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Empirical Likelihood Ratio Test for the Epidemic Change Model

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Abstract: Change point problem has been studied extensively since 1950s due to its broad applications in many fields such as finance, biology and so on. As a special case of the multiple change point problem, the epidemic change point problem has received a lot of attention especially in medical studies. In this paper, a nonparametric method based on the empirical likelihood is proposed to detect the epidemic changes of the mean after unknown change points. Under some mild conditions, the asymptotic null distribution of the empirical likelihood ratio test statistic is proved to be the extreme distribution. The consistency of the test is also proved. Simulations indicate that the test behaves comparable to the other available tests while it enjoys less constraint on the data distribution. The method is applied to the Standford heart transplant data and detects the change points successfully.

Key words: Consistency, empirical likelihood ratio, epidemic change point, extreme distribution.

1. Introduction

1.1 The Change Point Problem

Change point problems can be encountered in many applied fields such as finance, biology, geology etc. In statistics, a change point can be viewed as a place or time point such that the observations follow different distributions before and after that point. Multiple change points can be detected similarly. Page (1954, 1955) introduced a simple process to detect a single change. Since then, change point problems have received extensive attentions. For instance, Chernoff and Zacks (1964), Gardner (1969), Hawkins (1992) studied the testing and estimation of a change in the mean of a normal model. Hsu (1977), Inclán (1993) studied change point problem for the variance. Worsley (1986) provided confidence regions and tests for a change-point in a sequence of exponential models.

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The change point problem for the regression model has been studied by Krishnaiah and Miao (1988). Kim and Siegmund (1989) proposed a likelihood ratio test to detect a single change-point in a simple linear regression model. Chen and Gupta (1997) studied the change points for the variance while the mean is constant for the univariate normal model using information approach. Chen and Gupta (2000) extended their results to the multivariate normal models and discussed the testing and detection of change points for some continuous distributions besides the normal distribution such as the exponential distributions, and also for some discrete distributions such as the gamma distribution and the binomial distributions by using likelihood ratio test (LRT), Bayesian approach and information approach. Ning and Gupta (2009) considered the generalized lambda distribution change point model (GLDCM) and applied it to detect the variations of the copy numbers of DNA. Ning (2011) proposed a nonparametric method to detect the mean change in a linear trend.

The special multiple change points problem is that of the epidemic change point problem, which can be described as the change of the parameters at some unknown location such that $\theta_1 = \cdots = \theta_p = \theta_{q+1} = \cdots = \theta_n = \alpha$ and $\theta_{p+1} = \cdots = \theta_q = \beta$, where p, q, α, β are unknown. The epidemic change point problem is of great practical interest with applications in many fields such as medical studies. Readers can refer to Levin and Kline (1985), Yao (1993), Ramanayake (1998), Ramanayake and Gupta (2003, 2004) and Guan (2004) for more details.

1.2 The Empirical Likelihood Method

Consider independently and identically distributed d-dimensional observations, say $x_1, \cdots, x_n$, from an unknown population distribution F. The main idea of empirical likelihood methods proposed and systematically developed by Owen (1990,1991) is to place an unknown probability mass at each observation. Let $p_i = P(X = x_i)$ and the empirical likelihood function of p is defined as

$$L(F) = \prod_{i=1}^{n} p_i.$$

It is clear that $L(F)$ subject to the constraints

$$p_i \geq 0 \text{ and } \sum_i p_i = 1$$

is maximized at $p_i = 1/n$, i.e., the likelihood $L(F)$ attains its maximum n^{-n} under the full nonparametric model. When a population parameter θ identified by $Em(X, \theta) = 0$ is of interest, the empirical log-likelihood maximum when θ has the true value θ_0 is obtained subject to the additional constraint

$$\sum x_i p_i = \theta_0.$$

The empirical log-likelihood ratio (ELR) statistic to test $\theta = \theta_0$ is given by

$$R(\theta_0) = \max\{\sum_i \log np_i : p_i \geq 0, \sum p_i = 1, \sum p_i m(x_i, \theta_0) = 0\}.$$

Owen (1988) showed, similar to the likelihood ratio test statistic in a parametric model setup, $-2\log R(\theta_0) \to \chi_r^2$ in distribution as $n \to \infty$ under the null model $\theta = \theta_0$ with mild regular conditions, where r is the dimension of $m(x, \theta)$.

The empirical likelihood method as outlined above can be extended to triangular arrays that are independent or nearly independent but not necessarily identically distributed (see Owen, 2001).

2. Methods

2.1 Epidemic Hypothesis of the Mean

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent random variables in $\mathbb{R}^d$ from a common distribution family F. We want to test the null hypothesis of no change in the mean against the epidemic change in the mean. That is, we test the following hypotheses.

$$H_0 : \mu_i = \mu_0,\ i = 1, 2, \cdots, n$$

$$H_1 : \exists p, q \in \mathbb{Z}^+,\ \text{such that } 1 < p < q < n \text{ and}$$

$$\mu_i = \begin{cases} \mu_0, & \text{if } i \leq p, \\ \mu^* = \mu_0 + \delta, & \text{if } p < i \leq q, \\ \mu_0, & \text{if } q < i \leq n. \end{cases}$$

where μ_0 and δ are both unknown and $\delta > 0$. Ramanayake and Gupta (2003) studied this problem for the exponential family. They proposed the LRT statistic and investigate its asymptotic properties. In this paper, an empirical likelihood ratio test without the constraint of the data distribution to detect the epidemic changes is proposed in Section 2. The null distribution of the test statistic and the consistency of the test are proved in Section 3. Simulations conducted in Section 4 indicate that the ELR test is comparable with other existing tests. The method is applied to a real data in Section 5. Some discussions are provided in Section 6.

