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L-Moments Estimations for Mixture of Weibull Distributions

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Abstract: Mixture of Weibull distributions has wide application in modeling of heterogeneous data sets. The parameter estimation is one of the most important problems related to mixture of Weibull distributions. In this paper, we propose a *L*-moment estimation method for mixture of two Weibull distributions. The proposed method is compared with maximum likelihood estimation (MLE) method according to the bias, the mean absolute error, the mean total error and completion time of the algorithm (time) by simulation study. Also, applications to real data sets are given to show the flexibility and potentiality of the proposed estimation method. The comparison shows that, the proposed method is better than MLE method.

Key words: EM algorithm, heterogeneous data, *L*-moment, mixture distribution, MLE.

1. Introduction

The mixture distributions have provided a mathematical-based approach to the statistical modeling of a wide variety of random phenomena. The mixture distributions are useful and flexible models to analyze random durations in a possibly heterogeneous population. In many applications, available data can be considered as the data coming from a mixture population of two or more distributions. Therefore mixture distributions play a vital role in many practical applications. For example, direct applications of finite mixture models are in fisheries research, economics, medicine, psychology, paleoanthropology, botany, agriculture, zoology, life testing and reliability, among others. Indirect applications include outliers, Gaussian sums, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric density estimation.

Recently the mixture of Weibull distributions has been recognized as a suitable model for heterogeneous datasets. Different methods are used to estimate

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the parameter of the mixture of Weibull distributions. Mendenhall and Hader (1958) considered an n components mixture Weibull distribution. They derived the maximum likelihood estimates for the scale and mixing parameters, assuming that the shape parameters are known. Kao (1959) and Jiang and Murthy (1995) proposed graphical procedures to decide the appropriateness of a two components mixture Weibull distribution. A method for estimating the parameters of mixture distributions using sample moments has been outlined by Paul R. Rider (1961) who considered special case of the mixture of Weibull distributions. Falls (1970) attempted to find the five parameters of a two components mixture Weibull distribution by the method of moments. Olsson (1979) directly searched the maximum of the log-likelihood function of the Mixture Weibull distribution through the Nelder-Mead Simplex Procedure. Kaylan and Harris (1981) derived MLEs for the first Mixed Weibull. Beetz (1982) estimated the parameters of a mixture Weibull distribution by fitting the mixed probability density to the experimental histogram using the maximum likelihood method. Cheng and Fu (1982) proposed a weighted least squares method for estimating the parameters of a mixture of two Weibulls when the data are grouped postmortem. Sinha (1986) gave an iterative procedure to obtain the MLE of a two-Weibull mixture for postmortem data. Ashour (1987) considered the problem of maximum likelihood estimation with five unknown parameters of the mixture Weibull distribution for multistage censored type-I sample. Jiang and Kececioglu (1992) presented an algorithm for estimating the parameters of a Weibull mixture model with the right censored data using the method of maximum likelihood. Ahmad and Abdulrahman (1994) presented a procedure for finding the maximum likelihood estimates of the parameters of a mixture of two Weibull distributions. Ling *et al.* (2009) established parameter estimation methods for the mixture Weibull model using nonlinear least squares theory; and Quasi-Newton method have been used to solve the optimization problem.

In this paper we will describe an EM algorithm for L-moment estimation of two components mixture Weibull distribution. Also, we provide a comprehensive comparison of the L-moment and maximum likelihood estimation methods for the two component mixture Weibull distribution. We use the following criteria for comparison: the bias, the mean absolute error, the mean total error and completion time of the algorithm (time).

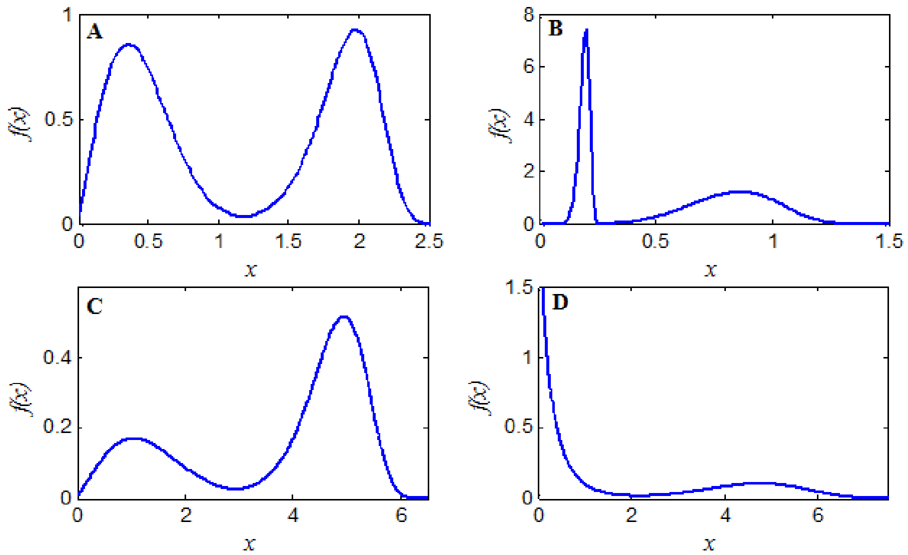
The structure of the paper is as follows. In Section 2, we defined density function, survival function and hazard function of two components mixtures Weibull distribution and we show plots of two component mixture Weibull distributions for different parameter values. In Section 3, we first proposed L-moment estimators for two components mixture Weibull distributions. Also, we presented the maximum likelihood estimations of two components mixture Weibull distribution. A numerical comparison of the L-moment estimation and the maximum likelihood estimation methods by simulation study is given in Section 4. In Section 5, the proposed L-moment estimation method for two components mixture Weibull distribution is applied to illustrative examples based on heterogeneous survival real data sets successfully. Finally, some conclusions are noted in Section 6.

