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An Empirical Study on Implied GARCH Models

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Abstract: An empirical study is employed to investigate the performance of implied GARCH models in option pricing. The implied GARCH models are established by either the Esscher transform or the extended Girsanov principle. The empirical P-martingale simulation is adopted to compute the options efficiently. The empirical results show that: (i) the implied GARCH models obtain accurate standard option prices even the innovations are conveniently assumed to be normal distributed; (ii) the Esscher transform describes the data better than the extended Girsanov principle; (iii) significant model risk arises when using implied GARCH model with non-proper innovations in exotic option pricing.

Key words: Empirical martingale simulation, Esscher transform, extended Girsanov principle, implied GARCH model, option pricing.

1. Introduction

Conditional heteroscedastic models such as the ARCH and the GARCH models (Engle, 1982; Bollerslev, 1986) play an important role in time series modeling and financial derivative pricing. Many studies have indicated that GARCH models outperform the well-known Black-Scholes (BS) model (Black and Scholes, 1973) in financial derivative pricing, see for example, Rosenberg and Engle (1994; 1995), Duan (1995) and Hagerud (1996). In the literature, many different types of GARCH models with different innovations have been proposed to depict the financial time series data. However, the model that best fits historical data does not necessarily have its minimum mean squared error between the option prices observed in the market and the prices derived from the GARCH model. Therefore, the implied method was proposed to improve the pricing performance (Fofana and Brorsen, 2001; Yung and Zhang, 2003).

The implied GARCH model is obtained by matching the GARCH option prices with the market "plain vanilla" option values under certain loss function.

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This concept is similar to the implied volatility function model in the BS framework. However, investigation of the pricing performance when using normal or heavy-tailed innovations in the implied GARCH models is still lacking. Furthermore, based on different economic considerations, many different risk-neutral measures for GARCH models have been proposed to compute the no-arbitrage prices of financial derivatives. However, it is still uncertain which economic consideration is preferred by most investors if GARCH models are believed to depict the market behavior well. In this current work, an empirical study is employed to investigate whether the implied GARCH models can obtain accurate option prices even when a simple type of GARCH model is used. Also, we examine whether different innovations or different risk-neutral measures would significantly affect the implied GARCH option pricing. Moreover, it is important to know whether the implied method under the GARCH framework is suitable for any financial derivative.

One of the challenges of computing GARCH option prices is to define a risk-neutral measure. Since a GARCH model is a discrete-time and continuous-state model, thus the market described by the model is incomplete and the risk-neutral measure is not unique. Duan (1995) considered the derivation of risk-neutral GARCH model with normal innovations (denoted by GARCH-N). However, many empirical studies show that the financial returns are conditional leptokurtic (Bollerslve, 1987; Baillie and Bollerslev, 1989; Hsieh, 1989; Baillie and DeGennaro, 1990; Wang, Christopher, Christopher and James, 2001; Siu, Tong, and Yang, 2004). Thus researchers have considered GARCH models with leptokurtic innovations. For example, the GARCH model with standardized t innovations (Bollerslev, 1987), generalized exponential innovations (Nelson, 1991), shifted-gamma innovations (Siu, Tong and Yang, 2004) and double-exponential innovations (Huang, 2011; Huang and Guo, 2011) have been discussed. To evaluate the financial derivatives in GARCH models with leptokurtic innovations, the Esscher transform (Gerber and Shiu, 1994) and the extended Girsanov principle (Elliott and Madan, 1998) are two popular change of measure processes used in practice (Badescu and Kulperger, 2008). In this study, we consider the cases of normal and double exponential innovations. The corresponding change of measure processes of these models are derived under the Esscher transform and the extended Girsanov principle in Section 2.

Another technical issue arises from the computational efficiency of the GARCH option pricing because there is usually no closed-form solution to obtain the no-arbitrage price under the GARCH framework. The standard Monte-Carlo simulation usually develops heavy computational burden when higher accuracy is required. In order to tackle this problem, Duan and Simonato (1998) proposed an empirical martingale simulation (EMS) method to compute option prices more efficiently by generating random paths of the underlying assets from the risk-neutral model. Since an explicit expression of the risk-neutral model may be difficult to obtain in a complex model, Huang (2011) proposed an empirical P-martingale simulation (EPMS) method, which extends the EMS from the risk-neutral framework to the dynamic P measure. The strong consistency of the EPMS method is established and its efficiency is demonstrated by simulation studies. Therefore, the EPMS method is employed to construct the implied GARCH model and compute the option prices in this study.

In the empirical study, S&P 500 index options from January 2, 2003 to June 30, 2009 are adopted to conduct several comparison studies of the implied GARCH models between different innovations and between the two change of measure processes. The results show that the implied GARCH models are capable of computing accurate plain vanilla option prices even the innovations are conveniently assumed to be normal distributed. In addition, the change of measure process derived by the Esscher transform describes the data better than that derived by the extended Girsanov principle. Moreover, significant model risk arises when using implied GARCH model with non-proper innovations for exotic option pricing, which is consistent with the simulation results in Hull and Suo (2002) for the BS framework.