2.2 Empirical Likelihood Ratio Test

For a fixed (p, q), the empirical log-likelihood function is

$$l(\mu_0, \mu^*|p, q) = \sum_i \log u_i + \sum_j \log v_j,$$

where $i = 1, \cdots, p, q+1, \cdots, n$; $j = p+1, \cdots, q$, $u_i = P(X = x_i)$, $v_j = P(X = x_j)$ and X is a random variable. With the constraints $\sum_i u_i = \sum_j v_j = 1$, $l(p, q)$ reaches the maximum value at $u_i = (n-q+p)^{-1}$ and $v_j = (q-p)^{-1}$ by Lagrange multiplier method. Therefore, the empirical log-likelihood ratio function is

$$\mathrm{lr}(\mu_0, \mu^*|p, q) = \sum_i \log((n-q+p)u_i) + \sum_j \log((q-p)v_j).$$

Then, the profile empirical likelihood ratio function for given μ_0 and μ^* is written as

$$R(\mu_0, \mu^*|p, q) = \sup\{\mathrm{lr}(\mu_0, \mu^*|p, q) : \sum_i u_i = \sum_j v_j = 1, \sum_i u_i x_i = \mu_0, \sum_j v_j x_j = \mu^*\},$$

where $u_i \geq 0$ and $v_j \geq 0$. To test the hypothesis $\mu^* = \mu_0$ or equivalently test $\delta = 0$, the test statistic is defined as

$$\begin{aligned} Z_{n,p,q} &= -2 \sup_{\mu_0}\{R(\mu_0, \mu_0|p, q)\} \\ &= -2 \sup_{\mu_0}\{\sum_i \log((n-q+p)u_i) + \sum_j \log((q-p)v_j)\}. \end{aligned}$$

With the Lagrange multiplier method, we define

$$\begin{aligned} G(\mu_0, \lambda_1, \eta_1, \lambda_2, \eta_2, u_i, v_j) &= \sum_i \log((n-q+p)u_i) - n\lambda_1'(\sum_i u_i x_i - \mu_0) \\ + \eta_1(\sum_i u_i - 1) &+ \sum_j \log((q-p)v_j) - n\lambda_2'(\sum_j v_j x_j - \mu_0) + \eta_2(\sum_j v_j - 1), \end{aligned}$$

where $i = 1, \cdots, p, q+1, \cdots, n$; $j = p+1, \cdots, q$. By taking the first derivative of G with respect to u_i, we obtain

$$\begin{aligned} \frac{\partial G}{\partial u_i} &= \frac{1}{u_i} - \lambda_1'(x_i - \mu_0) + \eta_1 = 0 \\ \Longrightarrow u_i &= \frac{1}{n\lambda_1'(x_i - \mu_0) - \eta_1}. \end{aligned}$$

From the above equation, we have the following result immediately.

$$\sum u_i \frac{\partial G}{\partial u_i} = \sum u_i \{\frac{1}{u_i} - \lambda_1'(x_i - \mu_0) + \eta_1\} = 0$$
$$\Longrightarrow \eta_1 = -(n-q+p) \Longrightarrow u_i = \frac{1}{(n-q+p) + n\lambda_1'(x_i - \mu_0)}.$$

In the similar way, we also obtain

$$v_j = \frac{1}{(q-p) + n\lambda_2'(x_j - \mu_0)}.$$

Finally, we take the first derivative of G respect to μ_0,

$$\frac{\partial G}{\partial \mu_0} = \lambda_1 + \lambda_2 = 0 \Longrightarrow \lambda_2 = -\lambda_1.$$

Therefore,

$$v_j = \frac{1}{(q-p) - n\lambda_1'(x_j - \mu_0)}.$$

For convenience, we denote $\theta_{pq} = \frac{n-(q-p)}{n}$ and $\lambda_1 = \lambda$. It follows that

$$u_i = \frac{1}{n\theta_{pq} + n\lambda'(x_i - \mu_0)} = \frac{1}{n\theta_{pq}} \cdot \frac{1}{1 + \theta_{pq}^{-1}\lambda'(x_i - \mu_0)},$$
$$v_j = \frac{1}{n(1-\theta_{pq}) - n\lambda'(x_j - \mu_0)} = \frac{1}{n(1-\theta_{pq})} \cdot \frac{1}{1 - (1-\theta_{pq})^{-1}\lambda'(x_i - \mu_0)}.$$

Hence,

$$\begin{aligned} Z(\theta_{pq}, \lambda, \mu_0) &= (\lambda, \mu_0) \\ &= 2\{\sum_i \log(1 + \theta_{pq}^{-1}\lambda'(x_i - \mu_0) \\ &\quad + \sum_j \log(1 - (1-\theta_{pq})^{-1}\lambda'(x_j - \mu_0))\}. \end{aligned}$$

Define the score functions

$$\begin{aligned} \phi_1(\lambda, \mu_0) &= \frac{\partial Z(\theta_{pq}, \lambda, \mu_0)(\lambda, \mu_0)}{2\partial\lambda} \\ &= \sum_i \frac{\theta_{pq}^{-1}(x_i - \mu_0)}{1 + \theta_{pq}^{-1}\lambda'(x_i - \mu_0)} - \sum_j \frac{(1-\theta_{pq})^{-1}(x_j - \mu_0)}{1 - (1-\theta_{pq})^{-1}\lambda'(x_j - \mu_0)}, \end{aligned}$$

and

$$\begin{aligned}\phi_2(\lambda,\mu_0) &= \frac{\partial Z(\theta_{pq},\lambda,\mu_0)(\lambda,\mu_0)}{-2\lambda\partial\mu_0}\\ &= \sum_i \frac{\theta_{pq}^{-1}}{1+\theta_{pq}^{-1}\lambda'(x_i-\mu_0)} - \sum_j \frac{(1-\theta_{pq})^{-1}}{1-(1-\theta_{pq})^{-1}\lambda'(x_j-\mu_0)}.\end{aligned}$$

