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The Weibull-*G* Family of Probability Distributions

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Abstract: The Weibull distribution is the most important distribution for problems in reliability. We study some mathematical properties of the new wider Weibull-G family of distributions. Some special models in the new family are discussed. The properties derived hold to any distribution in this family. We obtain general explicit expressions for the quantile function, ordinary and incomplete moments, generating function and order statistics. We discuss the estimation of the model parameters by maximum likelihood and illustrate the potentiality of the extended family with two applications to real data.

Key words: Generalized distribution, lifetime, maximum likelihood estimation, order statistic, Weibull distribution.

1. Introduction

Numerous classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. For that reason, several methods for generating new families of distributions have been studied.

Some attempts have been made to define new families of probability distributions that extend well-known families of distributions and at the same time provide great flexibility in modeling data in practice. One such example is a broad family of univariate distributions generated from the Weibull distribution introduced by Gurvich *et al.* (1997), by extending the classical Weibull model. Its cumulative distribution function (cdf) is given by

$$G(x;\alpha,\boldsymbol{\xi}) = 1 - \exp[-\alpha H(x;\boldsymbol{\xi})], \quad x \in \mathcal{D} \subseteq \mathbb{R},\ \alpha > 0, \tag{1}$$

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where $H(x;\boldsymbol{\xi})$ is a non-negative monotonically increasing function depending on the parameter vector $\boldsymbol{\xi}$.

The corresponding probability density function (pdf) becomes

$$g(x;\alpha,\boldsymbol{\xi}) = \alpha\exp[-\alpha H(x;\boldsymbol{\xi})]\,h(x;\boldsymbol{\xi}),$$

where $h(x;\boldsymbol{\xi})$ is the derivative of $H(x;\boldsymbol{\xi})$. Different functions $H(x;\boldsymbol{\xi})$ in (1) include important statistical models such as: $H(x;\boldsymbol{\xi}) = x$ gives the exponential distribution; $H(x;\boldsymbol{\xi}) = x^2$ leads to the Rayleigh distribution; $H(x;\boldsymbol{\xi}) = \log(x/k)$ yields the Pareto distribution; $H(x;\boldsymbol{\xi}) = \beta^{-1}[\exp(\beta x) - 1]$ gives the Gompertz distribution.

Recently, Zografos and Balakrishnan (2009) proposed and studied a broad family of univariate distributions through a particular case of Stacy's generalized gamma distribution. Consider a continuous distribution G with density g, and further Stacy's generalized gamma density $f(x) = \gamma x^{\gamma\,\delta-1}e^{-x^\gamma}/\Gamma(\delta)$ for $x > 0$ and positive parameters γ and δ. Based on this density, by replacing x by $-\log[1-G(x)]$ and considering $\gamma = 1$, Zografos and Balakrishnan (2009) defined their family with cdf

$$F(x;\delta) = \gamma\{\delta, -\log[1-G(x)]\}, \quad x \in \mathcal{X} \subseteq \mathbb{R},\ \delta > 0,$$

where $\gamma(\delta, z) = \int_0^z t^{\delta-1}e^{-t}dt/\Gamma(\delta)$ denotes the incomplete gamma function and $\Gamma(\cdot)$ is the gamma function.

This pdf family is given by

$$f(x;\delta) = \frac{1}{\Gamma(\delta)}\{-\log[1-G(x)]\}^{\delta-1}\,g(x).$$

The Weibull distribution is a very popular model and it has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. It is generally adequate for modeling monotone hazard rates. In this paper, we introduce and study in generality a family of univariate distributions with two additional parameters, in the same vein as the extended Weibull (Gurvich *et al.*, 1997) and gamma (Zografos and Balakrishnan, 2009) families, using the Weibull generator applied to the odds ratio $G(x)/[1-G(x)]$. The term "generator" means that for each baseline distribution G we have a different distribution F. The main aim of this paper is to study a new family of distributions, with the hope it yields a "better fit" in certain practical situations. Additionally, we provide a comprehensive account of the mathematical properties of the proposed family of distributions.

This paper is unfolded as follows. In Section 2, we define the Weibull-G family of distributions. Section 3 provides some special distributions obtained by the Weibull generator. In Section 4, some general mathematical properties of the family are discussed. The formulas derived are manageable by using modern computer resources with analytic and numerical capabilities. In Section 5, the estimation of the model parameters is performed by the method of maximum likelihood. In Section 6, two illustrative applications based on real data are investigated. Finally, concluding remarks are presented in Section 7.