The rest of this study is organized as follows. In Section 2, we introduce the construction of the implied GARCH models. The change of measure processes obtained from the Esscher transform and the extended Girsanov principle in GARCH models with normal and double exponential innovations are derived. In addition, the EPMS method is introduced briefly. Section 3 conducts several empirical studies such as investigate the pricing effect caused by using different innovations or change of measure processes, and examine the evidence of the model risk in exotic option pricing. Conclusions are in Section 4, and theoretical proofs are given in the Appendix.

2. Implied GARCH Models

The traditional scheme for computing the GARCH option prices includes the following steps:

1. Fit the GARCH model by historical asset prices under the dynamic measure.

2. Transform the dynamic GARCH model into the risk-neutral GARCH model.

3. Compute the option prices by Monte-Carlo simulation method under the risk-neutral GARCH model.

However, the best fitting asset model in Step-1 does not necessarily have the minimum mean squared error between the option prices observed in the market and the prices derived from the GARCH model. Therefore, many studies suggest using the implied method to handle this problem (Fofana and Brorsen, 2001; Yung and Zhang, 2003). The parameters of an implied GARCH model are obtained by matching the prices derived from the model with the prices observed in the market under certain measures of hedging loss. In this study, we use the root mean squared error (RMSE) as the criterion to estimate the GARCH model parameters (denoted by $\widetilde{\boldsymbol{\beta}}$). That is,

$$\widetilde{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \text{RMSE}(\boldsymbol{\beta}) = \arg\min_{\boldsymbol{\beta}} \left\{ \frac{1}{N} \sum_{j=1}^{N} \left(C_j^{market} - C_j^{model}(\boldsymbol{\beta}) \right)^2 \right\}^{\frac{1}{2}},$$

where C_j^{market} denotes the j-th market plain vanilla option price, C_j^{model} denotes the corresponding GARCH model option price, $j = 1, \cdots, N$, and N is the total number of trading options.

Since the GARCH models considered in this study have leptokurtic innovations, two popular change of measure processes proposed in the literature are used to derive the corresponding risk-neutral measures. One is the Esscher transform (Gerber and Shiu, 1994) and the other one is the extended Girsanov principle (Elliott and Madan, 1998). The details of these two change of measure processes are given in Section 2.1.

However, if high accuracy in computing the option prices is required, then the standard Monte-Carlo simulation in Step-3 usually develops heavy computational burden. To resolve this, Huang (2011) proposed an EPMS method to compute the option prices more efficiently and established its strong consistency. The EPMS method generates the random paths of the underlying assets under the dynamic P measure, which is more flexible and convenient than the EMS method of Duan and Simonato (1998) under the risk-neutral framework particularly when the explicit expression of the risk-neutral model is not easy to obtain. The procedure of the EPMS method is illustrated in Section 2.4.

2.1 Change of Measure Processes

Here, we briefly introduce two change of measure processes derived by the Esscher transform and the extended Girsanov principle.

Esscher transform

Let S_t denote the price of the underlying asset at time t and $R_t = \log(S_t/S_{t-1})$ denote the log return process. The conditional moment generating function (mgf) of R_t given $\mathcal{F}_{t-1}$ is defined by $M_{R_t|\mathcal{F}_{t-1}}(z) = E_{t-1}(e^{zR_t})$, where $\mathcal{F}_{t-1}$ denotes the information set consisting of the prices of riskless bond and the underlying asset prior to time $t-1$ and the notation $E_{t-1}(\cdot)$ denotes the conditional expectation given $\mathcal{F}_{t-1}$ under the dynamic P measure. Let $f_t(\cdot)$ denote the conditional probability density function (pdf) of $R_t|\mathcal{F}_{t-1}$. Define a new conditional pdf as

$$f_t(x_t;\delta_t) = \frac{e^{\delta_t x_t} f_t(x_t)}{M_{R_t|\mathcal{F}_{t-1}}(\delta_t)} \tag{2.1}$$

with an extra parameter δ_t. If the parameter $\delta_t = \delta_t^*$ is chosen such that the discounted prices of the underlying asset are a martingale, that is, $S_{t-1} = E_{t-1}^{Q^{ess}}(e^{-r}S_t)$, where r is the riskless interest rate, and $E_{t-1}^{Q^{ess}}(\cdot)$ denotes the conditional expectation under the selected conditional pdf $f_t(x_t;\delta_t^*)$. Define a change of measure process $\{\Lambda_t^{ess}, t = 1, \cdots, T\}$ by

$$\Lambda_t^{ess} = \prod_{k=1}^{t} \frac{f_k(x_k;\delta_k^*)}{f_k(x_k)} = \prod_{k=1}^{t} \frac{e^{\delta_k^* x_k}}{M_{R_k|\mathcal{F}_{k-1}}(\delta_k^*)}, \tag{2.2}$$

where $f_k(x_k)$, $f_k(x_k;\delta_k^*)$, $M_{R_k|\mathcal{F}_{k-1}}(\delta_k^*)$ and δ_k^* are defined as above. The risk-neutral probability measure Q^{ess} under the Esscher transform is then defined by $dQ^{ess} = \Lambda_T^{ess}dP$. The economic foundation of the Esscher transform is to maximize the power utility function of the representative investor. For details, the reader is referred to Gerber and Shiu (1994).