Then $(\widehat{\lambda}(\theta_{pq}),\widehat{\mu}_0(\theta_{pq}))$ are determined by

$$\begin{aligned}\phi_1(\widehat{\lambda}(\theta_{pq}),\widehat{\mu}_0(\theta_{pq})) &= 0,\\ \phi_2(\widehat{\lambda}(\theta_{pq}),\widehat{\mu}_0(\theta_{pq})) &= 0.\end{aligned}$$

Therefore, we have

$$Z_{n,p,q} = Z(\theta_{pq},\widehat{\lambda}(\theta_{pq}),\widehat{\mu}_0(\theta_{pq})).$$

Since p, q are unknown, it is natural to use the maximally selected empirical likelihood ratio statistic which is defined as

$$Z_n^* = \max_{1<p<q<n}\{Z_{n,p,q}\},$$

and we will reject the null hypothesis with a significantly large value of Z_n^*. However, if p or q or $q-p$ is too small, the empirical likelihood estimators of $(\widehat{\lambda}(\theta_{pq}),\widehat{\mu}_0(\theta_{pq}))$ may not exist, that is, our test may not detect the change points occurring at the very beginning or the very end or two changes points which are too close. Therefore, we suggest the trimmed likelihood ratio statistic as

$$Z_n = \max_{k_0<p<q<n-k_1}\{Z_{n,p,q}\}. \tag{2.1}$$

As Perron and Vogelsang (1992) pointed out that k_0 and k_1 can be chosen arbitrarily. In our paper, we choose $k_0 = k_1 = 2[\log n]$ where $[x]$ means the largest integer not larger than x.

3. Asymptotic Distribution of Z_n

The main results in this section are similar to the results obtained by Csörgő and Horvath (1997) through the parametric likelihood ratio method.

Theorem 1. Suppose that $E_F||X||^3 < \infty$, and $E_F(XX')$ is positive definite. If H_0 is true, then we have

$$P(A(\log(t(n)))(Z_n)^{1/2} \le x + D_r(\log(t(n))) \longrightarrow \exp(-e^{-x}) \tag{3.1}$$

as $n \longrightarrow \infty$ for all x, where

$$A(x) = (2 \log x)^{1/2},$$
$$D_r(x) = 2 \log x + (r/2) \log \log x - \log \Gamma(r/2),$$
$$t(n) = \frac{n^2 + (2[\log n])^2 - 2n[\log n]}{(2[\log n])^2},$$

and r is the dimension of the parameter space.

Theorem 2. Under mild conditions, Assumption 1 and Assumption 2, if $\theta_{pq} \to \theta \in (0,1)$ as $n \to \infty$, the ELR test is consistent. That is, under the alternative hypothesis,

$$Z_n \to \infty \tag{3.2}$$

in probability.

Proofs of Theorem 1 and Theorem 2, and Assumptions 1 and 2 of Theorem 2 are given in the Appendix.

4. Simulation Results

In this section, we conduct three simulations to illustrate the behavior of the ELR test under different settings of the data distribution, and compare with some other available tests.

Simulation I. We compare the performance of the nonparametric ELR test developed in the previous section with some existing parametric change point tests against epidemic alternatives. Ramanayake and Gupta (2003) considered four different tests T_1, T_2, and T_3, T_4 under the assumption that the observations are exponentially distributed. The first two tests are based on the likelihood ratio test statistic, and the last two are modified LRT tests proposed by Aly and Bouzar (1992). Table 1 compares the powers of the ELR test developed in the previous section and these four tests. The simulations were done after 5000 repetition Monte Carlo experiments using a sample size of $n = 50$ and $\delta = 1, 3$ with a significance level $\alpha = 0.05$. We see in Table 1 that the ELR test performs very comparable to the other four tests T_1, T_2, T_3, and T_4 for selected values of (p, q), especially when $\delta = 1$, it has the higher power than at least two of those tests for all the selected change point locations. The ELR test achieves higher power for large values of δ. It also indicates that the difference $q - p$ between two change points is larger and the change points are closer to the center of the data, the performance of the test is better. The difficulty their tests may have is to choose the best test with the highest power according to the different locations of change points, which are not known usually in real cases. Before using these tests, the validity of exponential assumptions also have to be checked. The ELR test obviously avoids these possible difficulties. The Type I error of our test is 0.032 from the simulations, which is well controlled within the given nominal level.

Table 1: Comparison of powers; exponential distribution; $n = 50, \alpha = 0.05$

		$\delta = 1$					$\delta = 3$				
p	q	ELR	T1	T2	T3	T4	ELR	T1	T2	T3	T4
8	24	0.346	0.246	0.365	0.421	0.259	0.852	0.686	0.947	0.964	0.905
8	40	0.661	0.613	0.391	0.367	0.244	0.956	0.986	0.961	0.915	0.889
12	24	0.365	0.303	0.289	0.349	0.238	0.902	0.801	0.879	0.916	0.847
12	40	0.685	0.678	0.443	0.438	0.273	0.966	0.996	0.979	0.969	0.927
16	24	0.283	0.264	0.181	0.233	0.174	0.848	0.722	0.700	0.777	0.708
16	40	0.522	0.632	0.449	0.472	0.280	0.960	0.991	0.983	0.981	0.933
20	28	0.240	0.305	0.180	0.225	0.160	0.832	0.789	0.701	0.778	0.703
20	40	0.463	0.524	0.429	0.477	0.293	0.924	0.969	0.977	0.979	0.932
24	32	0.262	0.287	0.184	0.227	0.156	0.832	0.766	0.694	0.771	0.706
24	40	0.390	0.381	0.369	0.425	0.261	0.913	0.876	0.954	0.967	0.912
28	36	0.260	0.214	0.184	0.230	0.171	0.864	0.635	0.696	0.783	0.719
28	44	0.253	0.131	0.367	0.425	0.267	0.858	0.335	0.952	0.966	0.906
32	40	0.303	0.048	0.097	0.110	0.101	0.810	0.066	0.883	0.927	0.856