2. The Weibull-G Family of Distributions

Consider a continuous distribution G with density g and the Weibull cdf $F(x) = 1 - e^{-\alpha\, x^{\beta}}$ (for $x > 0$) with positive parameters α and β. Based on this density, by replacing x with $G(x)/\overline{G}(x)$ [$\overline{G}(x) = 1 - G(x)$], we define the cdf family by

$$F(x;\alpha,\beta,\boldsymbol{\xi}) = \int_0^{\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}} \alpha\,\beta\, t^{\beta-1} e^{-\alpha\, t^{\beta}} dt$$
$$= 1 - \exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right]^{\beta}\right\}, \quad x \in \mathcal{D} \subseteq \mathbb{R};\ \alpha,\beta > 0, \qquad (2)$$

where $G(x;\boldsymbol{\xi})$ is a baseline cdf, which depends on a parameter vector $\boldsymbol{\xi}$. The family pdf reduces to

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = \alpha\,\beta\, g(x;\boldsymbol{\xi}) \frac{G(x;\boldsymbol{\xi})^{\beta-1}}{\overline{G}(x;\boldsymbol{\xi})^{\beta+1}} \exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right]^{\beta}\right\}. \qquad (3)$$

Henceforth, let G be a continuous baseline distribution. For each G distribution, we define the Weibull-G (Wei-G for short) distribution with two extra parameters α and β defined by the pdf (3). A random variable X with pdf (3) is denoted by $X\ \sim$Wei-$G(\alpha,\beta,\boldsymbol{\xi})$. The additional parameters induced by the Weibull generator are sought as a manner to furnish a more flexible distribution. If $\beta = 1$, it corresponds to the exponential-generator.

An interpretation of the Wei-G family of distributions can be given as follows (Cooray, 2006) in a similar context. Let Y be a lifetime random variable having a certain continuous G distribution. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $G(x;\boldsymbol{\xi})/\overline{G}(x;\boldsymbol{\xi})$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Weibull model with scale α and shape β. We can write

$$\Pr(Y \le x) = \Pr\left(X \le \frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right) = F(x;\alpha,\beta,\boldsymbol{\xi}),$$

which is given by (2).

The hazard rate function of the Wei-G family is given by

$$\tau(x;\alpha,\beta,\boldsymbol{\xi}) = \frac{\alpha\,\beta\,g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})^{\beta-1}}{\overline{G}(x;\boldsymbol{\xi})^{\beta+1}} = \frac{\alpha\,\beta\,G(x;\boldsymbol{\xi})^{\beta-1}}{\overline{G}(x;\boldsymbol{\xi})^{\beta}}\tau(x;\boldsymbol{\xi}),$$

where $\tau(x;\boldsymbol{\xi}) = g(x;\boldsymbol{\xi})/\overline{G}(x;\boldsymbol{\xi})$. The multiplying quantity $\alpha\,\beta\,G(x;\boldsymbol{\xi})^{\beta-1}/\overline{G}(x;\boldsymbol{\xi})^{\beta}$ works as a corrected factor for the hazard rate function of the baseline model. (2) can deal with general situations in modeling survival data with various shapes of the hazard rate function. Table 1 lists $G(x;\boldsymbol{\xi})/\overline{G}(x;\boldsymbol{\xi})$ and the corresponding parameters for some special distributions.

Table 1: Distributions and corresponding $G(x;\boldsymbol{\xi})/\overline{G}(x;\boldsymbol{\xi})$ functions

Distribution	$G(x;\boldsymbol{\xi})/\overline{G}(x;\boldsymbol{\xi})$	$\boldsymbol{\xi}$
Uniform $(0 < x < \theta)$	$x/(\theta - x)$	θ
Exponential $(x > 0)$	$e^{\lambda x} - 1$	λ
Weibull $(x > 0)$	$e^{\lambda x^{\gamma}} - 1$	(λ, γ)
Fréchet $(x > 0)$	$(e^{\lambda x^{\gamma}} - 1)^{-1}$	(λ, γ)
Half-logistic $(x > 0)$	$(e^{x} - 1)/2$	$\varnothing$
Power function $(0 < x < 1/\theta)$	$[(\theta\,x)^{-k} - 1]^{-1}$	(θ, k)
Pareto $(x \geq \theta)$	$(x/\theta)^{k} - 1$	(θ, k)
Burr XII $(x > 0)$	$[1 + (x/s)^{c}]^{k} - 1$	(s, k, c)
Log-logistic $(x > 0)$	$[1 + (x/s)^{c}] - 1$	(s, c)
Lomax $(x > 0)$	$[1 + (x/s)]^{k} - 1$	(s, k)
Gumbel $(-\infty < x < \infty)$	$\{\exp[\exp(-(x-\mu)/\sigma)] - 1\}^{-1}$	(μ, σ)
Kumaraswamy $(0 < x < 1)$	$(1 - x^{a})^{-b} - 1$	(a, b)
Normal $(-\infty < x < \infty)$	$\Phi((x-\mu)/\sigma)/(1 - \Phi((x-\mu)/\sigma))$	(μ, σ)

3. Examples

In this section, we give some examples of the Wei-G family of distributions. The pdf (3) will be most tractable when the cdf $G(x;\boldsymbol{\xi})$ and the pdf $g(x;\boldsymbol{\xi})$ have simple analytic expressions. These sub-models generalize several important existing distributions in the literature; for example, Phani, exponential power, Chen, among others distributions.

