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ZOGRAFOS BALAKRISHNAN POWER LINDLEY DISTRIBUTION

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ABSTRACT

In this paper Zografos Balakrishnan Power Lindley (ZB-PL) distribution has been obtained through the generalization of Power Lindley distribution using Zografos and Balakrishnan (2009) technique. For this technique, density of upper record values exists as their special case. Probability density (pdf), cumulative distribution (cdf) and hazard rate function (hrf) of the proposed distribution are obtained. The probability density and cumulative distribution function are expanded as linear combination of the density and distribution function of Exponentiated Power Lindley (EPL) distribution. This expansion is further used to study different properties of the new distribution. Some mathematical and statistical properties such as asymptotes, quantile function, moments, mgf, mean deviation, renyi entropy and reliability are also discussed. Probability density (pdf), cumulative distribution (cdf) and hazard rate (hrf) functions are graphically presented for different values of the parameters. In the end Maximum Likelihood Method is used to estimate the unknown parameters and application to a real data set is provided a. It has been observed that the proposed distribution provides superior fit than many useful distributions for given data set.

Keywords: Gamma distribution, Upper record values density function, Power Lindley, Exponentiated Power Lindley, Maximum likelihood estimate

1. Introduction

Statistical distributions are commended in all the profession for modelling and statistical analysis of the lifetime data (Shanker et al (2015)). The survival data always remain an important aspect of statistical analysis as it can be categorized on the bases of their hazard rate. One is known the other can be automatically determined (Clark et al. (2003)). In the start Weibull distribution had been widely used to model survival studies but it had a flaw that it cannot model survival data base on non-monotone hazard rate (Cordeiro et al. (2018)). Therefore, in 1958 Lindley distribution was presented as an alternative to Weibull.

Lindley originally proposed Lindley distribution in the background of Baye's and Fiducial inference. Ghitany et al. (2008) had clearly studied Lindley and its application and illustrated that it is much flexible than the Exponential distribution but it had a flaw that it gives better fit to model only positively skewed data.

Therefore, Ghitany et al (2013) proposed the Power Lindley distribution and illustrated that it can also model negatively skewed data. They applied Power transformation of $x = y^{1/\alpha}$ to obtain probability density function of Power Lindley distribution (PLD):

$$g(x)=\frac{\alpha\beta^2}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta z^{\alpha}} \qquad x>0,\alpha,\beta>0 \tag{1}$$

However, its survival function is:

$$S(x)=\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}} \tag{2}$$

It cumulative density function was given by Warahena-Liyanage and Pararai (2014):

$$G(x)=1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}} \tag{3}$$

Many author has studied the application on the tensile strength data analysis (Ghitany et al. (2013)), estimation of reliability of stress-strenght system (Ghitany et al. (2015)) and reliability estimation in the context of Bayesian for reliability parameters (Makhdom et al. (2016)) etc.

Real world phenomena are frequently defined by using statistical distributions. Such increase in their worth has made compulsory for the statisticians to introduce more flexible distributions to model the real life data. One way of generating the new families of continuous distribution is by using different techniques to generalized the existing ones. The distinctive feature shared by the "generalized distributions" is that they have additional parameters than their arbitrary baseline distributions.These additional parameters effect the shape and skewness of the existing distributions to increase their flexibility.

Jones (2004) proposed a technique for generalizing the broad family of univariate distributions by using a Beta distribution . Following the line of Jones, in Zografos and Balakrishnan (2009) defined a new technique using the special case of Stacy's generalized gamma distribution when ($\gamma = 1$) instead of Beta distribution. The new generated distributions are commonly known as "Gamma-G" or "Zografos-Balakrishnan-G" distribution whereas "G" denotes the baseline distributions.

For any baseline distribution, Zografos and Balakrishnan (2009) states the Gamma-generated distribution with probability density function

$$f(x) = \frac{1}{\Gamma(a)}\{-\log[1 - G(x)]\}^{a-1} g(x) \quad (4)$$

and cumulative density function:

$$F(x) = \frac{\gamma(a, -\log[1 - G(x)])}{\Gamma(a)} \quad (5)$$

With an introduction of an extra shape parameter a. Whereas $a > 0$, for $x \in \mathbb{R}$, g(x) and G(x) are the baseline pdf and the distribution function, $\Gamma(a) = \int_0^{-\infty} u^{a-1} \exp(-u)\, du$ represents the complete gamma function and $\gamma(a, w) = \int_0^{w} u^{a-1} \exp(-u) du$ denotes the lower incomplete gamma function.

The density function of upper record values exists as their special case. In this technique, they presented the method of generalizing the distribution as well as a technique to extract record densities from record value work and generalized them so that they can emerge as a new family of distribution in their own.

A lot of distributions has been generalized using this method. Cordeiro et al. (2013) studied Gamma-lomax distributions with its properties. In Cordeiro (2015) also explore Gamma Modified Weibull distribution. Ramos et al. (2013) consider the Log-Logistic distribution and Silva (2013) considered the Extended Frechet distribution to apply this method. Lima and Corderio (2015) extended the family of Normal distribution. However, Zografos and Balakrishnan (2009) and Nadarajah et al. (2015) studied some general properties of Zografos and Balakrishnan-G family of distribution.

In this paper we have introduced Zografos-Balakrishnan Power Lindley (ZB-PL) distribution. The new distribution has constant, bathtub, increasing and decreasing hazard function as well as it has greater flexibility than many useful distributions.

The rest of the paper is ordered as: Section 2 contains density, distribution and hazard rate functions of Zografos Balakrishnan Power Lindley distribution. Section 3 comprises of quantile function. Section 4 covers some statistical and mathematical properties. Section 5 consists of Renyi entropy. Section 6 composed of reliability function. Section 7 deals with maximum likelihood estimates of the parameters of the given distribution. Finally, an application of real dataset is provided in section 8. Section 9 presents the conclusion.