2. Two Component Mixture Weibull Distribution

The probability density function of two components mixture Weibull distribution is defined mathematically as

$$f(x|\omega) = \pi \frac{\alpha_1}{\beta_1} \left(\frac{x}{\beta_1}\right)^{\alpha_1 - 1} e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}} + (1-\pi) \frac{\alpha_2}{\beta_2} \left(\frac{x}{\beta_2}\right)^{\alpha_2 - 1} e^{-\left(\frac{x}{\beta_2}\right)^{\alpha_2}}, \tag{1}$$

where $0 < \pi < 1$, $\alpha_i > 0$, $\beta_i > 0$ are mixture weight, shape and scale parameters of subpopulation i respectively and $\omega = (\pi, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is called the parameter vector of two components mixture Weibull distribution. Plots of density of two components mixture Weibull distribution for different parameter values are given in Figure 1.



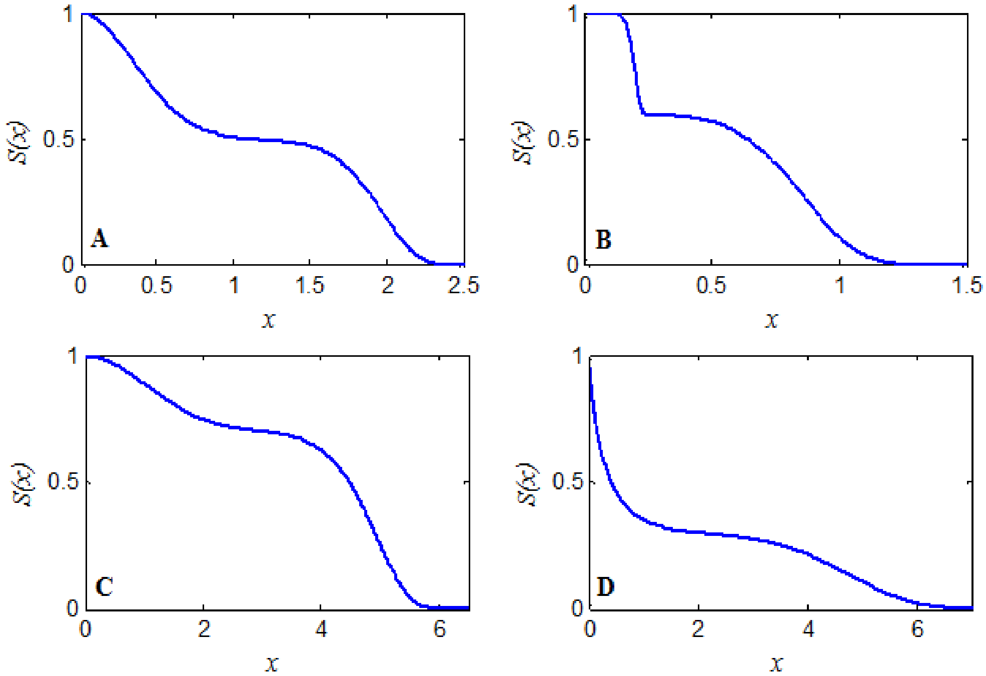
A:	$\alpha_1 = 2$	$\beta_1 = 0.5$	$\alpha_2 = 10$	$\beta_2 = 2$	$\pi = 0.5$
B:	$\alpha_1 = 5$	$\beta_1 = 0.9$	$\alpha_2 = 10$	$\beta_2 = 0.2$	$\pi = 0.6$
C:	$\alpha_1 = 2$	$\beta_1 = 1.5$	$\alpha_2 = 10$	$\beta_2 = 5$	$\pi = 0.3$
D:	$\alpha_1 = 0.8$	$\beta_1 = 0.3$	$\alpha_2 = 5$	$\beta_2 = 5$	$\pi = 0.7$

Figure 1: Plots of density of two components mixture Weibull distribution for different parameter values

The survival function $S(x|\omega)$ of two components mixture Weibull distribution is given as follows:

$$S(x|\omega) = \pi e^{-(\frac{x}{\beta_1})^{\alpha_1}} + (1-\pi)e^{-(\frac{x}{\beta_2})^{\alpha_2}}. \quad (2)$$

Plots of survival function of two components mixture Weibull distribution for different parameter values are given Figure 2.