3.1 Weibull-Uniform Distribution

As a first example, suppose that the parent distribution is uniform in the interval $(0,\theta)$, $\theta > 0$. Then, $g(x;\theta) = 1/\theta$, $0 < x < \theta < \infty$ and $G(x;\theta) = x/\theta$.

The Weibull-Uniform (WU) has cdf given by

$$F_{\mathrm{WU}}(x;\alpha,\beta,\theta) = 1 - \exp\left[-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}\right], \quad 0 < x < \theta < \infty,$$

where $\alpha, \beta > 0$. This distribution is known in the literature as the Phani distribution, see Phani (1987). The corresponding pdf is

$$f_{\mathrm{WU}}(x;\alpha,\beta,\theta) = \frac{\theta\,\alpha\,\beta}{(\theta - x)^2}\left(\frac{x}{\theta - x}\right)^{\beta-1}\exp\left[-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}\right], \quad 0 < x < \theta < \infty.$$

3.2 Weibull-Weibull Distribution

As a second example, consider the power function distribution with density and distribution functions (for $x > 0$) given by $g(x;\lambda,\gamma) = \lambda\,\gamma\,x^{\gamma-1}e^{-\lambda\,x^{\gamma}}$, $\lambda, \gamma > 0$ and $G(x;\lambda,\gamma) = 1 - e^{-\lambda\,x^{\gamma}}$, respectively. Then, the Wei-Weibull (WW) distribution has cdf given by

$$F_{\mathrm{WW}}(x;\alpha,\beta,\lambda,\gamma) = 1 - \exp\left[-\alpha\,(e^{\lambda x^{\gamma}} - 1)^{\beta}\right], \quad x > 0.$$

The WW distribution includes the exponential power (Smith and Bain, 1975) distribution when $\beta = 1$ and $\alpha = 1$. Further, for $\beta = 1$ and $\lambda = 1$, we obtain the Chen (Chen, 2000) distribution. If $\beta = \gamma = 1$ and $\alpha = \theta/\lambda$ ($\theta > 0$), we obtain the Gompertz (Gompertz, 1895) distribution. The corresponding pdf is

$$f_{\mathrm{WW}}(x;\alpha,\beta,\lambda,\gamma) = \alpha\,\beta\,\lambda\,\gamma\,x^{\gamma-1}(1 - \mathrm{e}^{-\lambda x^{\gamma}})^{\beta-1}\exp\left\{\lambda\,\beta\,x^{\gamma} - \alpha\,(\mathrm{e}^{\lambda x^{\gamma}} - 1)^{\beta}\right\}, \quad x > 0. \tag{4}$$

3.3 Weibull-Burr XII Distribution

Let us consider the parent Burr XII distribution with pdf and cdf given by $g(x) = c\,k\,s^{-c}x^{c-1}[1 + (x/s)^c]^{-k-1}$, $s, k, c > 0$ and $G(x) = 1 - [1 + (x/s)^c]^{-k}$, respectively. Then, the Wei-BXII (WBXII) distribution has cdf given by

$$F_{\mathrm{WBXII}}(x;\alpha,\beta,s,k,c) = 1 - \exp\left\{-\alpha\,[(1 + (x/s)^c)^k - 1]^{\beta}\right\}, \quad x > 0.$$

The WBXII distribution includes the generalized power Weibull (Nikulin and Haghighi, 2006) distribution when $\alpha = \beta = 1$. The corresponding pdf (for $x > 0$) becomes

$$f_{\mathrm{WBXII}}(x;\alpha,\beta,s,k,c) = \frac{\alpha\,\beta c\,k\,s^{-c}x^{c-1}}{1 + (x/s)^c}\exp\left\{-\alpha\,[(1 + (x/s)^c)^k - 1]^{\beta}\right\} \times [(1 + (x/s)^c)^k - 1]^{\beta-1}. \tag{5}$$