2. Zograforos-Balakrishnan Power Lindley

In this section, the density function of Zografos-Balakrishnan Power Lindley is presented. Graphical representation of pdf and cdf and hazard rate function is also given. The proposed distribution is also expanded as a linear combination of Exponentiated Power Lindley (EPL) distribution

By inserting the $g(x)$ (1) and $G(x)$ (3) as a baseline distribution's pdf and cdf respectively, in (4) and (5), after simplification we obtained the Zografos-Balakrishnan Power Lindley (ZB-PL) distribution with probability density function as:

$$f(x) = \frac{\alpha\beta^2}{\Gamma(a)(\beta+1)}(1 + x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left\{-log\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)+\beta x^{\alpha}\right\}^{a-1} \tag{6}$$

With one scale (β) and two shape (α, a) parameter, whereas $a > 0$, $\alpha > 0$, $\beta > 0$.

Distribution function as:

$$F(x) = \frac{\gamma\left(a, -log\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)+\beta x^{\alpha}\right)}{\Gamma(a)} \tag{7}$$

And Hazard rate function as:

$$h(x) = \frac{\alpha\beta^2\,(1+x^{\alpha})\;x^{\alpha-1}\;e^{-\beta x^{\alpha}}\left\{-log\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)+\beta x^{\alpha}\right\}^{a-1}}{(\beta+1)\left\{\Gamma(a)-\gamma\left(a, -log\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)+\beta x^{\alpha}\right)\right\}} \tag{8}$$

$\gamma(a, \mathrm{w}) = \int_0^{\mathrm{w}} \mathrm{u}^{a-1}\mathrm{e}^{-\mathrm{u}}\,\mathrm{du}$, $\Gamma(a, \mathrm{w}) = \int_{\mathrm{w}}^{\infty} \mathrm{u}^{a-1}\mathrm{e}^{-\mathrm{u}}\mathrm{du}$ and $\Gamma(a) = \int_0^{\infty} \mathrm{u}^{a-1}\mathrm{e}^{-\mathrm{u}}\mathrm{du}$ are lower incomplete, upper incomplete upper and complete gamma function respectively.

Special case:

Suppose, we have upper record values i-e $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ ... generated from sequence of iid continuous random variables from population with probability density function $g(x)$ and distribution function $G(x)$, then for parameter "a" being a positive integer, (6) is the density function of upper record values distribution rising from the sequence of independently identically distributed Power Lindley random variable exist as the special case of ZB-PL distribution.

As well as, if we consider that Z follow gamma distribution i.e. $Z \sim G(a, 1)$ than a logarithmic transformation of baseline distribution $Z = -log\,(1 - F(x))$ transform the variable $X = F^{-1}(1 - e^{-z})$ has a probability density function (6).

For large value of x, the hazard rate function of the ZB-PL distribution behaved as its density function whereas for small values of x, it behaved (proportional to) as Exponentiated Power Lindley distribution.

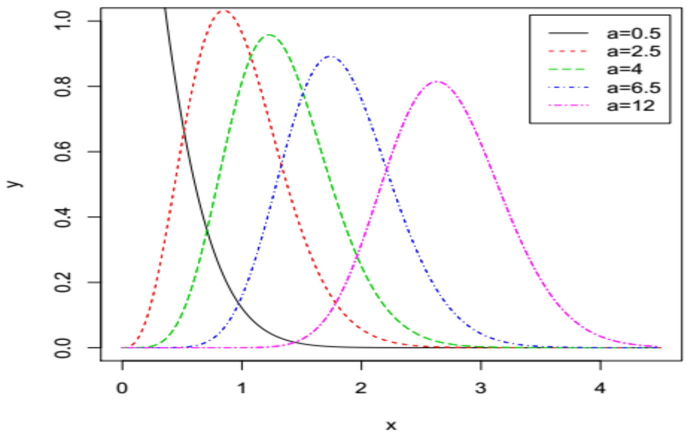

Figure 1. Probability density function: $\alpha = 1.5, \beta = 3$

In Figure 1, we have plotted pdf for distinct values of the parameter "a". By increasing the values of additional shape parameter (a), the curve changed its behaviour from right skewed to symmetric and then left skewed. As a result, its left tail start becoming heavier than right tail. The parameter also affects the spreadness of the curve as it seems to cover wider area under curve. The probability mass function moves from centre and tail to the shoulder of the curve.

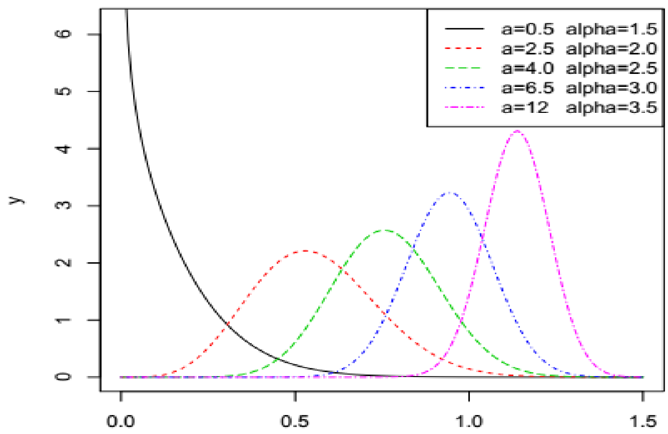

Figure 2. Probability density function: $\beta = 2$

In Figure 2 the values of both the shape parameter "a" and "α" (alpha) are varying, the curve of the pdf behaves highly positively skewed, approximately symmetric and negatively skewed for different values. The decrease in the spreadness shows that the probability mass is focus around the centre and the probability mass seems to move from shoulder to the centre and left tail of the curve.