Simulation II. Yao (1993) compared the powers of five different statistics under the normal distribution setting including (1) Levin & Kline' statistic by Levin and Kline (1985), which they suggested using the maximum likelihood estimate of μ_0 to replace the unknown true value of μ_0 in the log-likelihood ratio statistic; (2) the semi-likelihood ratio by Siegmund (1986), where he derived the asymptotic approximation for a significance level under certain assumptions; (3) a generalized likelihood ratio statistic suggested by Siegmund (1985,1986) under the situation μ_0 and δ are unknown. Later he (1988) developed the large deviation approximations for the significance level; (4) the score-like statistic; and (5) the recursive residual by Brown *et al.* (1975) which was initially used to test the change point in a linear model. They are corresponding to Z_1, Z_2, Z_3, Z_4 and Z_5 in the above table, to detect the epidemic mean change in a normal distribution with the constant variance. Yao also suggested the choice of the better statistic according to the difference between change points with respect to the sample size. We compare the ELR test with these five statistics under the same distribution settings in Table 2. From the table, we can see that when $q - p = 6, 10$ and the increment $\delta = 0.8, 1.2$, the ELR test performs better than the other five tests with respect to all the differences. When $\delta = 1.2$ and 1.6, the power of the ELR is higher than the five tests at the differences 6 and 10, but lower than the others at the differences 20 and 30 within an acceptable range. The results indicate that the ELR test is a comparable candidate for detecting the epidemic mean change. The Type I error is also well controlled within a given nominal level.

Table 2: Comparison of the powers; normal distribution; $n = 60, \alpha = 0.05$

δ	$q-p$	ELR	Z_1 $\delta_0 = 0.2$	Z_2 $\delta_0 = 0.2$	Z_3 $m_0 = 1$ $m_1 = 59$	Z_3 $m_0 = 6$ $m_1 = 54$	Z_4	Z_5 $m_0 = 6$ $p+q=n$
0.8	6	0.46	0.18	0.17	0.20	0.21	0.15	0.32
1.2		0.61	0.37	0.34	0.43	0.47	0.31	0.58
1.6		0.77	0.64	0.57	0.73	0.76	0.52	0.82
0.8	10	0.61	0.37	0.35	0.30	0.40	0.32	0.48
1.2		0.83	0.73	0.68	0.67	0.75	0.65	0.80
1.6		0.94	0.95	0.93	0.93	0.95	0.91	0.96
0.8	20	0.69	0.68	0.68	0.54	0.65	0.67	0.47
1.2		0.89	0.96	0.96	0.91	0.95	0.96	0.78
1.6		0.93	1.00	1.00	1.00	1.00	1.00	0.96
0.4	30	0.31	0.24	0.28	0.17	0.25	0.27	0.28
0.8		0.62	0.73	0.77	0.60	0.69	0.77	0.63
1.2		0.76	0.97	0.98	0.95	0.97	0.98	0.91
0		0.014	0.052	0.053	0.048	0.053	0.048	0.056

Simulation III. We conduct the simulations of powers with various distributions and different sample sizes. Table 3 displays the simulated power of the ELR test when the underlying distributions are poisson and binomial distributions with various combinations of change locations, different increment of means and different sample sizes. From the table, we observe that, in general, the power gets higher when the epidemic change point pair is closer to the center, and the difference $q - p$, that is, the proportion of the data with an epidemic change is larger. It illustrates that the ELR test is more sensitive to detect such changes. We notice that when the increment δ, that is, the magnitude of the epidemic change is larger, the ELR test performs better. When the sample size increases, the power of the ELR test also increases. The Type I errors for both cases are controlled within the given nominal level.

Table 3: Power of the ELR; $\alpha = 0.05$

$n = 50$		$\delta = 1$		$\delta = 2$	
p	q	Poisson	Binomial	Poisson	Binomial
No change		0.027	0.026	0.027	0.026
8	16	0.427	0.511	0.717	0.893
8	38	0.713	0.730	0.923	0.943
12	20	0.523	0.617	0.733	0.887
12	32	0.630	0.690	0.793	0.893
16	26	0.553	0.671	0.790	0.913
16	36	0.593	0.707	0.827	0.883
24	32	0.427	0.571	0.743	0.887
24	34	0.480	0.674	0.781	0.907
32	40	0.437	0.467	0.733	0.875
$n = 70$		$\delta = 1$		$\delta = 2$	
p	q	Poisson	Binomial	Poisson	Binomial
No change		0.018	0.023	0.018	0.023
8	18	0.715	0.437	0.883	0.877
8	28	0.855	0.567	0.968	0.883
18	26	0.730	0.473	0.823	0.852
18	38	0.833	0.663	0.925	0.932
24	32	0.697	0.513	0.860	0.933
24	40	0.783	0.663	0.960	0.973
30	38	0.715	0.577	0.828	0.943
30	46	0.813	0.643	0.968	0.961
42	50	0.718	0.487	0.863	0.883
42	58	0.772	0.687	0.945	0.930

5. Analysis of Stanford Heart Transplant Data

We apply the ELR procedure for detecting epidemic changes on the Stanford heart transplant data taken from "The Statistical Analysis of Failure Time Data 2nd Edition" by Kalbfleisch and Prentice (2002), Appendix A, pp. 387-389. This data set originally consists of 103 subjects indexed according to 35 known age groups. The average survival time is computed for each age group. One purpose of studying this data is to evaluate the effect of heart transplantation on subsequent survival. Kalbfleisch and Prentice (2002) fit the data by the hazard function and argued that the heart transplant is more beneficial for the age group under 46. This data was also studied by Ramanayake and Gupta (2003) by using epidemic change point model under the exponential assumption. We test the same epidemic change point hypothesis that $H_0 : \delta = 0$ vs $H_A : \delta > 0$. The estimated epidemic change points were $p = 7$ and $q = 23$ corresponding to ages 29 and 48 years old, respectively. It indicates that the heart transplant is more beneficial to the age group between 29 and 48 than to the age group below 29 and the age group above 48. This result is the same as the ones obtained by the authors above using parametric tests developed with the assumption that the observations have an exponential distribution. It is also similar to the result obtained by Kalbfleisch and Prentice (2002). The resulting test statistic $Z_n = 11.85$ is significant with p-value 0.03184 from Theorem 1. The left graph in Figure 1 shows the scatterplot of the data, and the right graph shows the values of the empirical likelihood ratio corresponding to all the possible combinations of p and q. The highest peak in the graph corresponds to the largest ELR values with the change points $p = 7$ and $q = 23$.