3.4 Weibull-Normal Distribution

The last example refers to the normal distribution. The Wei-normal (WN) density is obtained from (3) by taking $G(\cdot)$ and $g(\cdot)$ to be the cdf and pdf of the normal $\mathrm{N}(\mu, \sigma^2)$ distribution. Then, the WN distribution has cdf given by

$$F_{\mathrm{WN}}(x; \alpha, \beta, \mu, \sigma) = 1 - \exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}, \quad -\infty < x < \infty,$$

where $-\infty < \mu < \infty, \sigma > 0$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. For $\mu = 0$ and $\sigma = 1$, we obtain the standard WN distribution. The corresponding pdf is

$$f_{\mathrm{WN}}(x; \alpha, \beta, \mu, \sigma) = \frac{\alpha\,\beta\,\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma}\,\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\},$$
$$-\infty < x < \infty.$$

Figure 1 illustrates possible shapes of the density functions for some Weibull-G distributions.

4. Mathematical Properties

Despite the fact that the Wei-G cdf and pdf require mathematical functions that are widely available in modern statistical packages, frequently analytical and numerical derivations take advantage of power series for the pdf. By using the power series for the exponential function, we obtain

$$\exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right]^{\beta}\right\} = \sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\left[\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right]^{k\,\beta}.$$

Inserting this expansion in (2), we have

$$f(x; \alpha, \beta, \boldsymbol{\xi}) = \alpha\,\beta\,g(x;\boldsymbol{\xi})\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\frac{G(x;\boldsymbol{\xi})^{\beta(k+1)-1}}{\overline{G}(x;\boldsymbol{\xi})^{\beta(k+1)+1}}. \tag{6}$$

Now, using the generalized binomial theorem, we can write

$$\overline{G}(x;\boldsymbol{\xi})^{-[\beta(k+1)+1]} = \sum_{j=0}^{\infty}\frac{\Gamma(\beta(k+1)+j+1)}{j!\,\Gamma(\beta(k+1)+1)}\,G(x;\boldsymbol{\xi})^{j}. \tag{7}$$

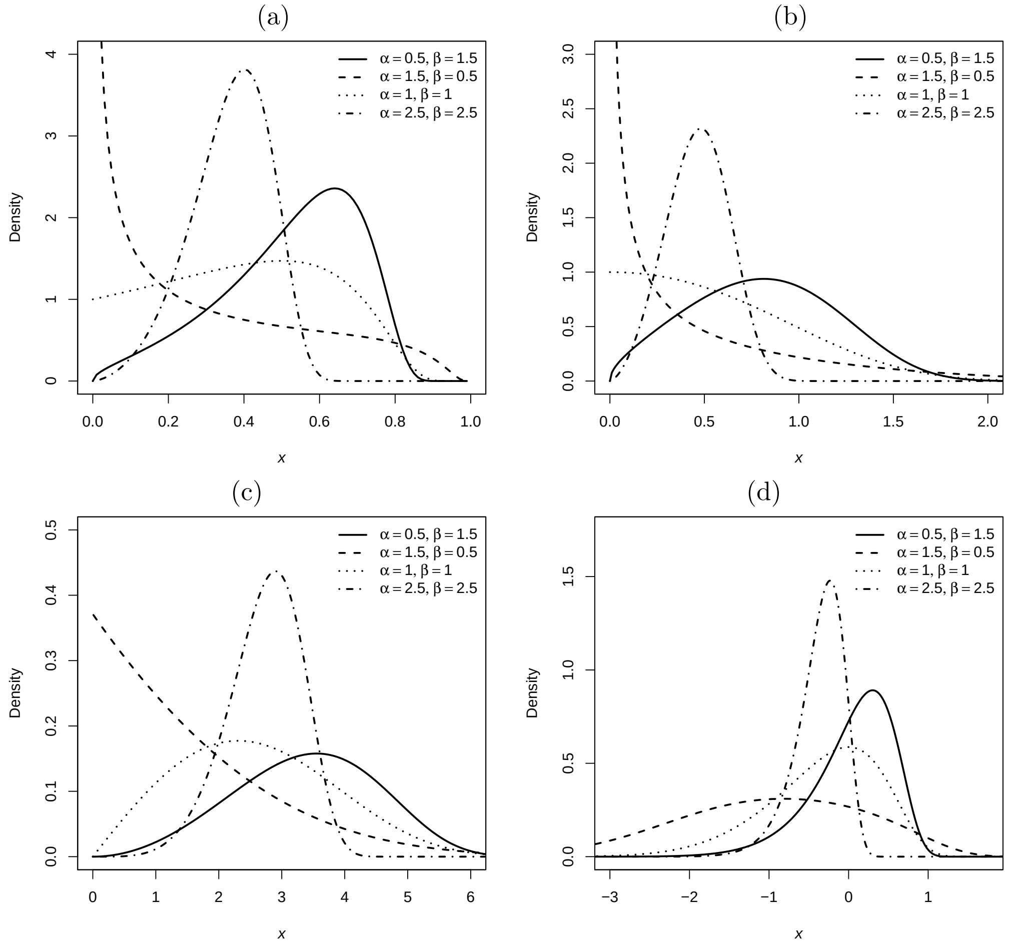

Figure 1: (a) WU$(\alpha, \beta, 1)$, (b) WW$(\alpha, \beta, 1, 1)$, (c) WBXII$(\alpha, \beta, 7, 3, 2)$ and (d) WN$(\alpha, \beta, 0, 1)$ density functions