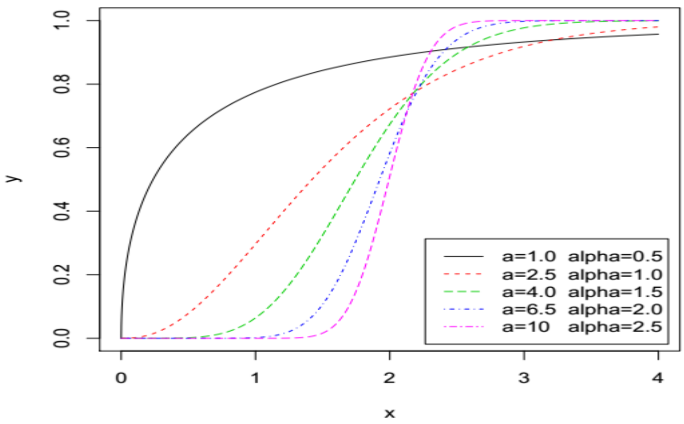

Figure 3. Cumulative distribution function: $\beta = 2$

Figure 3 shows the cdf for small values of shape parameters. The cdf curve cumulates to one for large values of x whereas by increasing the values of both the shape parameter "a" and "α" (alpha) the curve of the cdf cumulates to 1 for small values of the x.

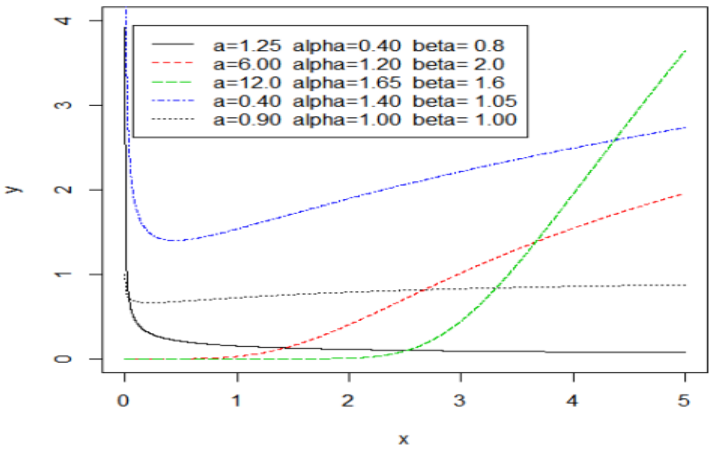

Figure 4. Hazard rate function

In Fig: (4) the hazard rate function display different shape and varies monotonically depending upon the value of the parameter:

a) $a < 1, \alpha, \beta = 1$ or $\alpha < 1, a, \beta = 1$ hrf is constant.

b) $a > 1$, α, $\beta < 1$ or $a, \alpha, \beta < 1$ hrf is monotonically decreasing.

c) $a < 1, \alpha, \beta > 1$ hrf is bathtub shape.

d) $a, \alpha, \beta > 1$ hrf is monotonically increasing.

2.1 Expansion of Density and Distribution Function

By using the concept of Exponentiated distributions, Corderio et al (2015) expanded the Gamma Lomax Distribution and Nadarajah et al. (2015) studied several general properties of ZB-G distribution. As many properties of Gamma-G distribution can be readily obtained through those of Exp-G distribution.

In this section, the proposed distribution is expanded as a linear combination of Exponentiated Power Lindley (EPL) distribution with parameter $(a + k)$.

According to Corderio et al (2015), a random variable X will follow Exp-G distribution with parameter a > 0 for any baseline distribution function G(x), if its pdf:

$$f_a^*(x) = aG(x)^{a-1}g(x) \tag{9}$$

and cdf is in the form:

$$F_a^*(x) = G(x)^a \tag{10}$$

It is clear from above equations that for large x values, for $a > 1$ and for $a < 1$ the component $aG(x)^{a-1}$is greater and smaller than 1 respectively. Similarly, for small values of x, the component $aG(x)^{a-1}$ is greater for $a < 1$ and smaller for $a > 1$..

Then, from (9) and (10) the EPL density function for x > 0 will be:

$$f_a^*(x) = \frac{a\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \quad \left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{a-1}$$

And the distribution function becomes:

$$F_a^*(x) = \left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{a}$$

$$\left\{-log\left(1-G(x)\right)\right\}^{a-1} = (a-1) \quad \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(a-1-j)}\binom{k+1-a}{k}$$
$$\left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{(a+k)-1} \tag{11}$$

Calculation of the constant $p_{j,k}$ can be obtained recursively by eq:

$$p_{j,k} = k^{-1}\sum_{m=1}^{k}\frac{(-1)^{m+1}[k-m(j+1)]}{m+1}p_{j,k-m}\ ,\ k=1,2,\ldots \text{ and } p_{j,0} = 1$$

As density function of ZB-PL obtained without any simplification is:

$$f(x) = \frac{\alpha\beta^2}{\Gamma(a)(\beta+1)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left[-log\left\{1-\left(1-\left(1+\frac{\beta}{\beta+1}x^\alpha\right)e^{-x^\alpha\beta}\right)\right\}\right]^{a-1} \tag{12}$$

Substituting values from (11) in (12):

$$f(x) = \frac{(a-1)\alpha\beta^2}{\Gamma(a)(\beta+1)}\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(a-1-j)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{(a+k)-1}$$

as a is positive integer so $\Gamma(a) = (a-1)\Gamma(a-1)$.

Multiplying and dividing with $(a+k)$ substituting value of (a) :

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)}\ \frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(a-1-j)}(a+k)$$
$$\left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{(a+k)-1}\ \frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \tag{13}$$

Consider,

$$b_k = \sum_{k=0}^{\infty} \frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)} \sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(a-1-j)} \tag{14}$$

From (9) we can rewrite the density of EPL ($f^*(x)$) with parameter (a+k) as:

$$f^*_{(a+k)}(x) = (a+k)G(x)^{(a+k)-1}g(x) \tag{15}$$

Whereas,

$$G(x)^{(a+k)-1} = \left\{1 - \left(1 + \frac{\beta x^\alpha}{\beta+1}\right) e^{-\beta x^\alpha}\right\}^{(a+k)-1}$$

$$g(x) = \frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}$$

By considering (14) and (15) we can write (13) as:

$$f(x) = \sum_{k=0}^{\infty} b_k\ f^*_{(a+k)}(x) \tag{16}$$

Integrating the above equation, we have:

$$F(x) = \sum_{k=0}^{\infty} b_k\ F^*_{(a+k)}(x) \tag{17}$$

Whereas $F^*_{(a+k)}(x)$ is the cdf of EPL distribution. From (16) and (17) it is clear that the pdf and cdf of the proposed distribution can be expressed as the one-dimensional combination of EPL's density and distribution function. The EPL and its properties were suggested and studied by Warahena-Liyanage and Pararai (2014).