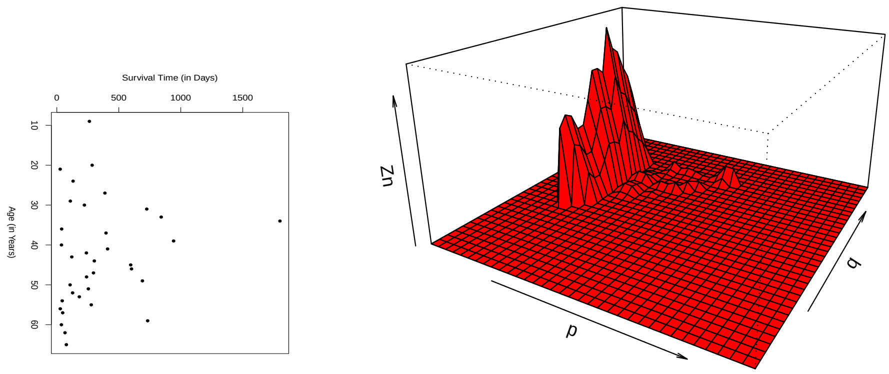

Figure 1: Left: scatterplot for survival time ordered by age of Stanford heart transplant data; right: empirical likelihood ratio values

6. Discussion

In this paper, we propose a test to detect the epidemic mean changes in the data based on the empirical likelihood method. The null distribution of the test statistic is derived under the mild conditions. The consistency of the test is also proved. Simulations are conducted for some continuous and discrete distributions corresponding to the various locations of change points. The results of the powers show that the ELR test performs well. We also compare our test with the LRT proposed by Ramanayake and Gupta (2003), in which they assumed the population distribution coming from the exponential family. Results indicate that the ELR test performs better than their tests in some scenarios, and comparable to the others. Simulations of the power of the ELR test under different distribution settings illustrate that the ELR test is robust to the data distribution. The power comparison under a normal model with the five statistics mentioned in Yao (1993) shows that the ELR behaves reasonable well and sometimes more competitive than the other tests with various differences between change point locations and the increment of the epidemic change. The ELR method is applied to the Stanford heart transplant data and detects two change points significantly.

The limitation of our test is that it may not detect the change points occurring at the very beginning or the very end of a data since the empirical estimators may not exist due to the properties of the empirical likelihood method. Modification of the test in order to handle such situations is the problem we are working on. In this paper, the proposed method only can detect one pair of change points occurring in an epidemic data. Therefore, the extension of the ELR test proposed in this paper to detect multiple pairs of change points occurring among the periodic epidemic data is one of the future research topics we would like to investigate.

Appendix

Proof of Theorem 1. We need the following lemmas. From the score functions, we let

$$g(\lambda) = \frac{1}{n}\sum \frac{x_i - \mu_0}{\theta_{pq} + \lambda'(x_i - \mu_0)} - \frac{1}{n}\sum \frac{x_j - \mu_0}{(1-\theta_{pq}) - \lambda'(x_j - \mu_0)} = 0.$$

Lemma 1. Under the condition of Theorem 1,

$$\widehat{\lambda}(\theta_{pq}) = \min(\theta_{pq}, 1-\theta_{pq}) O_p(\min(n-q+p, q-p))^{-1/2}.$$

proof. Let $\widehat{\lambda}(\theta_{pq}) = \rho\phi$, where $||\phi|| = 1$. Without any confusion, we simplify θ_{pq} as θ through the whole following part but we should remind that θ is related to p, q. Then

$$\begin{aligned}
0 &= ||\phi'|| \cdot ||g(\rho\phi)|| \geq |\phi' g(\rho\phi)| \\
&= \left| \frac{1}{n\theta}\sum_i \frac{\phi'(x_i-\mu_0)}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} - \frac{1}{n(1-\theta)}\sum_j \frac{\phi'(x_j-\mu_0)}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)} \right| \\
&= \left| \frac{1}{n\theta}\left[\sum_i \phi'(x_i-\mu_0) - \rho\sum_i \frac{\phi'(x_i-\mu_0)(x_i-\mu_0)'\phi/\theta^{-1}}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)}\right]\right. \\
&\quad \left. -\frac{1}{n(1-\theta)}\left[\sum_j \phi'(x_j-\mu_0) + \rho\sum_j \frac{\phi'(x_j-\mu_0)(x_j-\mu_0)'\phi/(1-\theta)^{-1}}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)}\right]\right|
\end{aligned}$$

$$= \left| \left(\frac{1}{n\theta} \sum_i \phi'(x_i - \mu_0) - \frac{1}{n(1-\theta)} \sum_j \phi'(x_j - \mu_0) \right) \right.$$
$$- \rho \left(\frac{1}{n\theta^2} \sum_i \frac{\phi'(x_i-\mu_0)(x_i-\mu_0)'\phi}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} \right.$$
$$\left. \left. + \frac{1}{n(1-\theta)^2} \sum_j \frac{\phi'(x_j-\mu_0)(x_j-\mu_0)'\phi}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)} \right) \right|$$
$$\geq \rho \left\{ \frac{1}{n\theta^2} \sum_i \frac{\phi'(x_i-\mu_0)(x_i-\mu_0)'\phi}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} + \frac{1}{n(1-\theta)^2} \sum_j \frac{\phi'(x_j-\mu_0)(x_j-\mu_0)'\phi}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)} \right\}$$
$$- \frac{1}{n\theta} \sum_i |\phi'(x_i-\mu_0)| - \frac{1}{n(1-\theta)} \sum_j |\phi'(x_j-\mu_0)|$$
$$= \rho \left\{ \frac{1}{n\theta^2} \sum_i \frac{(x_i-\mu_0)\phi'(x_i-\mu_0)'}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} + \frac{1}{n(1-\theta)^2} \sum_j \frac{(x_j-\mu_0)\phi'(x_j-\mu_0)'}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)} \right\}$$
$$- \frac{1}{n\theta} \left| \sum_{l=1}^{p} e_l' \sum_i (x_i-\mu_0) \right| - \frac{1}{n(1-\theta)} \left| \sum_{l=1}^{p} e_l' \sum_j (x_j-\mu_0) \right|,$$