Inserting (7) in (6), the Wei-G density function can be expressed as an infinite linear combination of exponentiated-G (exp-G for short) density functions

$$f(x; \alpha, \beta, \boldsymbol{\xi}) = \sum_{j,k=0}^{\infty} \omega_{j,k}\, h_{\beta(k+1)+j-1}(x; \boldsymbol{\xi}), \tag{8}$$

where

$$\omega_{j,k} = \frac{(-1)^k\, \beta\, \alpha^{k+1}\, \Gamma(\beta(k+1)+j+1)}{k!\, j!\, [\beta(k+1)+j-1]\, \Gamma(\beta(k+1)+1)},$$

and

$$h_a(x; \boldsymbol{\xi}) = a\, g(x; \boldsymbol{\xi}) G(x; \boldsymbol{\xi})^{a-1}.$$

Thus, some mathematical properties of the Wei-G model can be obtained directly from those properties of the exp-G distribution. For example, the ordinary and incomplete moments and moment generating function (mgf) of the Wei-G distribution can be obtained immediately from those quantities of the exp-G distribution.

The Wei-G family of distributions is easily simulated from (2) as follows: if U has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$X = G^{-1}\left(\frac{T}{T+1}\right),$$

has the Wei-$G(\alpha,\beta,\boldsymbol{\xi})$ distribution, where $T = \{\log[1/(1-U)]^{1/\alpha}\}^{1/\beta}$.

The sth moment of X can be obtained from (8) as

$$E(X^s) = \sum_{j,k=0}^{\infty} \omega_{j,k}\, E(Z_{j,k}^s),$$

where $Z_{j,k}$ denotes the exp-G distribution with power parameter $\beta(k+1)+j-1$. Since the inner quantities in (8) are absolutely integrable, the incomplete moments and mgf of X can be written as

$$I_X(y) = \int_{-\infty}^{y} x^s f(x)dx = \sum_{j,k=0}^{\infty} \omega_{j,k}\, I_{j,k}(y),$$

where $I_{j,k}(y) = \int_{-\infty}^{y} x^s\, h_{\beta(k+1)+j-1}(x;\boldsymbol{\xi})dx$ and

$$M_X(t) = \sum_{j,k=0}^{\infty} \omega_{j,k}\, E(\mathrm{e}^{tZ_{j,k}}).$$

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They enter in the problems of estimation and hypothesis tests in a variety of ways. Therefore, we now discuss some properties of the order statistics for the proposed class of distributions. The pdf $f_{i:n}(x)$ of the ith order statistic for a random sample $X_1,\cdots,X_n$ from the Wei-G distribution is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}[1-F(x)]^{n-i},$$

and then

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x;\alpha,\beta,\boldsymbol{\xi}) \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \times \exp\left\{ -\alpha\,(n+k-i) \left[\frac{G(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\right]^{\beta} \right\},$$

where $f(\cdot)$ and $F(\cdot)$ are the density and cumulative functions of the Wei-G distribution, respectively.

Some results of this section can be obtained numerically in any symbolic software such as MAPLE (Garvan, 2002), MATLAB (Sigmon and Davis, 2002), MATHEMATICA (Wolfram, 2003), Ox (Doornik, 2007) and R (R Development Core Team, 2009). The Ox (for academic purposes) and R are freely distributed and available at http://www.doornik.com and http://www.r-project.org, respectively. The results are easily computed by taking in these sums a large positive integer value in place of ∞.