Extended Girsanov principle

Assume the discounted asset price process $\tilde{S}_t = S_t e^{-rt}$ satisfies the multiplicative decomposition, $\tilde{S}_t = \tilde{S}_0 A_t M_t$, where $A_t \equiv \prod_{k=1}^{t} E_{k-1}(\tilde{S}_k/\tilde{S}_{k-1})$ is a predictable process and $M_t = \tilde{S}_t/(\tilde{S}_0 A_t)$ is a positive martingale under the physical probability measure P. Define a change of measure density process $\{\Lambda_t^{egp}, t = 0, 1, \cdots, T\}$ by

$$\Lambda_t^{egp} = \prod_{k=1}^{t} \frac{\phi_k(\tilde{S}_k/\tilde{S}_{k-1})e^{u_k}}{\phi_k(e^{-u_k}\tilde{S}_k/\tilde{S}_{k-1})}, \tag{2.3}$$

which is a P-martingale, where $u_k = \log E_{k-1}(\tilde{S}_k/\tilde{S}_{k-1}) = \log E_{k-1}[\exp(R_k - r)]$ denotes the risk-premium of the underlying asset and $\phi_k(y)$ is the conditional density of $Y = M_k/M_{k-1} = e^{-u_k}\tilde{S}_k/\tilde{S}_{k-1} = \exp(R_k)/E_{k-1}[\exp(R_k)]$ given $\mathcal{F}_{k-1}$ under the physical measure P. The equivalent martingale measure Q^{egp} under the extended Girsanov change of measure is then defined by $dQ^{egp} = \Lambda_T^{egp}dP$. The economic foundation of the extended Girsanov principle is to minimize the adjusted hedging capital of investors hedging portfolio. For details, the reader is referred to Elliott and Madan (1998).

2.2 GARCH-N Model

In this section, we consider the case of a GARCH model with normal innovations, which is commonly used in practice. Since the GARCH(1,1) model is the most popular for financial applications (Fan and Yao, 2003) and to simplify the illustration, we consider the following GARCH(1,1) model for the log return process R_t,

$$\left\{ \begin{array}{l} R_t = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t, \quad \varepsilon_t \sim N(0,1), \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \end{array} \right. \tag{2.4}$$

Model (2.4) is also discussed by Duan (1995). The change of measure processes derived by the Esscher transform and the extended Girsanov principle for the GARCH-N model are obtained in the following proposition.

Proposition 2.1. In Model (2.4), the change of measure processes Λ_t^{ess} and Λ_t^{egp} derived by the Esscher transform and the extended Girsanov principle, respectively, satisfy

$$\Lambda_t^{ess} = \Lambda_t^{egp} = \prod_{k=1}^{t} \exp\Big\{ -\frac{1}{2\sigma_k^2}\Big(\lambda^2\sigma_k^2 + 2\lambda\sigma_k(R_k - r - \lambda\sigma_k + \frac{1}{2}\sigma_k^2)\Big)\Big\}.$$

Consequently, the risk-neutral measures Q^{ess} and Q^{egp} derived from the two approaches are the same.

Interestingly, Proposition 2.1 shows that the two change of measure processes Λ_t^{ess} and Λ_t^{egp} are identical although the economic consideration of the Esscher transform and extended Girsanov principle are different. Furthermore, from the proof of Proposition 2.1 in the Appendix, the conditional density of R_t given $\mathcal{F}_{t-1}$ defined in (2.1) can be obtained as

$$f_t(R_t;\delta_t^*) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\Big\{ -\frac{1}{2\sigma_t^2}(R_t - r + \frac{\sigma_t^2}{2})^2\Big\}$$

by choosing $\delta_t^* = -\lambda/\sigma_t$. That is, the log return R_t conditional on $\mathcal{F}_{t-1}$ is normal distributed with mean $r - \sigma_t^2/2$ and variance σ_t^2 under Q^{ess}. As a result, let $\xi_t = \varepsilon_t + \lambda$, and the risk-neutral GARCH-N counterpart of Model (2.4) under measure Q^{ess} (or Q^{egp}) is written as follows,

$$\left\{ \begin{array}{l} R_t = r - \frac{\sigma_t^2}{2} + \sigma_t\xi_t, \quad \xi_t \sim N(0,1), \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - \lambda)^2 + \beta\sigma_{t-1}^2, \end{array} \right.$$

which is the same as Duan (1995)'s result derived from the locally risk-neutral valuation relationship with an expected utility maximizer.

2.3 GARCH-DE

Next, we consider the following GARCH models with double exponential innovations (denoted by GARCH-DE henceforth):

$$\begin{cases} R_t = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t, & \varepsilon_t \sim DE(0,1), \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \end{cases} \tag{2.5}$$

where $DE(0,1)$ stands for the double exponential distribution with zero mean and unit variance. The corresponding change of measure processes Λ_t^{ess} of Model (2.5) are derived in the following.

Proposition 2.2. For Model (2.5), we have the following results:

(i) The change of measure process Λ_t^{ess} derived by the Esscher transform is

$$\Lambda_t^{ess} = \prod_{k=1}^{t} \left\{1 - \frac{(\delta_k^*\sigma_k)^2}{2}\right\} \exp\left\{\delta_k^*\Big(R_k - r - \lambda\sigma_k + \frac{\sigma_k^2}{2}\Big)\right\},$$

where $\delta_k^* = \frac{-\sigma_k + \sqrt{a_k\sigma_k^2 + 2(a_k-1)^2}}{\sigma_k(1-a_k)}$ and $a_k = \exp(\lambda\sigma_k - \frac{\sigma_k^2}{2})$.