3. Quantile Function

Quantile function of ZB-PL distribution is given by using inverse transformation method.

If $X \sim$Zografos-Balakrishnan Power Lindley distribution than its cdf will be:

$$F(x) = \frac{\gamma\left(a, -log\left(1+\frac{\beta x^\alpha}{\beta+1}\right)+\beta x^\alpha\right)}{\Gamma(a)} = u \tag{18}$$

whereas u is uniform on interval [0, 1]. Whereas $Q^{-1}(a,u)$ is the inverse function of $Q(a,x) = 1 - \frac{\gamma(a,\ x)}{\Gamma(a)}$.

If G(x) is the distribution function of baseline cdf (Power Lindley distribution) then by inverse transformation of (18), we have:

$$x = F^{-1}(u) = G^{-1}[1 - exp\{-Q^{-1}(a, 1-u)\}] \tag{19}$$

as it is the quantile function of Zografos-Balakrishnan-G distributions.

Quantile function of Zografos- Balakrishnan Power Lindley

Let consider the cdf of Power Lindley distribution,

$$G(x) = u$$

$$\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}} = 1-u$$

Multiply both sides with $e^{(-1-\beta)}$ and Setting;

$$z(x) = -1-\beta-\ \beta x^{\alpha} \tag{20}$$

After simplification we have:

$$z(x)e^{(z(x))} = -(\beta+1)e^{(-1-\beta)}\,1-u$$

$$z(x) = W\{-(\beta+1)e^{(-1-\beta)}\,1-u\}$$

W is the lambert function which is the inverse relation of the function of product logarithm i.e.

$$Z{=}f^{-1}(ze^{z}) = W(ze^{z})$$

Now substituting value of Z(x) from (20)

$$-1-\beta-\ \beta x^{\alpha} = W\{-(\beta+1)e^{(-1-\beta)}\,1-u\}$$

And simplify:

$$x = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W\{-(\beta+1)e^{(-1-\beta)}\,1-u\}\right]^{\frac{1}{\alpha}}$$

Now for quantile function of ZB-PL distribution substitute u with its value from (19) after simplification:

$$x = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W\{-(\beta+1)\ \ e^{(-1-\beta)}\,(exp\{-Q^{-1}(a,1-u)\})\}\right]^{\frac{1}{\alpha}} \tag{21}$$

$$M(x) = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W\left\{-(\beta+1)e^{(-1-\beta)}\left(exp\{-Q^{-1}(a,\frac{1}{2})\}\right)\right\}\right]^{\frac{1}{\alpha}} \tag{22}$$

4. Moments

In this section moments, mgf and mean deviation of ZB-PL distribution is derived. As it is illustrated in (16) that density of ZB-PL can be written as a linear combination of EPL distribution with parameter $(a+k)$. This result can be further used to study different properties of the newly proposed distribution.

Therefore, if X~ZB-PL and *Y(a+k)*~ EPL with $(a+k)$ parameter then, nth moment of X can be obtained as:

$$E(x^n) = \sum_{k=0}^{\infty} b_k\, E\left(y_{(a+k)}^n\right) \tag{23}$$

whereas,

$$b_k = \sum_{k=0}^{\infty} \frac{\binom{k+1-a}{k}}{(a+k)\Gamma(a-1)} \sum_{j=0}^{k} \frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(a-1-j)} \tag{24}$$

The nth moment of EPL distribution is given by Warahena-Liyanage and Pararai (2014):

$$E\left(y_{(a+k)}^n\right) = \frac{\alpha\beta^2(a+k)}{(\beta+1)} L_1(\alpha,\beta,(a+k),n) \tag{25}$$

However,

$$L_1(\alpha,\beta,(a+k),n) = \sum_{i=0}^{\infty}\sum_{l=0}^{i}\sum_{m=0}^{l+1} \frac{\binom{(a+k)-1}{i}\binom{i}{l}\binom{l+1}{m}(-1)^{-i}\beta^l\Gamma(m+n\alpha^{-1}+1)}{\alpha(\beta+1)^i[\beta(i+1)]^{m+n\alpha^{-1}+1}}$$

By inserting (25) in (23) we have the nth moment of ZB-PL.

$$E(x^n) = \frac{\alpha\beta^2(a+k)}{(\beta+1)} \sum_{k=0}^{\infty} b_k\, L_1(\alpha,\beta,(a+k),n) \tag{26}$$

4.1 Moment Generating Function (mgf)

If X~Zografos-Balakrishna-G distribution, then its moment generating function M(t) will be:

$$M(t) = \sum_{k=0}^{\infty} b_k M_{a+k}(t) \tag{27}$$

However, $\mathrm{M}_{\mathrm{a+k}}(\mathrm{t})$is the mgf of $\mathrm{Y}_{\mathrm{a+k}}(\mathrm{t})$~Exp-G distribution [Nadarajah et al. (2015)].

So, if X~ZB-PL distribution, Then $\mathrm{Y}_{\mathrm{a+k}}(\mathrm{t})$~ EPL. The mgf of EPL distribution is given by Warahena-Liyanage and Pararai (2014):

$$M_{a+k}(t) = \frac{\alpha\beta^2}{\beta+1} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_1(\alpha,\beta,(a+k),n) \tag{28}$$

whereas,

$$L_1(\alpha,\beta,(a+k),n)=\sum_{i=0}^{\infty}\sum_{l=0}^{i}\sum_{m=0}^{l+1}\frac{\binom{(a+k)-1}{i}\binom{i}{l}\binom{l+1}{m}(-1)^{-i}\beta^{l}\Gamma(m+n\alpha^{-1}+1)}{\alpha(\beta+1)^{i}[\beta(i+1)]^{m+n\alpha^{-1}+1}}$$

Substituting (28) in (27), the mgf of ZB-PL is:

$$M(t)=\frac{\alpha\beta^2}{\beta+1}\sum_{k=0}^{\infty}b_k\sum_{n=0}^{\infty}\frac{t^n}{n!}L_1(\alpha,\beta,(a+k),n) \quad (29)$$

whereas b_k is defined in (24).