where e_i and e_j are the unit vectors in the ith and jth coordinate directions. The negative sum of the last two expressions is of magnitude of $\min\{\theta, 1-\theta\} O_p(\min(n-q+p, q-p)^{-1/2})$ by the central limit theorem. For the first two terms

$$\rho \left\{ \frac{1}{n\theta^2} \sum \frac{\phi'(x_i-\mu_0)(x_i-\mu_0)'\phi}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} + \frac{1}{n(1-\theta)^2} \sum \frac{\phi'(x_j-\mu_0)(x_j-\mu_0)'\phi}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)} \right\}$$
$$= \rho \left\{ \frac{1}{\theta} \frac{1}{n-q+p} \sum \frac{\phi'(x_i-\mu_0)(x_i-\mu_0)'\phi}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} \right.$$
$$\left. + \frac{1}{1-\theta} \frac{1}{q-p} \sum \frac{\phi'(x_j-\mu_0)(x_j-\mu_0)'\phi}{1+\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)} \right\}$$
$$= \frac{1}{\theta} \frac{\rho\phi' S_{n-q+p}\phi}{1+\rho\phi'\theta^{-1}(x_i-\mu_0)} + \frac{1}{1-\theta} \frac{\rho\phi' S_{q-p}\phi}{1-\rho\phi'(1-\theta)^{-1}(x_j-\mu_0)}$$
$$\geq \frac{1}{\theta} \frac{\rho\phi' S_{n-q+p}\phi}{1+\rho\phi' z_1} + \frac{1}{1-\theta} \frac{\rho\phi' S_{q-p}\phi}{\rho\phi' z_2} \geq \frac{1}{\theta} \frac{\rho\phi' S_{n-q+p}\phi}{1+\rho\phi' z} + \frac{1}{1-\theta} \frac{\rho\phi' S_{q-p}\phi}{\rho\phi' z}$$

where

$$S_{n-q+p} = \frac{1}{n-q+p} \sum_i (x_i-\mu_0)(x_i-\mu_0)',$$

$$S_{q-p} = \frac{1}{q-p}\sum_j (x_j - \mu_0)(x_j - \mu_0)'.$$

We also have

$$z_1 = \max_{1<i\leq p, q+1\leq i<n} ||x_i - \mu_0||/\theta^{-1} = o_p((n\theta)^{1/2})/\theta^{-1},$$
$$z_2 = \max_{p+1\leq j\leq q} ||x_j - \mu_0||/(1-\theta)^{-1} = o_p((n(1-\theta))^{1/2})/(1-\theta)^{-1},$$

because of $E_F|X|^3 \leq \infty$ and $z = \max\{z_1, z_2\}$. We also note that

$$\phi' S_{n-q+p}\phi \geq \sigma_{p1} + o_p(1),$$
$$\phi' S_{q-p}\phi \geq \sigma_{p2} + o_p(1),$$

where σ_{p1} and σ_{p2} are the smallest eigenvalues of covariance matrices of $\{x_1, \cdots, x_p, x_{q+1}, \cdots, x_n\}$, $\{x_{p+1}, \cdots, x_q\}$ respectively. Therefore, we obtain

$$\widehat{\lambda}(\theta_{pq}) = \min\{\theta, 1-\theta\} O_p(n-q+p, q-p)^{-1/2}. \tag{6.1}$$

which completes the proof. For the convenience, we denote $\widehat{\lambda}(\theta_{pq}) = \widehat{\lambda}$. Therefore,

$$\begin{aligned}
g(\widehat{\lambda}) &= \frac{1}{n}\sum \frac{x_i - \mu_0}{\theta + \widehat{\lambda}'(x_i - \mu_0)} - \frac{1}{n}\sum \frac{x_j - \mu_0}{(1-\theta) - \widehat{\lambda}'(x_j - \mu_0)} \\
&= \frac{1}{n\theta}\sum \frac{x_i - \mu_0}{1 + \widehat{\lambda}'\theta^{-1}(x_i - \mu_0)} - \frac{1}{n(1-\theta)}\sum \frac{x_j - \mu_0}{1 - \widehat{\lambda}'(1-\theta)^{-1}(x_j - \mu_0)} \\
&= \frac{1}{n\theta}\sum (x_i - \mu_0)(1 + \theta^{-1}\gamma_i)^{-1} - \frac{1}{n(1-\theta)}\sum (x_j - \mu_0)(1 - (1-\theta)^{-1}\gamma_j) \\
&= \frac{1}{n\theta}\sum (x_i - \mu_0)\left(1 - \theta^{-1}\gamma_i + \frac{(\theta^{-1}\gamma_i)^2}{1 - \theta^{-1}\gamma_i}\right) \\
&\quad - \frac{1}{n(1-\theta)}\sum (x_j - \mu_0)\left(1 + (1-\theta)^{-1}\gamma_j + \frac{((1-\theta)^{-1}\gamma_i)^2}{1 + (1-\theta)^{-1}\gamma_i}\right) \\
&= \frac{1}{n\theta}\sum (x_i - \mu_0) - \frac{1}{n\theta^2}\sum (x_i - \mu_0)\gamma_i + \frac{1}{n\theta}\sum (x_i - \mu_0)\frac{(\theta^{-1}\gamma_i)^2}{1 - \theta^{-1}\gamma_i} \\
&\quad - \frac{1}{n(1-\theta)}\sum (x_j - \mu_0) - \frac{1}{n(1-\theta)^2}\sum (x_j - \mu_0)\gamma_j \\
&\quad - \frac{1}{n(1-\theta)}\sum (x_j - \mu_0)\frac{((1-\theta)^{-1}\gamma_j)^2}{1 - (1-\theta)^{-1}\gamma_j}.
\end{aligned}$$