5. Maximum Likelihood Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the new family of distributions from complete samples only. Let $x_1, \cdots, x_n$ be observed values from the Wei-G distribution with parameters α, β and $\boldsymbol{\xi}$. Let $\boldsymbol{\Theta} = (\alpha, \beta, \boldsymbol{\xi})^{\top}$ be the $p \times 1$ parameter vector. The total log-likelihood function for $\boldsymbol{\Theta}$ is given by

$$\ell(\boldsymbol{\Theta}) = n\,\log(\alpha) + n\,\log(\beta) + \sum_{i=1}^{n} \log[g(x_i;\boldsymbol{\xi})] - \alpha \sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta} + \beta \sum_{i=1}^{n} \log[H(x_i;\boldsymbol{\xi})] - \sum_{i=1}^{n} \log[G(x_i;\boldsymbol{\xi})] - \sum_{i=1}^{n} \log[\overline{G}(x_i;\boldsymbol{\xi})],$$

where $H(x;\boldsymbol{\xi}) = G(x;\boldsymbol{\xi})/\overline{G}(x;\boldsymbol{\xi})$. The components of the score function $U(\boldsymbol{\Theta}) = (U_\alpha, U_\beta, U_{\boldsymbol{\xi}})^{\top}$ are

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta},$$

$$U_\beta = \frac{n}{\beta} - \alpha \sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta} \log[H(x_i;\boldsymbol{\xi})] + \sum_{i=1}^{n} \log[H(x_i;\boldsymbol{\xi})],$$

and

$$U_{\boldsymbol{\xi}_k} = -\alpha\,\beta\sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta-1}\partial H(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k + \beta\sum_{i=1}^{n}\frac{\partial H(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k}{H(x_i;\boldsymbol{\xi})}$$
$$+\sum_{i=1}^{n}\frac{\partial g(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k}{g(x_i;\boldsymbol{\xi})} - \sum_{i=1}^{n}\frac{\partial G(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k}{G(x_i;\boldsymbol{\xi})} - \sum_{i=1}^{n}\frac{\partial \overline{G}(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k}{\overline{G}(x_i;\boldsymbol{\xi})}.$$

Setting U_α, U_β and $U_{\boldsymbol{\xi}}$ equal to zero and solving the equations simultaneously yields the MLE $\widehat{\boldsymbol{\Theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\boldsymbol{\xi}})^\top$ of $\boldsymbol{\Theta} = (\alpha, \beta, \boldsymbol{\xi})^\top$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

For interval estimation on the model parameters, we require the observed information matrix

$$J(\boldsymbol{\Theta}) = -\begin{pmatrix} U_{\alpha\alpha} & U_{\alpha\beta} & | & U_{\alpha\boldsymbol{\xi}}^\top \\ U_{\beta\alpha} & U_{\beta\beta} & | & U_{\beta\boldsymbol{\xi}}^\top \\ -- & -- & -- & -- \\ U_{\alpha\boldsymbol{\xi}} & U_{\beta\boldsymbol{\xi}} & | & U_{\boldsymbol{\xi}\boldsymbol{\xi}} \end{pmatrix},$$

whose elements are

$$U_{\alpha\alpha} = -\frac{n}{\alpha^2},$$
$$U_{\alpha\beta} = -\sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^\beta \log[H(x_i;\boldsymbol{\xi})],$$
$$U_{\alpha\boldsymbol{\xi}_k} = -\beta\sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta-1}H_k^{'}(x_i;\boldsymbol{\xi}),$$
$$U_{\beta\beta} = -\frac{n}{\beta^2} - \alpha\sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^\beta \left\{\log[H(x_i;\boldsymbol{\xi})]\right\}^2,$$
$$U_{\beta\boldsymbol{\xi}_k} = \sum_{i=1}^{n}\frac{H_k^{'}(x_i;\boldsymbol{\xi})}{H(x_i;\boldsymbol{\xi})} - \alpha\,\beta\sum_{i=1}^{n} H_k^{'}(x_i;\boldsymbol{\xi})H(x_i;\boldsymbol{\xi})^{\beta-1}\log[H(x_i;\boldsymbol{\xi})]$$
$$-\alpha\sum_{i=1}^{n} H_k^{'}(x_i;\boldsymbol{\xi})H(x_i;\boldsymbol{\xi})^{\beta-1},$$

