when applied to the uncertain topology of covert networks where ties may be deliberately concealed.

Supplementary Material

Code: The code for this research consists of approximately 160 R Markdown files. The files include R code for the results and graphics as well as data files resulting from the linear programming algorithm in (4), (5), and (6).

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