4.2 Mean Deviation

Mean deviation about mean and median for x can be written as:

$$E_1(x)=2\mu_1'F(\mu_1')-2m_1(\mu_1') \quad (30)$$

$$E_2(x)=\mu_1'-2m_1(M) \quad (31)$$

Respectively. Whereas, M is the median of x given in (22), $F(\mu_1')$ can be calculated from (7) and $\mu_1'=E(x)$ can be obtained from (26) by substituting n=1.

Whereas, $m_1(z)$ is the first incomplete moment is defined as:

$$m_1(z)=\int_{-\infty}^{z}xf(x)dx$$

Using results from (16) we can write $m_1(z)$ as:

$$m_1(z)=\sum_{k=0}^{\infty}b_k\ M_k(z) \quad (32)$$

While, b_k is defined in (24). However,

$$M_k(z)=\int_0^z xf'_{(a+k)}(x)\,dx$$

As $f'_{(a+k)}(x)$ is the density function of EPL, so:

$$M_k(z)=\int_0^z(a+k)\left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{(a+k)-1}\ \frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}dx$$

Using the binomial expansion:

$$(1-z)^{\gamma-1}=\ \sum_{i=0}^{\infty}\binom{\gamma-1}{i}(-1)^i z^i \quad (33)$$

whereas γ is real non-integer and $|z|$.By simplification we get:

$$M_k(z)=\frac{(a+k)\alpha\beta^2}{\beta+1}\sum_{i=0}^{\infty}\binom{(a+k)-1}{i}(-1)^i\quad\int_0^z\left(\frac{1+\beta(1+x^\alpha)}{\beta+1}\right)^i\quad(1+x^\alpha)x^{\alpha-1}e^{-i\beta x^\alpha}e^{-\beta x^\alpha}dx$$

$$= \frac{(a+k)\alpha\beta^2}{\beta+1} \sum_{i=0}^{\infty} \binom{(a+k)-1}{i} \frac{(-1)^i}{(\beta+1)^i} \int_0^z (1+\beta(1+x^\alpha))^i \quad (1+ x^\alpha) x^{\alpha-1} e^{-\beta x^\alpha (i+1)} dx$$

Using the binomial power series (1.110) from book of Gradstein and Ryzik (1965):

$$(1+z)^\gamma = \sum_{i=0}^{\gamma} \binom{\gamma}{i} z^i \tag{34}$$

Using binomial expansion series (34), we get:

$$M_k(z) = \frac{(a+k)\alpha\beta^2}{\beta+1} \sum_{i=0}^{\infty} \binom{(a+k)-1}{i} \frac{(-1)^i}{(\beta+1)^i} \sum_{l=0}^{i} \binom{i}{l} \beta^l \int_0^z (1+ x^\alpha)^{l+1} \; x^{\alpha-1} e^{-\beta x^\alpha (i+1)} dx$$

$$= \frac{(a+k)\alpha\beta^2}{\beta+1} \sum_{i=0}^{\infty} \binom{(a+k)-1}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{l+1} \binom{l+1}{m} \frac{\beta^l (-1)^i}{(\beta+1)^i} \int_0^z x^{m\alpha+\alpha-1} e^{-\beta x^\alpha (i+1)} dx$$

Let

$$s = \beta x^\alpha (i+1), 0 < x < z \text{ then } 0 < s < \beta z^\alpha (i+1)$$

$$x = \left(\frac{s}{\beta(i+1)}\right)^{1/\alpha}$$

$$dx = \frac{1}{\alpha} \left(\frac{s}{\beta(i+1)}\right)^{\frac{1}{\alpha}-1} \frac{ds}{\beta(i+1)}.$$

Now,

$$M_k(z) = \frac{(a+k)\alpha\beta^2}{\beta+1} \sum_{i=0}^{\infty} \binom{(a+k)-1}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{l+1} \binom{l+1}{m} \frac{\beta^l (-1)^i}{(\beta+1)^i} \frac{1}{\alpha\beta(i+1)^{m+1}} \int_0^{\beta z^\alpha (i+1)} s^m e^{-s} ds$$

$\int_0^{\beta z^\alpha (i+1)} s^m e^{-s} ds$ is the upper incomplete gamma function, so:

$$M_k(z) = \frac{(a+k)\alpha\beta^2}{\beta+1} \sum_{i=0}^{\infty} \binom{(a+k)-1}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{l+1} \binom{l+1}{m} \frac{\beta^l (-1)^i}{(\beta+1)^i} \frac{\gamma(m+1, \beta z^\alpha (i+1))}{\alpha\beta(i+1)^{m+1}} \tag{35}$$

By substituting (35) in (32) we have first incomplete moment:

$$m_1(z) = \frac{(a+k)\alpha\beta^2}{\beta+1} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{i} \sum_{m=0}^{l+1} \binom{(a+k)-1}{i} \binom{i}{l} \binom{l+1}{m} \frac{b_k \beta^l (-1)^i}{(\beta+1)^i} \frac{\gamma(m+1, \beta z^\alpha (i+1))}{\alpha\beta(i+1)^{m+1}} \tag{36}$$

Using (30), (31) and (36) we can acquire mean deviations of ZB-PL distribution.