and

$$
\begin{aligned}
U_{\xi_k\xi_l} &= \alpha\,\beta \sum_{i=1}^{n} H''_{kl}(x_i;\boldsymbol{\xi}) H(x_i;\boldsymbol{\xi})^{\beta-1} \\
&\quad -\alpha\,\beta(\beta-1) \sum_{i=1}^{n} H'_k(x_i;\boldsymbol{\xi}) H'_l(x_i;\boldsymbol{\xi}) H(x_i;\boldsymbol{\xi})^{\beta-2} \\
&\quad +\beta \sum_{i=1}^{n} \frac{H''_{kl}(x_i;\boldsymbol{\xi})}{H(x_i;\boldsymbol{\xi})} - \beta \sum_{i=1}^{n} \frac{H'_k(x_i;\boldsymbol{\xi}) H'_l(x_i;\boldsymbol{\xi})}{H(x_i;\boldsymbol{\xi})^2} - \sum_{i=1}^{n} \frac{G''_{kl}(x_i;\boldsymbol{\xi})}{G(x_i;\boldsymbol{\xi})} \\
&\quad + \sum_{i=1}^{n} \frac{G'_k(x_i;\boldsymbol{\xi}) G'_l(x_i;\boldsymbol{\xi})}{G(x_i;\boldsymbol{\xi})^2} - \sum_{i=1}^{n} \frac{\overline{G}''_{kl}(x_i;\boldsymbol{\xi})}{\overline{G}(x_i;\boldsymbol{\xi})} + \sum_{i=1}^{n} \frac{\overline{G}'_k(x_i;\boldsymbol{\xi}) \overline{G}'_l(x_i;\boldsymbol{\xi})}{\overline{G}(x_i;\boldsymbol{\xi})^2} \\
&\quad + \sum_{i=1}^{n} \frac{g''_{kl}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})} - \sum_{i=1}^{n} \frac{g'_k(x_i;\boldsymbol{\xi}) g'_l(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})^2},
\end{aligned}
$$

where $t'_k(\cdot;\boldsymbol{\xi}) = \partial t(\cdot;\boldsymbol{\xi})/\partial \boldsymbol{\xi}_k$ and $t''_{kl}(\cdot;\boldsymbol{\xi}) = \partial^2 t(\cdot;\boldsymbol{\xi})/\partial \boldsymbol{\xi}_k \partial \boldsymbol{\xi}_l$.

6. Applications

The first set consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. These data have also been analyzed by Smith and Naylor (1987).

For these data, we fit the Weibull-exponential (WE) distribution defined in (4) with $\beta = 1$. Its fit is also compared with the widely known exponentiated Weibull (EW) (Mudholkar and Srivastava, 1993) and exponentiated exponential (EE) (Gupta and Kundu, 1999) models with corresponding densities:

$$
\begin{aligned}
&\text{EW}: f_{\text{EW}}(x;\alpha,\beta,\lambda) = \alpha\,\beta\,\lambda^{\beta}\,x^{\beta-1}\mathrm{e}^{-(\lambda x)^{\beta}}\left(1-\mathrm{e}^{-(\lambda x)^{\beta}}\right)^{\alpha-1}, \quad x>0, \\
&\text{EE}: f_{\text{EE}}(x;\alpha,\lambda) = \alpha\,\lambda\,\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}, \quad x>0,
\end{aligned}
$$

where $\alpha > 0, \beta > 0$ and $\lambda > 0$.

The second data set were used by Birnbaum and Saunders (1969) and correspond to the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The data are: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.

For these data, we fit the WBXII distribution defined in (5) and compare it with the Weibull-log-logistic (WLL) (for $x > 0$) and the beta Burr XII (BBXII) (for $x > 0$) (Paranaíba *et al.*, 2011) models with corresponding densities:

$$\begin{aligned}
&\text{WLL} : f_{\text{WLL}}(x; \alpha, \beta, s, c) \\
&= \frac{\alpha\,\beta\,c\,s^{-c} x^{c-1} \exp\left\{-\alpha\left[(1 + (x/s)^c) - 1\right]^{\beta}\right\}\left[(1 + (x/s)^c) - 1\right]^{\beta-1}}{1 + (x/s)^c}, \\
&\text{BBXII} : f_{\text{BBXII}}(x; a, b, s, k, c) \\
&= \frac{c\,k\,s^{-c} x^{c-1}}{B(a, b)}\left[1 + (x/s)^c\right]^{-(kb+1)}\{1 - \left[1 + (x/s)^c\right]^{-k}\}^{a-1},
\end{aligned}$$

where $a, b, \alpha, \beta, s, c, k > 0$ and $B(a, b)$ is the beta function.

The MLEs of the model parameters (with standard errors in parentheses) and the Akaike information criterion (AIC) for the WE, WBXII and the other models are listed in Table 2. The fitted densities for the first and second data sets are displayed in Figures 2 and 3 (together with the data histogram), respectively. These results illustrate the potentiality of the WE and WBXII distributions and the importance of the two additional parameters.