(ii) The change of measure process Λ_t^{egp} derived by the extended Girsanov principle is

$$\Lambda_t^{egp} = \prod_{k=1}^{t} \exp\left\{ - \frac{\sqrt{2}}{\sigma_k}\Big(\Big|R_k - r - \log(1 - \frac{\sigma_k^2}{2})\Big| - \Big|R_k - r - \lambda\sigma_k + \frac{\sigma_k^2}{2}\Big|\Big)\right\}$$

provided by $\sigma_k^2 < 2$.

Remark 2.1. Although the simple case of a GARCH model with one ARCH parameter and one GARCH parameter and a particular type of mean equation is discussed in Sections 2.2 and 2.3, the derivations of the change of measure processes given in Proposition 2.1 and 2.2 can be extended to more general GARCH models, like GARCH(p, q) and threshold GARCH models, by adjusting the volatility parts, σ_t, in the formulae of Λ_t^{ess} and Λ_t^{egp}. Furthermore, the derivation of the change of measure processes can be extended directly to other types of mean equation.

Unlike the result in Proposition 2.1 for the GARCH-N model, the change of measure processes Λ_t^{ess} and Λ_t^{egp} in Proposition 2.2 are no longer equal. Huang and Guo (2011) established the derivation of the risk-neutral GARCH models under Q^{egp} in a general setting. Hence, by applying Proposition 3.1 of Huang and Guo (2011), the risk-neutral GARCH models with double exponential innovations considered in this study under Q^{egp} can be obtained. On the other hand, under Q^{ess}, first note that the mgf of a double exponential distribution with mean μ and variance $2b^2$ is $M(z) = e^{z\mu}/(1-z^2b^2)$ for $|z| < 1/b$. Next, by straightforward computation, the mgf of R_t conditional on $\mathcal{F}_{t-1}$ under Q^{ess} is

$$M_{R_t|\mathcal{F}_{t-1}}(z;\delta_t^*) = \exp\Big\{z(r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2)\Big\}\Big(\frac{1-(\delta_t^*\sigma_t)^2/2}{1-(z+\delta_t^*)^2\sigma_t^2/2}\Big), \tag{2.6}$$

where δ_t^* is defined in Proposition 2.2 (i). Recall that in GARCH-N, the conditional distribution of R_t given $\mathcal{F}_{t-1}$ under Q^{ess} belongs to the same distribution family as in the dynamic model. However, this is not the case in the GARCH-DE model since the mgf in (2.6) is not related to a double exponential distribution. As a result, it is not a trivial task to obtain an explicit expression of the risk-neutral GARCH-DE model under Q^{ess}. And the traditional Monte Carlo simulation method and the empirical martingale simulation method of Duan and Simonato (1998) under the risk-neutral framework can not be applied in this situation. Huang (2011) proposed an EPMS method to solve this problem by computing the no-arbitrage GARCH option prices more efficiently under the dynamic P measure with the help of the change of measure processes obtained in Propositions 2.1 and 2.2. Details are in the next section.

2.4 Empirical P-martingale simulation

Let Y_T denote a Radon-Nikodým derivative of a $\tilde{P}$ measure with respect to the dynamic P measure. Thus $E(Y_T) = 1$ and $d\tilde{P} = Y_T dP$. Define $Y_t = E(Y_T|\mathcal{F}_t)$, $0 \le t < T$, where $\mathcal{F}_t$ denotes the information set up to time t. Then Y_t is called the Radon-Nikodým derivative process (or a change of measure process) and is a martingale process under the P measure (abbreviated as P-martingale). The objective of the proposed EPMS method is to guarantee that the simulated processes of the discounted underlying asset prices and the change of measure values are both empirical P-martingale. The procedure is as follows:

1. Generate the paths of stock prices $\widehat{S}_{t,i}$, for $i = 1, \cdots, n$ and $t = 1, \cdots, T$, by the naive Monte Carlo method.

2. Let $Y_0^* = \widehat{Y}_0 = Y_0 = 1$ and define the empirical martingale change of measure process $Y_{t,i}^*$, $i = 1, \cdots, n$, iteratively by

$$Y_{t,i}^* = \frac{W_i(t,n)}{W_0(t,n)}, \tag{2.7}$$

where $W_i(t,n) = \frac{Y_{t-1,i}^*}{\widehat{Y}_{t-1,i}}\widehat{Y}_{t,i}$, $W_0(t,n) = \frac{1}{n}\sum_{i=1}^n W_i(t,n) = \frac{1}{n}\sum_{i=1}^n \frac{Y_{t-1,i}^*}{\widehat{Y}_{t-1,i}}\widehat{Y}_{t,i}$ and $\widehat{Y}_{t,i} = Y_t(\widehat{S}_{u,i}\ ,\ 0 \le u \le t)$.