A:	$\alpha_1 = 2$	$\beta_1 = 0.5$	$\alpha_2 = 10$	$\beta_2 = 2$	$\pi = 0.5$
B:	$\alpha_1 = 5$	$\beta_1 = 0.9$	$\alpha_2 = 10$	$\beta_2 = 0.2$	$\pi = 0.6$
C:	$\alpha_1 = 2$	$\beta_1 = 1.5$	$\alpha_2 = 10$	$\beta_2 = 5$	$\pi = 0.3$
D:	$\alpha_1 = 0.8$	$\beta_1 = 0.3$	$\alpha_2 = 5$	$\beta_2 = 5$	$\pi = 0.7$

Figure 2: Plots of survival functions of two components mixture Weibull distribution for different parameter values

The hazard function $h(x|\omega)$ of two components mixture Weibull distribution is given as follows:

$$h(x|\omega) = \frac{\pi\frac{\alpha_1}{\beta_1}\left(\frac{x}{\beta_1}\right)^{\alpha_1-1}e^{-(\frac{x}{\beta_1})^{\alpha_1}} + (1-\pi)\frac{\alpha_2}{\beta_2}\left(\frac{x}{\beta_2}\right)^{\alpha_2-1}e^{-(\frac{x}{\beta_2})^{\alpha_2}}}{\pi e^{-(\frac{x}{\beta_1})^{\alpha_1}} + (1-\pi)e^{-(\frac{x}{\beta_2})^{\alpha_2}}}. \quad (3)$$

Plots of hazard functions of two components mixture Weibull distribution for different parameter values are given in Figure 3.

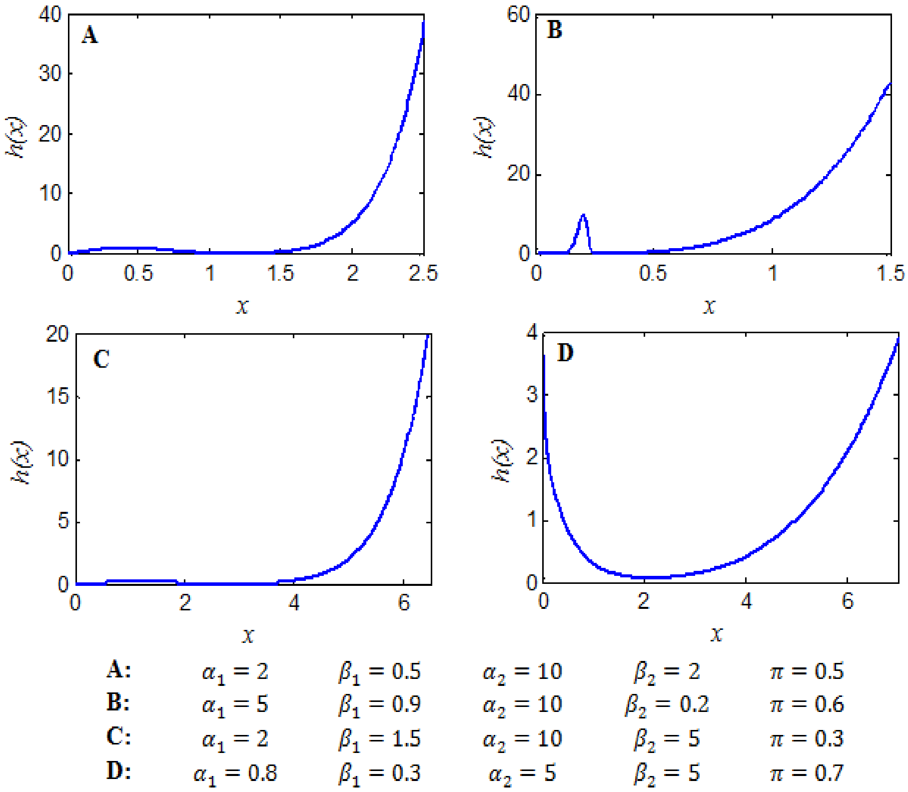


Figure 3: Plots of hazard functions of two components mixture Weibull distribution for different parameter values

3. Estimation

The Expectation-Maximization (EM) algorithm can be used to estimate the parameters of a mixture model distribution. EM algorithm is an iterative algorithm with two steps: an expectation step and a maximization step. In the EM framework, observed data $x_1, \cdots, x_n$ is viewed as being incomplete, as the associated component-label vector $\boldsymbol{z}$ is not available. The EM algorithm is applied to the mixture of distributions by treating $\boldsymbol{z}$ as missing data. In the E-step, it is determined elements of component-label vector $\boldsymbol{z}$. In the M-step, model parameters are calculated using the values of the component-label vector calculated in the previous E-step.

3.1 L-Moment Estimation

The new proposed estimator for two components mixture Weibull distribution is based L-moments. These are defined in terms of order statistics and have their origin in Hosking (1990). Kundu and Raqab (2005) have been applied for the generalized Rayleigh distribution, Abdul-Moniem (2007) have been applied for the exponential distributions and Teimouri *et al.* (2013) have been applied for the Weibull distribution. In this study, the proposed L-moment estimators for Weibull distribution of Teimouri *et al.* (2011) is adapted for two components mixture Weibull distribution.