Table 2: MLEs of the parameters (standard errors in parentheses) and AIC of the WE and WBXII models for the two data sets

Application	Model	Estimates					AIC
	WE(α, β, λ)	0.0148	2.8796	1.0178			34.8
		(0.0598)	(2.0488)	(1.1954)			
First data set	EW(α, β, λ)	0.6712	7.2846	0.5820			35.4
		(0.2489)	(1.7070)	(0.0292)			
	EE(α, λ)	31.349	2.6116				68.8
		(9.5198)	(0.2380)				
	WBXII(α, β, s, k, c)	100.24	0.6383	151.42	0.0024	13.230	920.6
		(191.96)	(0.3306)	(12.817)	(0.0067)	(5.6938)	
Second data set	WLL(α, β, s, c)	19.9507	0.3786	235.96	15.8459		933.2
		(13.726)	(0.2336)	(27.321)	(9.7957)		
	BBXII(a, b, s, k, c)	123.07	59.095	233.13	2.2139	0.7180	924.0
		(0.1292)	(60.873)	(296.00)	(1.4763)	(0.1160)	

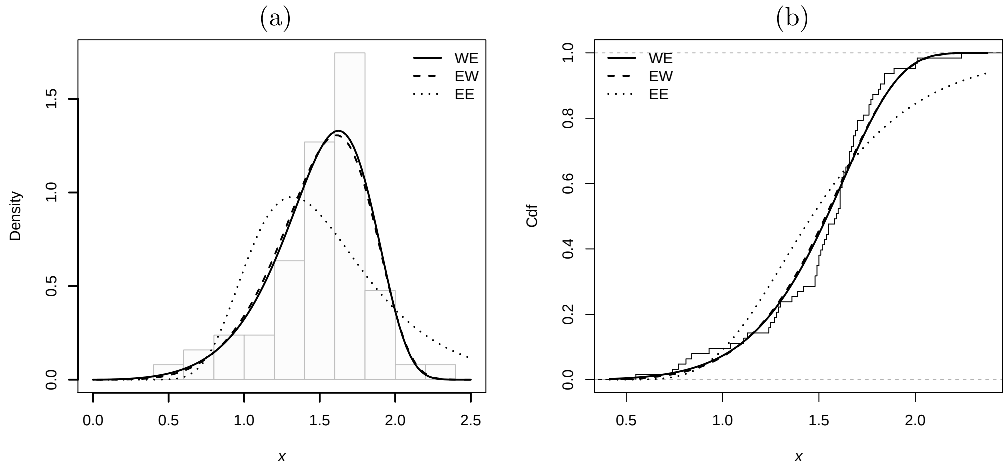

Figure 2: Estimated (a) pdf and (b) cdf for the WE, EW and EE models for failure times data

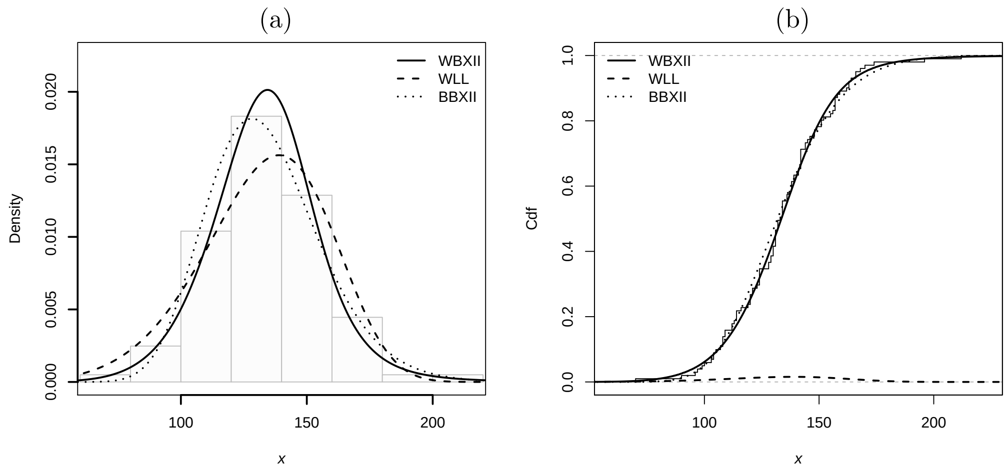

Figure 3: Estimated (a) pdf and (b) cdf for the WBXII, WLL and BBXII models for the second data set

7. Concluding Remarks

Following the contents of the classes of extended Weibull (Gurvich *et al.*, 1997) and gamma (Zografos and Balakrishnan, 2009) families of distributions, we derive general mathematical properties of a new wider Weibull family of distributions. This generator can extend several widely known distributions such as the uniform, Weibull, Burr XII and Weibull distributions. The Weibull-G density function can be expressed as a mixture of exponentiated-G density functions. This mixture representation is important to derive several structural properties of this family in full generality. Some of them are provided such as the ordinary and incomplete moments, quantile function and order statistics. For each baseline distribution G, our results can be easily adapted to obtain its main structural properties. The estimation of the model parameters is approached by the method of maximum likelihood and the observed information matrix is derived. We fit some Weibull-G distributions to two real data sets to demonstrate the potentiality of this family. We hope this generalization may attract wider applications in statistics.

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