3. Let $S_0^* = \widehat{S}_0 = S_0$ and defined the empirical martingale stock prices $S_{t,i}^*$, $i = 1, \cdots, n$, iteratively by

$$S_{t,i}^* = S_0 \frac{Z_i(t,n)}{Z_0(t,n)}, \tag{2.8}$$

where $Z_i(t,n) = \frac{S_{t-1,i}^*}{\widehat{S}_{t-1,i}} \widehat{S}_{t,i}$ and $Z_0(t,n) = \frac{e^{-rt}}{n} \sum_{i=1}^n Z_i(t,n) Y_{t,i}^*$.

Note that the two processes $Y_{t,i}^*$ and $S_{t,i}^*$ defined in (2.7) and (2.8), respectively, are themselves functions of the sample size n and satisfy

$$\frac{1}{n} \sum_{i=1}^n Y_{t,i}^* = Y_0 = 1,$$

and

$$\frac{1}{n} \sum_{i=1}^n e^{-rt} S_{t,i}^* Y_{t,i}^* = S_0,$$

for any integer n and $t = 1, \cdots, T$. That is, $Y_{t,i}^*$ and $e^{-rt} S_{t,i}^*$ are the so-called "emprical P-martingale processes". Huang (2011) proved that the derivative prices obtained by these two empirical P-martingale processes also converge to the theoretical value.

Combining the results obtained in Section 2.2-2.4, one can establish the implied GARCH-N and implied GARCH-DE models under the change of measure processes Λ^{ess} and Λ^{egp}. In the next section, we investigate the pricing performances of these implied GARCH models and conduct several comparison studies between different innovations and between Λ^{ess} and Λ^{egp}.

3. Empirical Study

In this section, data description is given in Section 3.1, and Section 3.2 shows that the implied GARCH models are capable of obtaining accurate standard option prices. Section 3.3 demonstrates the test results of normal versus double exponential innovations and Λ^{ess} versus Λ^{egp}. Section 3.4 shows the pricing performance of the implied method in different economic periods, that is, recession and recovery periods. Section 3.5 investigates the empirical evidence of model risk in up-and-out option pricing.

3.1 Description of Data

The data used in this study are S&P 500 index European options traded on the Chicago Board Option Exchange (CBOE). The sample period extends from

January 2, 2003 to June 30, 2009. Figure 1 shows the time series plots of the indices and the corresponding log returns. The time period from December 2007 to June 2009 highlighted in Figure 1 denotes the recession period, as announced by the NBER's business circle Dating Committee when the U.S. economy reached a trough. Herein, the trading days are used as time measure, i.e., 252 trading days per year. The data set contains the information of Option Trading Date, Expiration Date, Spot Price, Strike Price, Best Bid and Best Offer, Trading Volume, BS implied volatility, Delta, Gamma, Vega, and Theta. The mid-point of the best bid and best offer quote is adopted to be the option price (Dumas, Fleming and Whaley, 1998), and the annual risk-free interest rate is set to be 2.79%, which is the average of the Daily Treasury Bill Rates in the sample period. Moreover, the following filters are applied to screen data:

1. Only call options are considered.
2. To ensure the option liquidity, we select the options with maturities T between 10 and 180 days, moneyness (K/S_0) between 0.9 and 1.1, and positive trading volume.
3. Similar to Badescu and Kulperger (2008), we consider the trading data for the last Wednesday of each month. If the Wednesday is a holiday, then we choose the Tuesday.

Hence, there are 78 days and 7,330 options traded in our data set. The average option price is 30.696.

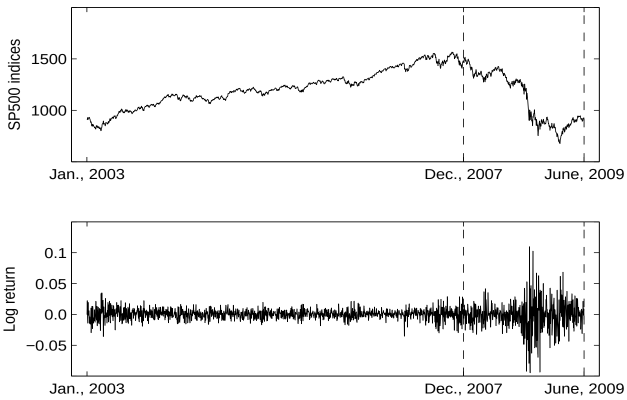

Figure 1: The time series plots of the S&P500 index and the corresponding log returns from January 2, 2003 to June 30, 2009 with recession period from December 2007 to June 2009

3.2 Implied GARCH Option Pricing

In this section, we consider six types of GARCH models: the GARCH-N, GARCH-DE^{ess}, GARCH-DE^{egp}, TGARCH-N, TGARCH-DE^{ess} and TGARCH-DE^{egp} models, where GARCH-DE^{ess} denotes the case of computing the GARCH-DE option prices by the change of measure process Λ_t^{ess}, etc., and TGARCH stands for the following threshold GARCH model:

$$\begin{cases} R_t = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t, \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \gamma I_{(\varepsilon_{t-1}<0)}\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2, \end{cases}$$