Hosking's (1990) idea works on the basis of computing the r-th moment of the i-th order statistic. Equating the sample L-moment to the population counterpart gives the L-moment estimate. The L-moment μ_r^0 is given by

$$\mu_r^0 = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} E(X_{r-k:r}), \tag{4}$$

for $r = 1, 2, 3, \cdots$, where C_i^n denotes the binominal coefficient $n!/(i!(n-i)!)$ and $X_{i:n}$ denotes the i-th order statistics in a sample of size n. The sample L-moments are defined

$$m_r^0 = \frac{1}{r}\sum_{i=1}^{n} \frac{\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} C_k^{n-i} C_{r-k-1}^{i-1}}{C_r^n} X_{i:n}. \tag{5}$$

Firstly in the E-Step, the component-label vector $\boldsymbol{z}$ is determined by random in the new proposed estimation method for two components mixture Weibull distribution. The mixture weight is computed according to the component-label vector $\boldsymbol{z}$ as follows

$$\hat{\pi} = \frac{\sum_{i=1}^{n} \hat{z}_i}{n}, \tag{6}$$

where $\hat{z}_i$ is element of component-label vector $\boldsymbol{z}$ and it is defined as follows

$$z_i = \begin{cases} 1, & \text{if } i\text{-th observation is element of first subpopulation,} \\ 0, & \text{otherwise.} \end{cases}$$

In the M-step, L-moment estimations (LE) of shape and scale parameters for first component in two components mixture Weibull distribution are given follows by respectively,

$$\hat{\alpha}_1^{\mathrm{LM}} = \frac{\ln(2)}{\ln\left(1 - m_{2(1)}^0 / m_{1(1)}^0\right)}, \tag{7}$$

$$\hat{\beta}_1^{\mathrm{LM}} = \frac{m_{1(1)}^0}{\Gamma\left(1/\hat{\alpha}_1^{\mathrm{LM}} + 1\right)}, \tag{8}$$

where $m^0_{1(1)}$ and $m^0_{2(1)}$ are sample L-moments for first subpopulation and defined as follows

$$m^0_{1(1)} = \frac{\sum_{i=1}^n \hat{z}_i x_{i:n}}{\sum_{i=1}^n \hat{z}_i}, \tag{9}$$

$$m^0_{2(1)} = \frac{2}{(\sum_{i=1}^n \hat{z}_i)^2 - \sum_{i=1}^n \hat{z}_i} \sum_{i=1}^n \hat{z}_i(\varphi_{i(1)} - 1)x_{i:n} - m^0_{1(1)}, \tag{10}$$

where $\varphi_{i(1)}$ is rank number of i-th observation for ascending ordered observations in first subpopulation. $\varphi_{i(1)}$ is obtained only for i-th observations $\hat{z}_i \geq 0.5$. $\varphi_{i(1)}$ is assumed 1 if $\hat{z}_i$ is smaller than 0.5.

LE of shape and scale parameters for second component in two component mixture Weibull distribution are given as follows by respectively,

$$\hat{\alpha}_2^{\mathrm{LM}} = \frac{\ln(2)}{\ln\left(1 - m^0_{2(2)}/m^0_{1(2)}\right)}, \tag{11}$$

$$\hat{\beta}_2^{\mathrm{LM}} = \frac{m^0_{1(2)}}{\Gamma\left(1/\hat{\alpha}_2^{\mathrm{LM}} + 1\right)}, \tag{12}$$

where $m^0_{1(2)}$ and $m^0_{2(2)}$ are sample L-moments for second subpopulation and defined as follows;

$$m^0_{1(2)} = \frac{\sum_{i=1}^n (1-\hat{z}_i) x_{i:n}}{n - \sum_{i=1}^n \hat{z}_i}, \tag{13}$$

$$m^0_{2(2)} = \frac{2}{(n - \sum_{i=1}^n \hat{z}_i)^2 - n + \sum_{i=1}^n \hat{z}_i} \sum_{i=1}^n (1-\hat{z}_i)(\varphi_{i(2)} - 1)x_{i:n} - m^0_{1(2)}, \tag{14}$$

where $\varphi_{i(2)}$ is rank number of i-th observation for ascending ordered observations in second subpopulation. Then estimation of parameters according to initial values of missing observation vector $\boldsymbol{z}$, $\hat{z}_i$ values are updated by

$$\hat{z}_i = \frac{\hat{\pi} f(x_i|\hat{\alpha}_1^{\mathrm{LM}}, \hat{\beta}_1^{\mathrm{LM}})}{\hat{\pi} f(x_i|\hat{\alpha}_1^{\mathrm{LM}}, \hat{\beta}_1^{\mathrm{LM}}) + (1-\hat{\pi}) f(x_i|\hat{\alpha}_2^{\mathrm{LM}}, \hat{\beta}_2^{\mathrm{LM}})}. \tag{15}$$

The EM algorithm can be stopped if $|\ln L^{(t)} - \ln L^{(t+1)}| \leq \mathrm{tol}$, where tol is the desired tolerance. $\ln L$ is denoted logarithmic likelihood value and defined as follows

$$\ln L = \sum_{i=1}^n \ln(\hat{\pi} f(x_i|\hat{\alpha}_1^{\mathrm{LM}}, \hat{\beta}_1^{\mathrm{LM}}) + (1-\hat{\pi}) f(x_i|\hat{\alpha}_2^{\mathrm{LM}}, \hat{\beta}_2^{\mathrm{LM}})), \tag{16}$$

where $f(x_i)$ is density function of two parameter Weibull distribution.