5. Renyi Entropy

The variation or uncertainty exists in r.v X is measured through entropy. Renyi (by Renyi (1961)) and Shannon (by Shannon (1948)) are the basic entropies. Here we will discuss only Renyi entropy as Shannon entropy exists as its special case for v →1 so:

$$I_R(v) = \frac{1}{1-v}\ log \int_{-\infty}^{\infty} f^v(x)dx \tag{37}$$

As using probability density function of distribution without simplification:

$$\int_{-\infty}^{\infty} f^v(x)dx = \int_{-\infty}^{\infty}\left(\frac{\alpha\beta^2}{\Gamma(a)(\beta+1)}\right)^v \left((1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\right)^v \left[-log\left\{1-\left(1-\left(1+\frac{\beta}{\beta+1}x^\alpha\right)e^{-x^\alpha\beta}\right)\right\}\right]^{va-v} dx \tag{38}$$

According to Nadarajah et al. (2015), for:

$$\left\{-\,log\left(1-G(x)\right)\right\}^{va-v} = (va-v)\sum_{k=0}^{\infty}\binom{k+v-va}{k} \sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(va-v-j)}\left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{v(a+k)-v} \tag{39}$$

As

$$p_{j,k} = k^{-1}\sum_{m=1}^{k}\frac{(-1)^{m+1}[k-m(j+1)]}{m+1}p_{j,k-m}$$

for k $= 1, 2, \dots$ and $p_{j,0} = 1$.

whereas G(x) is the cdf Power Lindely distribution (3)

Now substituting (39) in (38) we have:

$$\int_{-\infty}^{\infty} f^v(x)dx = (va-v)\sum_{k=0}^{\infty}\binom{k+v-va}{k}\ \sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\ p_{j,k}}{(va-v-j)}\left(\frac{\alpha\beta^2}{\Gamma(a)(\beta+1)}\right)^v\ I_k \tag{40}$$

However;

$$I_k = \int_0^{\infty}\left\{1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right\}^{v(a+k)-v}\ \left((1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\right)^v dx$$

As Shannon entropy exist as the special case of Renyi entropy for v → 1:

So,using binomial expansion (33):

$$I_k = \sum_{i=0}^{\infty}\binom{v(a+k)-v}{i}(-1)^i\int_0^{\infty}\left(1+\frac{\beta x^\alpha}{\beta+1}\right)^i\left(e^{-\beta x^\alpha}\right)^i\ \left((1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\right)^v dx$$

Using the binomial series (34):

$$I_k = \sum_{i=0}^{\infty} \binom{\nu(a+k)-\nu}{i} \sum_{l=0}^{i} \binom{i}{l} \frac{\beta^l (-1)^i}{(\beta+1)^l} \int_0^{\infty} (1 + x^{\alpha})^{\nu} (x^{\alpha})^l (x^{\alpha-1})^{\nu} \; e^{-\beta x^{\alpha}(i+\nu)} dx$$

Again, using binomial series (34):

$$I_k = \sum_{i=0}^{\infty} \binom{\nu(a+k)-\nu}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{\nu} \binom{\nu}{m} \frac{\beta^l (-1)^i}{(\beta+1)^l} \int_0^{\infty} x^{m\alpha+\alpha l+\alpha\nu-\nu} \; e^{-\beta x^{\alpha}(i+\nu)} dx \qquad (41)$$

Let y = $\beta x^{\alpha}(i+\nu)$,

$$x = \left(\frac{y}{\beta(i+\nu)}\right)^{1/\alpha} \quad as\ 0 < x < \infty \quad then\ 0 < y < \infty$$

$$dx = \frac{1}{\alpha}\left(\frac{y}{\beta(i+\nu)}\right)^{\frac{1}{\alpha}-1} \frac{dy}{\beta(i+\nu)}$$

so,

$$I_k = \sum_{i=0}^{\infty} \binom{\nu(a+k)-\nu}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{\nu} \binom{\nu}{m} \frac{\beta^l(-1)^i}{(\beta+1)^l} \frac{1}{\alpha(\beta(i+\nu))^{m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)}} \int_0^{\infty} (y)^{m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)-1} e^{-y} dy$$

As $\int_0^{\infty} (y)^{m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)-1} e^{-y} dy$ is gamma function so;

$$I_k = \sum_{i=0}^{\infty} \binom{\nu(a+k)-\nu}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{\nu} \binom{\nu}{m} \frac{\beta^l(-1)^i}{(\beta+1)^l} \frac{\Gamma\left(m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)\right)}{\alpha(\beta(i+\nu))^{m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)}} \qquad (42)$$

From (42), (40) and (37) we have renyi entropy as:

$$I_R(\nu) = -\frac{\nu log\Gamma(a)}{1-\nu} + \frac{1}{1-\nu} log \left\{ \begin{matrix} (\nu a - \nu) \sum_{k=0}^{\infty} \binom{k+\nu-\nu a}{k} \sum_{j=0}^{k} \frac{(-1)^{j+k} \binom{k}{j} \; p_{j,k}}{(\nu a - \nu - j)} \\ \sum_{i=0}^{\infty} \binom{\nu(a+k)-\nu}{i} \sum_{l=0}^{i} \binom{i}{l} \sum_{m=0}^{\nu} \binom{\nu}{m} \; W \end{matrix} \right\}$$

whereas,

$$W=\left(\frac{\alpha\beta^2}{\Gamma(a)(\beta+1)}\right)^{\nu}\frac{\Gamma\left(m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)\right)}{\alpha\left(\beta(i+\nu)\right)^{m+l+\nu-\left(\frac{\nu-1}{\alpha}\right)}}.$$

6. Reliability

If we have two independent r.v $X_1' \sim$ ZB-PL (a_1) and $X_2' \sim$ZB-PL (a_2) then the reliability will be $R = \text{Prob}\,(X_2' < X_1')$. As the density function of ZB-PL is the linear combination of EPL $(a_i + k)$ i=1,2. According to Nadarajah et al. (2015) and the reference of results obtain from (16) and (17) therefore, we can redefine Reliability as:

$$R = \sum_{j,k=0}^{\infty} d_{jk} \tag{43}$$

The linear combination of the reliability of EPL variable.