where α_0, α_1, β_1 and γ are nonnegative numbers, $\alpha_1 + \beta_1 + \gamma/2 < 1$, $I_{(\cdot)}$ is an indicator function, and ε_t's are i.i.d. random variables with zero mean and unit variance. As mentioned in Remark 2.1, the change of measure process for TGARCH models with normal or double exponential innovations can be obtained analogously to the standard GARCH models. We use these six implied GARCH models to fit the data set introduced in Section 3.1 by the EPMS method and compute the corresponding RMSEs. Figure 2 shows the boxplots of RMSEs of the implied GARCH models. The medians of the RMSEs of the six implied GARCH models are 2.091, 2.081, 2.185, 0.813, 0.782 and 0.958, respectively. Comparing the magnitudes of these medians to the values reported in the literature (for example, Badescu and Kulperger, 2008, Table 7), the implied GARCH models are capable of computing accurate plain vanilla option prices. Moreover, it is not surprising to find that the implied TGARCH models outperform the implied GARCH models since an extra parameter γ is added to the GARCH models. One of the interesting findings is that not only are the values of the RMSEs of the implied TGARCH models smaller than their implied GARCH counterparts,

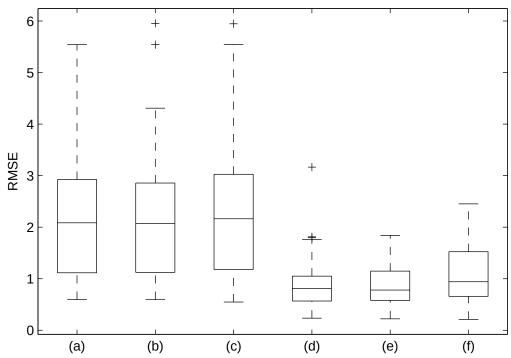

Figure 2: The boxplots of the RMSEs of the implied (a) GARCH-N, (b) GARCH-DE^{ess}, (c) GARCH-DE^{egp}, (d) TGARCH-N, (e) TGARCH-DE^{ess} and (f) TGARCH-DE^{egp} models

but also the dispersions of the RMSEs of the implied TGARCH models are significantly less than the implied GARCH models. This reveals that the parameter γ is not only useful to depict the asymmetric property of the dynamics of the data but also plays a crucial role in option pricing. Another interesting finding in Figure 2 is that the implied TGARCH-N model, implied TGARCH-DE^{ess} and implied TGARCH^{egp} models seem to be comparable, which implies that the simple normal innovations are enough to depict the market behavior in the implied GARCH framework, and the two change of measure processes do not make difference in GARCH option pricing. In next section, we examined the above phenomenon by a statistical test.

3.3 Testing the Pricing Performances

We investigate whether there are significant differences in option pricing between the normal and double exponential innovations or between the change of measure processes Λ^{ess} and Λ^{egp}. The case of the implied GARCH-N model versus the implied GARCH-DE^{ess} is used to be an example for illustration. Let $d_1, \cdots, d_{78}$ be the differences of the RMSEs between the implied GARCH-N model and the implied GARCH-DE^{ess} model for the 78 selected trading days. Further let M_d be the median of $d_1, \cdots, d_{78}$. By using the sign test for testing $H_0 : M_d = 0$ versus $H_1 : M_d > 0$, the corresponding p-value is less than 10^{-4}, which implies that the implied GARCH-DE^{ess} model describes the data significantly better than the implied GARCH-N model. Herein, $A \succ B$ denotes that Model A performs significantly better than Model B and $A \approx B$ denotes that the two models are comparable. After conducting the pairwise tests of the implied GARCH models considered in Section 3.2, we have the following result: TGARCH-DE^{ess} $\approx$ TGARCH-N $\succ$ TGARCH-DE^{egp} $\succ$ GARCH-DE^{ess} $\succ$ GARCH-N $\succ$ GARCH-DE^{egp}. Therefore, the change of measure process Λ^{ess} describes the data significantly better than Λ^{egp}. Interestingly, the implied TGARCH-N model is comparable to the implied TGARCH-DE^{ess} model. In next section, we further examine the pricing performances of the implied GARCH models in different economic periods.

3.4 Recession Period versus Recovery Period

In this section, we investigate the pricing performances of the implied GARCH models in the recession and recovery periods. By the report of New NBER Research published at September 20, 2010, the recession period during our study sample is from December 2007 to June 2009. As shown in Figure 1, the dynamics of the S&P500 index fluctuates widely in the recession period. Therefore, we divide the data into two groups accordingly and investigate the option pricing performance of the implied method in the recession and recovery periods. Figure 3 gives the boxplots of the RMSEs of the implied GARCH models in both periods. These boxplots show that the implied models are capable of obtaining good option prediction in both periods since the medians of the implied GARCH models are less than 3 and the medians of the implied TGARCH models are less than 1.2. Moreover, by using the testing procedure in Section 3.3, we have TGARCH-DE^{ess} $\approx$ TGARCH-N $\succ$ TGARCH-DE^{egp} $\succ$ GARCH-DE^{ess} $\succ$ GARCH-N $\succ$ GARCH-DE^{egp} in the recovery period and TGARCH-N $\succ$ TGARCH-DE^{ess} $\succ$ TGARCH-DE^{egp} $\succ$ GARCH-N $\approx$ GARCH-DE^{ess} $\succ$ GARCH-DE^{egp} in the recession period. The comparing results in the recovery period is the same as those in Section 3.3. However, for the data in the recession period, the implied TGARCH-N model performs better than others. Consequently, it seems that the heavy-tailed property of the innovations does not contribute significantly to implied TGARCH option pricing if the asymmetry property has been characterized. To examine this argument, the exotic option pricing by using the implied method is considered in next section.