3.2 Maximum Likelihood Estimation

Several authors have discussed the problems associated with the maximum likelihood estimation (MLE) method. The main difficulty of the method is the lack of analytical tractability and the need for iterative computational methods. EM algorithm is widely used for the estimation of parameters of the mixture Weibull distribution. In the E-step, the starting values of component-label vector $\boldsymbol{z}$ are taken as randomly. In the M-step, it requires the maximization of the log-likelihood function according to the values of the component-label vector determined in the previous E-step.

In the M-step, an iterative method must be used to estimate of the shape parameters of the mixture Weibull distribution. Newton-Raphson method can be used for the estimation of the shape parameters. The maximum likelihood estimations of α_1 and α_2 in the $(r+1)$-th iteration of Newton-Raphson method is defined by

$$\hat{\alpha}_{k,(r+1)} = \hat{\alpha}_{k,r} + \frac{A_{k,r} + (1/\hat{\alpha}_{k,r}) - (C_{k,r}/B_{k,r})}{(1+\hat{\alpha}_{k,r}^2) + (B_{k,r}D_{k,r} - C_{k,r}^2)/B_{k,r}^2}, \quad k = 1, 2, \tag{17}$$

where

$$A_{k,r} = \frac{\sum_{i=1}^n \hat{z}_{k,i} \ln x_i}{\sum_{i=1}^n \hat{z}_{k,i}}, \; B_{k,r} = \sum_{i=1}^n \hat{z}_{k,i} x_i^{\hat{\alpha}_{k,r}},$$

$$C_{k,r} = \sum_{i=1}^n \hat{z}_{k,i} x_i^{\hat{\alpha}_{k,r}} \ln x_i, \quad \text{and} \quad D_{k,r} = \sum_{i=1}^n \hat{z}_{k,i} x_i^{\hat{\alpha}_{k,r}} (\ln x_i)^2.$$

Once the shape parameters are estimated, then the scale parameters can be estimated as follows:

$$\hat{\beta}_k = \left(\frac{\sum_{i=1}^n \hat{z}_{k,i} x_i^{\hat{\alpha}_k}}{\sum_{i=1}^n \hat{z}_{k,i}} \right)^{1/\hat{\alpha}_k}, \quad k = 1, 2. \tag{18}$$

In the formulas, the belonging probability $z_{k,i}$, which is the probability that the unit belongs to the k-th subpopulation $(k = 1, 2)$ and $z_{k,i}$ is given by

$$\hat{z}_{1,i} = \frac{\hat{\pi} f(x_i|\hat{\alpha}_1, \hat{\beta}_1)}{\hat{\pi} f(x_i|\hat{\alpha}_1, \hat{\beta}_1)(1-\hat{\pi}) f(x_i|\hat{\alpha}_2, \hat{\beta}_2)}, \tag{19}$$

$$\hat{z}_{2,i} = 1 - \hat{z}_{1,i}, \tag{20}$$

where $f(x_i)$ is density function of two parameter Weibull distribution.

4. Simulation

Here, we compare the performances of the MLE and the LE for the two component mixture Weibull distribution. A simulation study with 10000 samples each of size $n = 30, 50, 100$ are randomly generated from the two components mixture Weibull distribution with different values of parameters. No restriction was imposed on the maximum number of iterations and convergence was assumed when the absolute differences between successive estimates were less than 10^{-4}. The average of 10000 MLEs and L-moments, their standard error is denoted by av and se respectively. We make all computations using Matlab program. The simulation results are given in Table 1.

Table 1 shows that convergence can be achieved for all cases in this simulation study. This emphasizes the numerical stability of the EM algorithm. The values of mean and standard error suggest that the EM estimates performed consistently. Standard errors of LEs and MLEs decrease when the sample size increases. Simulation results show that the L-moment estimation method works well, and the estimation performance is satisfied.

$$\text{Bias}(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i - \theta), \tag{21}$$

$$\text{MAE} = \frac{1}{10000}\sum_{i=1}^{10000}|\hat{\theta}_i - \theta|, \tag{22}$$

$$\text{MTE} = \frac{1}{10000}\sum_{i=1}^{10000}\sum_{j=1}^{5}\frac{(\hat{\theta}_{ij} - \theta_j)^2}{\theta_j}, \tag{23}$$

where $\hat{\theta}_i$ the estimator of θ in the i-th replication and $\hat{\theta}_{ij}$ the estimator of θ_j in the i-th replication. Larger values of the bias, the mean-absolute error and the mean-total error correspond to less efficient estimators. The comparison results according to bias and the mean-absolute error are given in Table 2. LE method is better than MLE method for all sample size and cases in terms of bias and the mean-absolute error in simulation study.

The comparison results according to the mean total error and time are given in Table 3. It can be deduced from the Table 3 that LE method is better than MLE method for all sample size and cases according to the mean total error and time. The comparisons of LE and MLE in terms of total error and time according to sample size are given Figure 4. As seen from these figures, LE method is better than MLE method.