Whereas:

$$d_{j,k}=\frac{\binom{k+1-a_1}{k}}{(a_1+k)\Gamma(a_1-1)}\frac{\binom{j+1-a_2}{j}}{(a_2+j)\Gamma(a_2-1)}\left[\sum_{i=0}^{k}\frac{(-1)^{i+k}\binom{k}{i}\,p_{i,k}}{(a_1-1-i)}\right]\left[\sum_{i=0}^{i}\frac{(-1)^{i+j}\binom{j}{i}\,p_{i,j}}{(a_2-1-i)}\right]$$

However, $R_{j,k}=$ Prob$(Y_j' < Y_k')$ then $Y_j' \sim$ Exp- Power Lindley $(a_2 + j)$ and $Y_k' \sim$ Exp-Power Lindley $(a_1 + k)$. As $F^*(a_2 + j)$ is the cdf of Y_j' and $f^*(a_1 + k)$ is the pdf of Y_k'.

As:

$$R_{j,k} = \int_0^{\infty} F^*(a_2 + j)f^*(a_1 + k)dx$$

$$R_{j,k} = \frac{(a_1+k)\alpha\beta^2}{\beta+1}\int_0^{\infty}\left\{1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right\}^{(a_2+j)}$$

$$(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left\{1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right\}^{(a_1+k)-1}dx$$

$$=\frac{(a_1+k)\alpha\beta^2}{\beta+1}\int_0^{\infty}\left\{1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right\}^{(a_2+j+a_1+k)-1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}dx$$

Using the binomial expansion (33):

$$R_{j,k}=\frac{(a_1+k)\alpha\beta^2}{\beta+1}\sum_{l=0}^{\infty}\binom{(a_2+j+a_1+k)-1}{l}\frac{(-1)^l}{(\beta+1)^l}$$

$$\int_0^{\infty}(1+\beta(1+x^{\alpha}))^l(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}(l+1)}dx$$

Using binomial series (34)

$$=\frac{(a_1+k)\alpha\beta^2}{\beta+1}\sum_{l=0}^{\infty}\binom{(a_2+j+a_1+k)-1}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta^m(-1)^l}{(\beta+1)^l}\int_0^{\infty}(1+x^{\alpha})^{m+1}x^{\alpha-1}e^{-\beta x^{\alpha}(l+1)}dx$$

Again, using binomial series (34):

$$= \frac{(a_1+k)\alpha\beta^2}{\beta+1} \sum_{l=0}^{\infty} \binom{(a_2+j+a_1+k)-1}{l} \sum_{m=0}^{l} \binom{l}{m} \sum_{n=0}^{m+1} \binom{m+1}{n} \frac{\beta^m(-1)^l}{(\beta+1)^l} \int_0^{\infty} x^{n\alpha+\alpha-1} e^{-\beta x^{\alpha}(l+1)} dx$$

Let $v = \beta x^{\alpha}(l+1)$

$x = \left(\frac{v}{\beta(l+1)}\right)^{1/\alpha}$ as $0 < x < \infty$ then $0 < v < \infty$

$dx = \frac{1}{\alpha}\left(\frac{v}{\beta(l+1)}\right)^{\frac{1}{\alpha}-1} \frac{dv}{\beta(l+1)}$

so,

$$R_{j,k} = \frac{(a_1+k)\alpha\beta^2}{\beta+1} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{n=0}^{m+1} \binom{(a_2+j+a_1+k)-1}{l} \binom{l}{m} \binom{m+1}{n} \frac{\beta^m(-1)^l}{\alpha(\beta+1)^l} \frac{\int_0^{\infty} v^n e^{-v} dv}{(\beta(l+1))^{n+1}}$$

Here $\Gamma(n+1) = \int_0^{\infty} v^n e^{-v} dv$ is the complete gamma function. Therefore, reliability between independent variable is:

$$R_{j,k} = \frac{(a_1+k)\alpha\beta^2}{\beta+1} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{n=0}^{m+1} \binom{(a_2+j+a_1+k)-1}{l} \binom{l}{m} \binom{m+1}{n} \frac{\beta^m(-1)^l}{\alpha(\beta+1)^l} \frac{\Gamma(n+1)}{(\beta(l+1))^{n+1}} \tag{44}$$

By substituting of (44) in (43) the reliability between two independent ZB-PL random variables is:

$$R = \frac{(a_1+k)\alpha\beta^2}{\beta+1} \sum_{j,k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{n=0}^{m+1} \binom{(a_2+j+a_1+k)-1}{l} \binom{l}{m} \binom{m+1}{n} \frac{d_{jk}\, \beta^m(-1)^l \Gamma(n+1)}{\alpha(\beta+1)^l(\beta(l+1))^{n+1}} \tag{45}$$

7. Maximum Likelihood Estimation

In this section the method of MLE is used to estimate unknown parameters of ZB-PL distribution. Suppose we have a random sample of size n such that $x_i \sim$ ZB-PL (a, α, β) $i = 1, \dots, n$ than the likelihood function will be:

$$L(\underline{\theta}) = \left(\frac{\alpha\beta^2}{\Gamma(a)(\beta+1)}\right)^n e^{-\beta \sum_{i=1}^{n} x_i^{\alpha}} \prod_{i=1}^{n} \left[(1+x_i^{\alpha}) x_i^{\alpha-1} \left\{-log\left(1+\frac{\beta x_i^{\alpha}}{\beta+1}\right)\beta x_i^{\alpha}\right\}^{a-1}\right]$$

Log of likelihood function:

$$log\, L = nlog\, \alpha + 2n\, log\, \beta - nlog\, \Gamma(a) - nlog(\beta+1) + \sum_{i=1}^{n} log\,(1+x_i^{\alpha})$$
$$+(\alpha-1)\sum_{i=1}^{n} log\, x_i - \beta \sum_{i=1}^{n} x_i^{\alpha} + (a-1)\sum_{i=0}^{1} log\left\{-log\left(1+\frac{\beta x_i^{\alpha}}{\beta+1}\right)\beta x_i^{\alpha}\right\}$$

Differentiating with respect to parameters a, α, and β we have:

$$\frac{\partial \log L}{\partial a} = n\psi(a) + \sum_{i=1}^{n} \log\left\{-\log\left(1+\frac{\beta x_i^{\alpha}}{\beta+1}\right)\beta x_i^{\alpha}\right\} \tag{46}$$

However, $\psi(a) = \Gamma'(a)/\Gamma(a)$, is a digamma function.