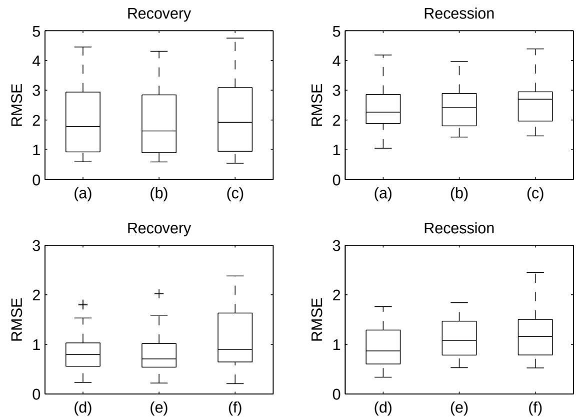

Figure 3: The left and right panels are the boxplots of the RMSE during the recovery and recession periods, respectively, for (a) GARCH-N, (b) GARCH-DE^{ess}, (c) GARCH-DE^{egp}, (d) TGARCH-N, (e) TGARCH-DE^{ess} and (f) TGARCH-DE^{egp} models

3.5 Exotic Option Pricing

As demonstrated in the previous sections, the implied TGARCH-N model is an adequate model for pricing plain vanilla options. Further note that the risk-neutral TGARCH-N models derived by Λ^{ess} and Λ^{egp} processes are the same, which means that practitioners do not have to worry about the selection of the change of measure processes if the innovations are assumed to be normal distributed. Therefore, the implied model with normal innovations seems to be a convenient choice for derivative pricing in practice. In the following, we show that this convenient way might not be suitable in exotic option pricing.

Exotic options are traded over the counter (OTC). In general, pricing exotic options is more complicated than pricing plain vanilla options since their payoffs depend on the history (or path) of the underlying asset prices. Hull and Suo (2002) discovered a model risk of the implied volatility model under the BS framework in barrier option pricing when the true model is assumed to be a stochastic volatility model. Recently, Huang and Guo (2011) demonstrated the model risk of an implied GARCH-N model in assessing barrier and lookback options when the underlying assets are generated from a GARCH-t model. Both simulation studies show that there exists significant model risk when pricing exotic options with an implied model. In this study, we investigate the empirical evidence of model risk in pricing exotic options.

Suppose that the stock prices follow a GARCH model with heavy-tailed innovations. We are interested in investigating the model risk of using the implied GARCH model with normal innovations in exotic option pricing. For example, let the true model be GARCH-DE^{ess} but an implied GARCH-N model is employed to evaluate the plain vanilla and up-and-out call options. The payoff of an up-and-out call is defined by $\max(S_T - K, 0) I_{\{\max_{0\le t\le T} S_t \le B\}}$, where B is the barrier price and $I_{\{\cdot\}}$ is an indicator function. Denote the prices of plain vanilla call options derived by the implied GARCH-N and GARCH-DE^{ess} models by $C^{(1)}$ and $C^{(2)}$, respectively. Similarly, $UOC^{(1)}$ and $UOC^{(2)}$ denote the corresponding up-and-out call option prices. The real data introduced in Section 3.1 are used and the implied GARCH models obtained in Section 3.2 are employed to compute the option prices by the EPMS method with 10,000 sample paths. The values of $C^{(1)}$, $C^{(2)}$, $UOC^{(1)}$ and $UOC^{(2)}$ are computed for the 78 trading days. Then, we compute the following mean squared losses:

$$MSL_{2,1} = \frac{1}{78}\sum_{i=1}^{78}\Big(\max\Big(\frac{C_i^{(2)}}{C_i^{(1)}}, \frac{C_i^{(1)}}{C_i^{(2)}}\Big) - 1\Big)^2,$$

and

$$MSL^*_{2,1} = \frac{1}{78}\sum_{i=1}^{78}\Big(\max\Big(\frac{UOC_i^{(2)}}{UOC_i^{(1)}}, \frac{UOC_i^{(1)}}{UOC_i^{(2)}}\Big) - 1\Big)^2.$$

If there exists severer model risk in exotic option pricing than in plain vanilla case, then $MSL^*_{2,1}$ would be significantly greater than $MSL_{2,1}$. Table 1 shows the values of $MSL^*_{i,j}/MSL_{i,j}$ with $S_0/K = 0.9, 1, 1.1$ and $(T, B) = (63, S_0/0.85)$, where $(i,j) = (2,1), (3,1), (5,4), (6,4)$ stand for the cases of GARCH-DE^{ess}

versus GARCH-N, GARCH-DE^{egp} versus GARCH-N, TGARCH-DE^{ess} versus TGARCH-N, and TGARCH-DE^{egp} versus TGARCH-N, respectively. Apparently, the values of MSL^* are all significantly greater than their MSL counterparts, especially when the option is at-the-money ($S_0/K = 1$) or in-the-money ($S_0/K = 1.1$). Moreover, the results of the cases with $(T, B) = (126, S_0/0.85)$, $(63, S_0/0.95)$ and $(126, S_0/0.95)$ are all similar to those shown in Table 1. Hence, we do not report them here for saving the space. From the above finding, the up-and-out option prices derived from implied GARCH-DE model (or TGARCH-DE model) are very different from those obtained by the implied GARCH-N model (or TGARCH-N model), which is consistent with the simulation findings of Hull and Suo (2002) and Huang and Guo (2011).