Table 1: The means and standard errors of the LEs and MLEs for different parameter values of two components mixture Weibull distribution

	LM						MLE					
	$n=30$		$n=50$		$n=100$		$n=30$		$n=50$		$n=100$	
	av	*se*	*av*	*se*	*av*	*se*	*av*	*se*	*av*	*se*	*av*	*se*
$\alpha_1=2$	2.081323	0.004798	2.049037	0.003544	2.031524	0.002403	2.216218	0.004927	2.127906	0.003583	2.074295	0.002404
$\beta_1=0.5$	0.497394	0.000699	0.497628	0.000542	0.499118	0.000385	0.496964	0.000698	0.497534	0.000542	0.50022	0.000386
$\alpha_2=10$	10.66727	0.025988	10.42321	0.019033	10.2518	0.013093	11.25505	0.026075	10.74773	0.018413	10.40004	0.012302
$\beta_2=2$	2.000667	0.000549	2.000434	0.000429	2.000188	0.000305	1.997212	0.000547	1.999171	0.000427	1.999902	0.000305
$\pi=0.5$	0.500473	0.000137	0.500392	0.000105	0.500209	0.0000772	0.50118	0.000127	0.50118	0.000101	0.500895	0.000075
$\alpha_1=5$	5.123577	0.012329	5.07235	0.008586	5.062538	0.005464	4.739122	0.016464	4.915733	0.012651	5.084579	0.006744
$\beta_1=0.9$	0.841407	0.001911	0.889519	0.000896	0.900699	0.000251	0.688621	0.003054	0.791166	0.002439	0.885699	0.000994
$\alpha_2=10$	10.4881	0.031432	10.31149	0.021957	10.13559	0.01442	9.175303	0.033266	9.540661	0.02466	9.85516	0.014844
$\beta_2=0.2$	0.271254	0.002196	0.213642	0.001003	0.200136	0.000080	0.443577	0.003469	0.324845	0.00275	0.21713	0.0011
$\pi=0.6$	0.589103	0.000353	0.597152	0.000148	0.598954	0.000036	0.558013	0.000634	0.576274	0.00051	0.595522	0.000207
$\alpha_1=2$	2.173329	0.007047	2.122869	0.004963	2.089742	0.003237	2.373613	0.007621	2.229801	0.005289	2.158039	0.003413
$\beta_1=1.5$	1.479057	0.002894	1.488888	0.00216	1.493632	0.001537	1.551155	0.003054	1.531496	0.002329	1.519092	0.001649
$\alpha_2=10$	10.57953	0.021073	10.37231	0.015936	10.22191	0.011271	11.21601	0.023113	10.74375	0.016627	10.39128	0.011282
$\beta_2=5$	5.001856	0.001314	5.001583	0.000911	4.999961	0.00065	5.00245	0.001335	5.001256	0.000922	5.000399	0.000659
$\pi=0.3$	0.301037	0.000218	0.300216	0.000164	0.299465	0.000118	0.306527	0.00026	0.30428	0.000197	0.301528	0.000137
$\alpha_1=0.8$	0.790196	0.001529	0.794255	0.001162	0.799897	0.000813	0.844907	0.001687	0.815263	0.001206	0.794277	0.000815
$\beta_1=0.3$	0.297702	0.00096	0.297837	0.000751	0.298738	0.000541	0.293068	0.001053	0.294785	0.000769	0.296101	0.000542
$\alpha_2=5$	5.834155	0.020617	5.501588	0.014138	5.273661	0.00969	5.945987	0.023731	5.502117	0.015339	5.274589	0.010115
$\beta_2=5$	5.018941	0.003682	5.017814	0.002909	5.012562	0.00208	4.928248	0.003965	4.952286	0.003147	4.974199	0.002229
$\pi=0.7$	0.70148	0.000247	0.70104	0.000197	0.699875	0.00015	0.690747	0.000339	0.692484	0.000259	0.694446	0.000184

Table 2: The bias and the mean-absolute errors of the LEs and MLEs for different parameter values of two components mixture Weibull distribution