Denote $w_i = x_i - \mu_0$ and $w_j = x_j - \mu_0$. Then

$$\begin{aligned}
&g(\widehat{\lambda}) = 0 \Longrightarrow (\bar{w}_1 - \bar{w}_2) - \frac{1}{n\theta^2}\sum w_i w_i'\widehat{\lambda} - \frac{1}{n(1-\theta)^2}\sum w_j w_j'\widehat{\lambda} \\
&+ \frac{1}{n\theta}\sum (x_i - \mu_0)\frac{(\theta^{-1}\gamma_i)^2}{1 - \theta^{-1}\gamma_i} - \frac{1}{n(1-\theta)}\sum (x_j - \mu_0)\frac{((1-\theta)^{-1}\gamma_j)^2}{1 - (1-\theta)^{-1}\gamma_j} = 0.
\end{aligned}$$

The absolute value of the last two terms can be bounded by

$$\begin{aligned}
&\frac{1}{n\theta}\sum ||x_i-\mu_0||\left|\frac{(\theta^{-1}\gamma_i)^2}{1-\theta^{-1}\gamma_i}\right| \\
&= \frac{1}{n\theta}\sum ||x_i-\mu_0|||\gamma_i|^2\left|\frac{(\theta^{-1})^2}{1-\gamma_i\theta^{-1}}\right| \\
&= \frac{1}{n\theta}\sum ||x_i-\mu_0||^3||\widehat{\lambda}||^2|\theta^{-1}|\left|\frac{\theta^{-1}}{1-\gamma_i\theta^{-1}}\right| \\
&= o((n-q+p)^{1/2})\min\{\theta,1-\theta\}O_p(\min(n-q+p,q-p)^{-1})O_p(1) \\
&= \min\{\theta,1-\theta\}o_p(\min(n-q+p,q-p)^{-1/2}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{n(1-\theta)}\sum ||x_j-\mu_0||\left|\frac{((1-\theta)^{-1}\gamma_j)^2}{1-(1-\theta)^{-1}\gamma_i}\right| \\
&= \frac{1}{n(1-\theta)}\sum ||x_j-\mu_0|||\gamma_j|^2\left|\frac{(1-\theta^{-1})^2}{1-\gamma_j\theta^{-1}}\right| \\
&= \frac{1}{n(1-\theta)}\sum ||x_j-\mu_0||^3||\widehat{\lambda}||^2|(1-\theta)^{-1}|\left|\frac{(1-\theta)^{-1}}{1-\gamma_j(1-\theta)^{-1}}\right| \\
&= o((q-p)^{1/2})\min\{\theta,1-\theta\}O_p(\min(n-q+p,q-p)^{-1})O_p(1) \\
&= \min\{\theta,1-\theta\}o_p(\min(n-q+p,q-p)^{-1/2}),
\end{aligned}$$

Hence, we can rewrite $\widehat{\lambda}$, that is, $\widehat{\lambda}(\theta_{pq})$ as

$$\widehat{\lambda}(\theta_{pq}) = (\bar{w}_1-\bar{w}_2)\left(\frac{1}{n\theta^2}\sum w_iw_i' + \frac{1}{n(1-\theta)^2}\sum w_jw_j'\right)^{-1} + \beta, \tag{6.2}$$

where $||\beta|| = \min\{\theta,1-\theta\}o_p(\min(n-q+p,q-p)^{-1/2})$. Note that under the null hypothesis and the observations are i.i.d, we have

$$\begin{aligned}
&\frac{1}{\theta}\cdot\frac{1}{n\theta}\sum w_iw_i' = \frac{1}{\theta}(S+O_p((n\theta)^{-1/2})), \\
&\frac{1}{1-\theta}\cdot\frac{1}{n(1-\theta)}\sum w_jw_j' = \frac{1}{1-\theta}(S+O_p((n(1-\theta))^{-1/2})),
\end{aligned}$$

where $S = E_F(X-\mu_0)(X-\mu_0)'$. Therefore, λ can be rewritten as follows

$$\begin{aligned}
\widehat{\lambda}(\theta_{pq}) &= (\theta^{-1}S+\theta^{-1}O_p((n\theta)^{-1/2})+(1-\theta)^{-1} \\
&\quad + (1-\theta)^{-1}O_p((n(1-\theta))^{-1/2}))(\bar{w}_1-\bar{w}_2)+\beta \\
&= \theta(1-\theta)S^{-1}(\bar{w}_1-\bar{w}_2)+\tilde{\beta}, \qquad (6.3)
\end{aligned}$$

where $\tilde{\beta} = \min\{\theta, 1-\theta\} o_p(\min(n-q+p, q-p)^{-1/2})$. With two-term Taylor expansion at $(\widehat{\lambda}(\theta_{pq}), \widehat{\mu}_0(\theta_{pq})) = (0,0)$ and some algebraic calculations, we obtain

$$Z_{n,p,q} = n\theta(1-\theta)(\bar{w}_1 - \bar{w}_2)' S^{-1}(\bar{w}_1 - \bar{w}_2) + O_p(||\widehat{\lambda}(\theta_{pq})||/\min(\theta, 1-\theta)). \quad (6.4)$$

Lemma 2. Under the conditions of Theorem 1, for all the $\delta > 0$, we can find $C = C(\delta), T_0 = T_0(\delta)$ and $N = N(\delta)$such that for $T > T_0$ and $n > N$,

$$P\left(\max_{T \le n-(q-p) \le n-T} (m/\log\log m)^{1/2} ||\widehat{\lambda}(\theta_{pq})/\min(\theta, 1-\theta)|| > C\right) \le \delta,$$
$$P\left(n^{-1/2} \max_{T \le n-(q-p) \le n-T} (m||\widehat{\lambda}(\theta_{pq})/\min(\theta, 1-\theta)|| > C\right) \le \delta,$$

where $m = O_p(\min(n-q+p, q-p)^{-1/2})$.
proof. The proof is similar to the proof of Lemma 1.2.2 of Csörgő and Horvath (1997).