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i{}^{\alpha} \log x_i}{(1 + x_i{}^{\alpha})} + \sum_{i=1}^{n} log x_i$$

$$-(a-1) \sum_{i=1}^{n} \left[\frac{-\log\left(1 + \frac{\beta x_i{}^{\alpha}}{\beta+1}\right) \beta x_i{}^{\alpha} \ln x_i - \left(\frac{\beta x_i{}^{\alpha} \ln x_i}{1 + \beta(1 + x_i{}^{\alpha})}\right) \beta x_i{}^{\alpha}}{\left(\log\left(1 + \frac{\beta x_i{}^{\alpha}}{\beta+1}\right) \beta x_i{}^{\alpha}\right)} \right] \quad (47)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{2n}{\beta} - \frac{n}{(\beta+1)} - \sum_{i=1}^{n} x_i{}^{\alpha} - (a-1)$$

$$\sum_{i=1}^{n} \left[\frac{-log\left(1 + \frac{\beta x_i{}^{\alpha}}{\beta+1}\right) x_i{}^{\alpha} - \beta x_i{}^{\alpha} \left(\frac{x_i{}^{\alpha}(2\beta+1)}{(1 + \beta(1 + x_i{}^{\alpha}))(\beta+1)}\right)}{\left(log\left(1 + \frac{\beta x_i{}^{\alpha}}{\beta+1}\right) \beta x_i{}^{\alpha}\right)} \right] \quad (48)$$

As (46), (47) and (48) cannot be obtained in closed form therefore, to find MLE's by simultaneously solving the equations and using numerical method we can obtain the estimates.

8. Application

The newly proposed Zografos-Balakrishnan Power Lindley (ZBPL) distribution is applied to a real data set to exhibit applicability of the new model. This example deals with an uncensored data set containing a random sample about the remission time (in months) of 128 bladder cancer patient (Lee and Wang, 2003).

0.08,2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96,36.66,1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33,5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93,11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31,4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The density function of Lindley (L), Power Lindley (PL), Exponentiated Lindley (EL) and Weibull(W) are also fitted to compared with the new distribution.

Table 1 contains the descriptive of the data set. The Table 2 contains the values of the Estimates of the parameter, -2logL, Bayesian Information Criteria (BIC) and Akiake Information Criteria (AIC) for fitted models. The goodness of fit of the models is evaluated by comparing the values of -2log L, AIC and BIC for above mention data set.

The ZB-PL provides superior fit for the given dataset as compared to other models.

Table 1: Descriptive for Cancer Patient Data

Min	1st Quar.	Median	Mean	3rd Quar.	Max
0.080	3.348	6.395	9.366	11.838	79.050

Table 2 indicates the relative comparison of the model of ZB-PL with the other competitive models. The summary shows that the proposed model gives a good fit to the real data set. The AIC and BIC of the ZB-PL distribution is compared with other probability models. Smaller value of AIC and BIC indicates a good fit. Figure 5 presented below shows the boxplot, scatter plot, Total time test (TTT) plot and histogram with estimated density of the ZB-PL. The density curve of ZB-PL fits almost accurate on the histogram of data. The histogram for the estimated density support the results given in Table 2 which indicates the goodness of the new model.

Table 2: Estimates of Models for Cancer Patient Dataset

Model	α	β	a	AIC	BIC	-2logL
ZBPL	0.4922	1.6497	3.4939	827.46	836	821.46
PL	0.8303	0.2943	1	830.7	836.4	826.7
EL	1	0.1649	0.7334	836.6	842.3	832.6
L	1	0.1960	1	841.1	843.9	839.1
W	1.0474	0.0939	1	832.2	837.9	828.2

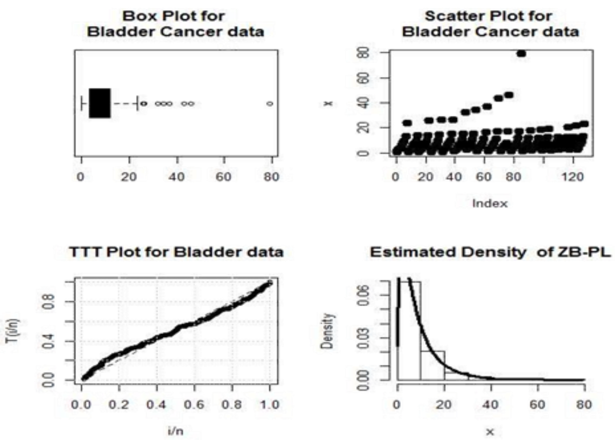

Figure 5. Graphs for ZB-PL

9. Conclusions

In this paper, Power Lindley distribution is generalized using Zografos and Balakrishnan technique (2009). The upper record value distribution arises from independently identically distributed Power Lindley random variable exist as a special case (for a = positive integral).

The density and distribution function is expanded as a linear combination of EPL distribution. This expansion is further used for studying different properties of the proposed distribution.

The probability distribution function (pdf), hazard rate function (hrf) and cumulative distribution function (cdf) are graphed. The hazard rate function depicts constant, monotonically decreasing, increasing and bathtub shape behaviour for different values of given parameters. However, pdf graphs indicate right skewed, approximately symmetric and negatively skewed behaviour. As the values of both the shape parameters increases, probability mass function of the distribution moves from right tail to shoulder and then to the centre and the left tail.

Different mathematical and statistical properties such as Asymptotes, Quantile function, Moments, Moment generating function, Mean deviation, Renyi Entropy, Reliability are studied. Maximum Likelihood Estimation equations are obtained. In the end application to a real dataset is provided. It has been noticed that the proposed distribution works better than many well-known distributions.

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