	LM						**MLE**					
	$n=30$		$n=50$		$n=100$		$n=30$		$n=50$		$n=100$	
	Bias	MAE	Bias	MAE	Bias	MAE	Bias	MAE	Bias	MAE	Bias	MAE
$\alpha_1=2$	0.0813	0.3618	0.0490	0.2723	0.0315	0.1884	0.2162	0.3927	0.1279	0.2853	0.0743	0.1938
$\beta_1=0.5$	-0.0026	0.0508	-0.0024	0.0434	-0.0009	0.0305	-0.0030	0.0556	-0.0025	0.0436	0.0002	0.0306
$\alpha_2=10$	0.6673	1.9841	0.4232	1.4803	0.2518	0.9922	1.2551	2.0908	0.7477	1.4891	0.4000	0.9935
$\beta_2=2$	0.0007	0.0429	0.0004	0.0343	0.0002	0.0240	-0.0028	0.0437	-0.0008	0.0345	-0.0001	0.0244
$\pi=0.5$	0.0005	0.0041	0.0004	0.0045	0.0002	0.0041	0.0012	0.0042	0.0012	0.0047	0.0009	0.0041
$\alpha_1=5$	0.1236	0.9308	0.0724	0.6571	0.0625	0.4339	-0.2609	1.3160	-0.0843	0.9045	0.0846	0.4751
$\beta_1=0.9$	-0.0586	0.0923	-0.0105	0.0384	0.0007	0.0195	-0.2114	0.2360	-0.1088	0.1323	-0.0143	0.0338
$\alpha_2=10$	0.4881	2.3084	0.3115	1.6782	0.1356	1.1340	-0.8247	2.5806	-0.4593	1.9535	-0.1448	1.1541
$\beta_2=0.2$	0.0713	0.0755	0.0136	0.0173	0.0001	0.0028	0.2436	0.2470	0.1248	0.1280	0.0171	0.0198
$\pi=0.6$	-0.0109	0.0113	-0.0028	0.0031	-0.0010	0.0012	-0.0420	0.0423	-0.0237	0.0238	-0.0045	0.0045
$\alpha_1=2$	0.1733	0.5004	0.1229	0.3699	0.0897	0.2514	0.3736	0.5781	0.2298	0.4175	0.1580	0.2827
$\beta_1=1.5$	-0.0209	0.2274	-0.0111	0.1733	-0.0064	0.1227	0.0512	0.2384	0.0315	0.1839	0.0191	0.1293
$\alpha_2=10$	0.5795	1.6227	0.3723	1.2470	0.2219	0.8900	1.2160	1.8963	0.7438	1.3680	0.3913	0.9150
$\beta_2=5$	0.0019	0.0936	0.0016	0.0725	0.0000	0.0516	0.0024	0.0955	0.0013	0.0733	0.0004	0.0522
$\pi=0.3$	0.0010	0.0099	0.0002	0.0093	-0.0005	0.0079	0.0065	0.0134	0.0043	0.0118	0.0015	0.0091
$\alpha_1=0.8$	-0,0098	0.1187	-0.0057	0.0922	-0.0001	0.0680	0.0449	0.1417	0.0153	0.1030	-0.0057	0.0714
$\beta_1=0.3$	-0,0023	0.0758	-0.0022	0.0595	-0.0013	0.0432	-0.0069	0.0780	-0.0052	0.0614	-0.0039	0.0439
$\alpha_2=5$	0.8342	1.4533	0.5016	1.0826	0.2737	0.7906	0.9460	1.7194	0.5021	1.1934	0.2746	0.8013
$\beta_2=5$	0.0189	0.2927	0.0178	0.2321	0.0126	0.1658	-0.0718	0.3170	-0.0477	0.2517	-0.0258	0.1762
$\pi=0.7$	0.0015	0.0136	0.0010	0.0124	-0.0001	0.0107	-0.0093	0.0204	-0.0075	0.0170	-0.0056	0.0131

Table 3: The mean total errors and times of the LEs and MLEs for different parameter values of two components mixture Weibull distribution

Method	n	**Population**							
		A		**B**		**C**		**D**	
		MTE	Time (sec)	MTE	Time (sec)	MTE	Time (sec)	MTE	Time (sec)
LM	30	0.67550	34.962	1.36978	37.427	0.63011	38.462	0.63770	88.102
MLE		0.69658	79.990	2.77831	108.053	0.75644	93.807	0.84495	190.621
LM	50	0.39000	37.229	0.62555	47.751	0.37212	42.322	0.37478	93.974
MLE		0.40029	86.182	1.68132	130.889	0.42311	105.352	0.45968	210.834
LM	100	0.17416	43.481	0.25725	51.014	0.18900	46.445	0.20349	97.743
MLE		0.19922	100.638	0.43177	166.194	0.20235	137.633	0.22374	241.886

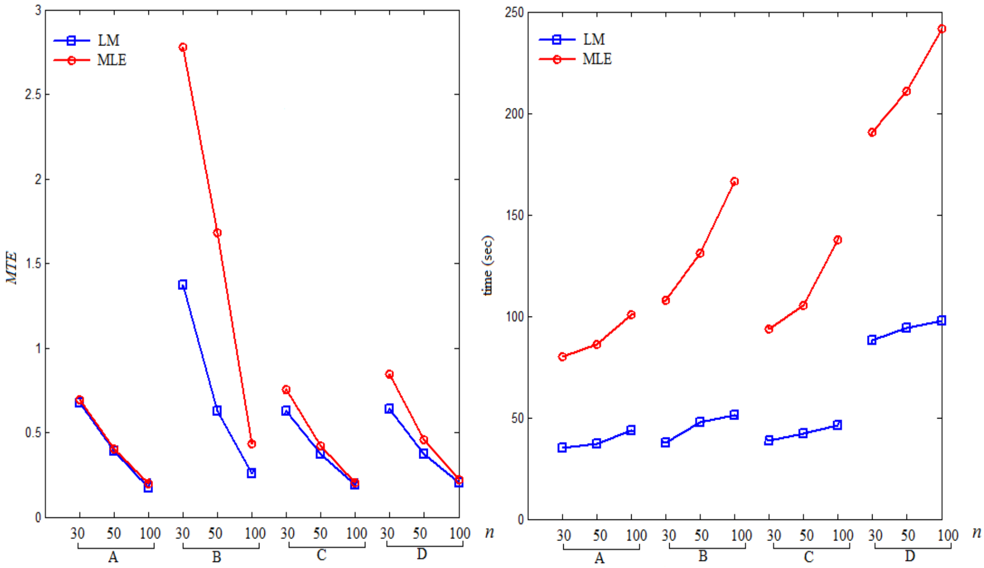


Figure 4: The comparisons of LE and MLE in terms of total error and time according to sample size

5. Application

Two examples of real data sets are used to illustrate the use of the proposed LE method for two components mixture Weibull distribution.