Lemma 3. Under the conditions of Theorem 1, for all $0 \le \alpha \le 1/2$ we have:

$$n^{\alpha} \max_{\theta \in \Theta_n} [\theta(1-\theta)]^{\alpha} |Z_{n,p,q} - R_{pq}| = O_p(1)$$
$$\max_{\theta \in \Theta_n} [\theta(1-\theta)]^{\alpha} |Z_{n,p,q} - R_{pq}| = O_p(n^{-1/2}(\log\log n)^{3/2}),$$

where

$$R_{pq} = n\theta(1-\theta)(\bar{w}_1 - \bar{w}_2)' S^{-1}(\bar{w}_1 - \bar{w}_2),$$

and

$$S = E_F(x_i - \mu_0)(x_i - \mu_0)',$$
$$\Theta_n = \{1 - \frac{q-p}{n} : k_0 \le p < q \le n - k_1\}.$$

proof. From (6.4), we have

$$Z_{n,p,q} = n\theta(1-\theta)(\bar{w}_1 - \bar{w}_2)' S^{-1}(\bar{w}_1 - \bar{w}_2) + O_p(||\widehat{\lambda}(\theta_{pq})||/\min(\theta, 1-\theta)).$$

Then apply Lemma 2 to complete the proof.

Proof of Theorem 1. Using Lemma 3 and argument similar to the proof of Theorem 1.3.1 of Csörgő and Horvath (1997), we can prove the theorem. Note in the proof of Theorem, we use Theorem A.3.4 instead of Corollary A.3.1 from Csörgő and Horvath (1997) because we will derive the null distribution of the trimmed test statistic Z_n.

Proof of Theorem 2. First we need the following notations. Define $\Delta_0 = \Delta_n = 0$ and $\Delta_{pq} = Z(p,q)/2n$. We also define $W = \theta + \lambda'(x-\mu)$, where $(\lambda(\theta), \mu(\theta))$ is the solution of the equations $g_1(\theta,\lambda,\mu) = 0$ and $g_2(\theta,\lambda(\theta),\mu(\theta)) = 0$, where g_1, g_2 are defined as follows. We also define

$$\begin{aligned}
dP &= \{\theta I\{\theta \le \theta_0\} + \theta_0 I\{\theta > \theta_0\}\}dF + (\theta-\theta_0)I\{\theta > \theta_0\}dG,\\
dQ &= \{(1-\theta)I\{\theta \ge \theta_0\} + (1-\theta_0)I\{\theta < \theta_0\}\}dG + (\theta-\theta_0)I\{\theta < \theta_0\}dF,\\
dR &= I\{\theta < \theta_0\}dF + I\{\theta > \theta_0\}dG,
\end{aligned}$$

where I is an indicator function, F, G are the distributions for the observations $\{x_1,\cdots,x_p,x_{q+1},\cdots,x_n\}$ and $\{x_{p+1},\cdots,x_q\}$ respectively, and we assume the true value of $\theta_{pq} = 1-(q-p)/n$ under the alternative hypothesis converges to θ_0 in probability. We let

$$\varsigma(\theta,\lambda,\mu) = \int \log W dP + \int \log(1-W)dQ - \theta\log\theta - (1-\theta)\log(1-\theta),$$

and then functions g_1, g_2 are defined as

$$\begin{aligned}
g_1(\theta,\lambda,\mu) &= \frac{\partial\varsigma(\theta,\lambda,\mu)}{\partial\lambda} = \int \frac{x}{W}dP - \int \frac{x}{1-W}dQ,\\
g_2(\theta,\lambda,\mu) &= \frac{\partial\varsigma(\theta,\lambda,\mu)}{\partial\mu} = \int \frac{1}{W}dP - \int \frac{1}{1-W}dQ.
\end{aligned}$$

We also let

$$\Omega = \begin{pmatrix} -\frac{\partial g_1}{\partial\lambda} & -\frac{\partial g_1}{\partial\mu} \\ -\frac{\partial g_2}{\partial\lambda} & -\frac{\partial g_2}{\partial\mu} \end{pmatrix}.$$

To prove the consistency of the test, we assume

Assumption 1. The following two integrals are finite

$$\begin{aligned}
&\int \frac{1}{W^2} + \frac{1}{(1-W)^2} d(F+G) < \infty,\\
&\int \frac{(x-\mu)(x-\mu)'}{W^2} + \frac{(x-\mu)(x-\mu)'}{(1-W)^2} d(F+G) < \infty.
\end{aligned}$$

Assumption 2. Ω is nonsingular for all the $\theta \in (0,1)$.

The existence of Ω can be guaranteed by the Assumption 1. With the Assumption 2, the matrix Ω can be easily proved to be positive definite, therefore,

$\zeta(\theta, \lambda(\theta), \mu(\theta))$ is the unique maximum of $\zeta(\theta, \lambda, \mu)$ for a fixed θ.

Suppose both assumptions hold. Under the alternative hypothesis, if $\theta_{pq} \to \theta \in (0,1)$,then we can show that the ELR test is consistent with the argument similar to the proof of Theorem 2 of Zou *et al.* (2007). It completes the proof of the Theorem 2.

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