Example 1: Fatigue lives dataset consists of 25 specimens of 2 different types. This data set was first discussed by Ling and Pan (1998) and later discussed by

Erişoğlu and Erol (2010).

Example 2: A data set consists of 100 failure times for oral irrigators dataset. This data set was first Colvert and Boardman (1976) and further discussed by Jiang and Murthy (1997) and Erişoğlu and Erol (2010).

It is known that Example 1 and 2 had two sub groups from previous studies. Therefore the number of components in the mixture models has been taken two in this study. Parameter estimates, time, lnL, and K-S goodness of fit testing results obtained using the LE and MLE methods for Example 1 and 2 are given Table 4. It can be seen that the LE method yields a value of K-S smaller than that of the MLE method. Therefore, LE method is better than MLE method in modeling for Example 1 and 2.

Table 4: Parameter estimates, standard errors of parameter estimates, time, $\ln L$, and K-S goodness of fit testing results obtained using the LE and MLE methods for Example 1 and 2

Example		**LE**	(std. error)	**MLE**	(std. error)
	$\hat{\alpha}_1$	3.7717	(0.2130)	4.0386	(0.2250)
	$\hat{\beta}_1$	11.1289	(0.2279)	11.0932	(0.2111)
	$\hat{\alpha}_2$	3.9497	(0.2838)	4.3134	(0.3039)
	$\hat{\beta}_2$	37.4515	(0.9118)	37.293	(0.8311)
I	$\hat{\pi}$	0.56	(0.0406)	0.56	(0.0405)
	Time(sec)	**0.266758**		0.428352	
	$\ln L$	-90.3087		**-90.2449**	
	K-S	**0.056456**		0.058768	
	(p-value)	**(0.9999983)**		(0.9999947)	
	$\hat{\alpha}_1$	1.263	(0.0174)	1.2428	(0.0171)
	$\hat{\beta}_1$	124.7848	(1.7549)	123.7296	(1.7585)
	$\hat{\alpha}_2$	6.3789	(0.1248)	6.4971	(0.1266)
	$\hat{\beta}_2$	515.7068	(2.1296)	515.3055	(2.0918)
II	$\hat{\pi}$	0.5918	(0.0102)	0.5918	(0.0101)
	Time(sec)	**0.281165**		1.400482	
	$\ln L$	-626.3724		**-626.3399**	
	K-S	**0.065859**		0.068412	
	(p-value)	**(0.788954)**		(0.748706)	

Figure 5 (a) shows the histograms of data sets and the fitted pdf curves of distribution according to parameter estimates obtained by using LE and MLE methods for Example 1 and 2. Figure 5 (b) shows a comparison of the empirical cumulative distribution function curves and the cumulative function curves of distributions according to parameter estimates obtained by using LE and MLE methods for Example 1 and 2.

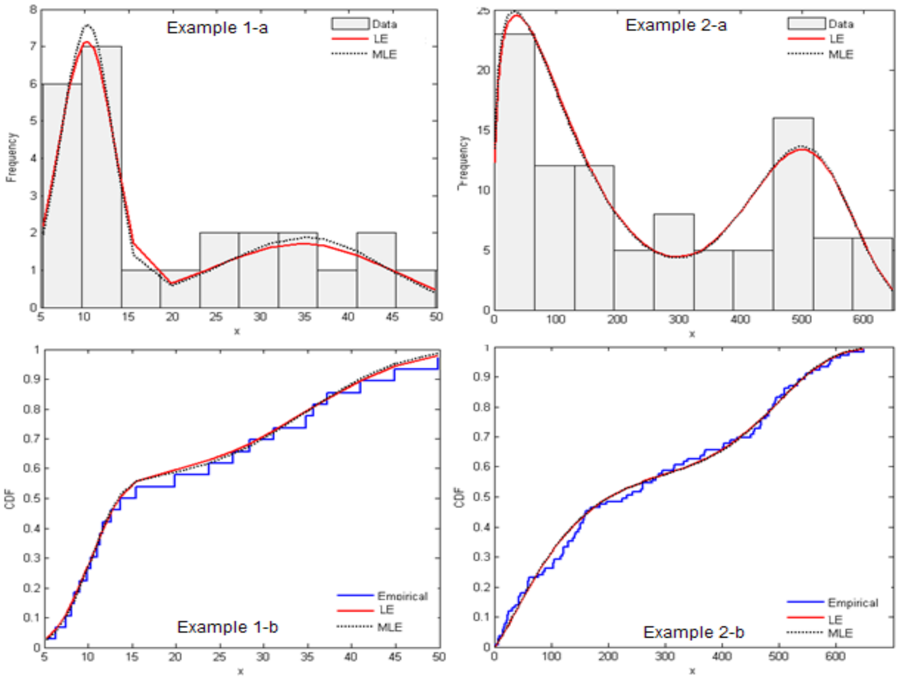


Figure 5: The (a) probability density functions and (b) cumulative distribution function curves for Example1 and 2

6. Conclusions

We have proposed a new method for two components mixture Weibull distribution based on L-moments. We have performed simulations to compare this method with MLE method. The simulation result shows that LE method is better than MLE method according to bias, the mean absolute error, the mean total error and completion time of the algorithm.

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