Table 1: The values of (a) $MSL^*_{2,1}/MSL_{2,1}$, (b) $MSL^*_{3,1}/MSL_{3,1}$, (c) $MSL^*_{5,4}/MSL_{5,4}$, and (d) $MSL^*_{6,4}/MSL_{6,4}$, where $K = S_0/0.9, S_0, S_0/1.1$ and $(T, B) = (63, S_0/0.85)$

S_0/K	0.9	1.0	1.1
(a) $MSL^*_{2,1}/MSL_{2,1}$	9.5	37.6	202.3
(b) $MSL^*_{3,1}/MSL_{3,1}$	2.2	31.8	157.5
(c) $MSL^*_{5,4}/MSL_{5,4}$	1.4	32.2	64.0
(d) $MSL^*_{6,4}/MSL_{6,4}$	1.6	13.4	41.0

Therefore, as shown in the previous sections, the model risk is not sever if the implied model with normal innovation is applied to evaluate plain vanilla options. However, when pricing exotic options, one can not just rely on the implied method. The effect of the innovation distribution on exotic option pricing can not be ignored.

4. Conclusion

In this study, we use the implied method to estimate the GARCH model parameters. Two popular change of measure processes and an efficient simulation method are employed in the investigation. The empirical results show that the implied GARCH models are capable of computing accurate plain vanilla option prices even the innovation is simply assumed to be normal distributed. In addition, the change of measure process derived by the Esscher transform describes the data better than that derived by the extended Girsanov principle. Moreover, the implied method under the GARCH framework with normal innovations might not be suitable for assessing complicated financial derivatives because significant model risk arises when using an implied GARCH model with non-proper innovations in exotic option pricing.

Appendix

Proof of Proposition 2.1

We first derive the change of measure process Λ_t^{ess} defined in (2.2) of the Esscher transform. Since the conditional distribution of R_t given $\mathcal{F}_{t-1}$ is $N(r + \lambda\sigma_t - \sigma_t^2/2, \sigma_t^2)$, the corresponding conditional mgf is

$$\mathrm{M}_{R_t|\mathcal{F}_{t-1}}(z) = e^{z(r+\lambda\sigma_t-\sigma_t^2/2)+z^2\sigma_t^2/2}.$$

Define a new conditional pdf $f_t(x_t;\delta_t)$ by (2.1) and choose the extra parameter $\delta_t = \delta_t^*$ to ensure $E_{t-1}^{Q^{ess}}(e^{-r}S_t) = S_{t-1}$ under $f_t(x_t;\delta_t^*)$, where $\delta_t^* = -\lambda/\sigma_t$. Therefore, by (2.2) we have

$$\Lambda_t^{ess} = \prod_{k=1}^{t} \exp\{-\frac{1}{2\sigma_k^2}[\lambda^2\sigma_k^2 + 2\lambda\sigma_k(R_k - r - \lambda\sigma_k + \frac{1}{2}\sigma_k^2)]\}.$$

Next, we derive the change of measure process of the extended Girsanov principle Λ_t^{egp} defined in (2.3). Since

$$\frac{\mathrm{M}_t}{\mathrm{M}_{t-1}} = \frac{\exp(R_t)}{E_{t-1}[\exp(R_t)]} = \exp\{\sigma_t\varepsilon_t - \frac{\sigma_t^2}{2}\},$$

thus the pdf of $Y = \mathrm{M}_t/\mathrm{M}_{t-1}$ is

$$\phi(y) = \frac{1}{y\sqrt{2\pi}\sigma_t}\exp\{-\frac{1}{2\sigma_t^2}(\log y + \frac{\sigma_t^2}{2})^2\}.$$

By (2.3), we have

$$\Lambda_t^{egp} = \prod_{k=1}^{t} \exp\{-\frac{1}{2\sigma_k^2}[\lambda^2\sigma_k^2 + 2\lambda\sigma_k(R_k - r - \lambda\sigma_k + \frac{1}{2}\sigma_k^2)]\}.$$

□

Proof of Proposition 2.2

We first derive the change of measure process Λ_t^{ess} defined in (2.2) of the Esscher transform. Since the conditional distribution of ε_t given $\mathcal{F}_{t-1}$ is double-exponentially distributed with conditional mgf $E_{t-1}(e^{s\varepsilon_t}) = (1 - s^2/2)^{-1}$ for $|s| < 2$, thus the corresponding conditional mgf of R_t given $\mathcal{F}_{t-1}$ is

$$\mathrm{M}_{R_t|\mathcal{F}_{t-1}}(z) = \frac{e^{z\mu_t}}{(1 - z^2\sigma_t^2/2)}.$$

Define a new conditional pdf $f_t(x_t;\delta_t)$ by (2.1) and choose the extra parameter $\delta_t = \delta_t^*$ to ensure $E_{t-1}^{Q^{ess}}(e^{-r}S_t) = S_{t-1}$ under $f_t(x_t;\delta_t^*)$ where

$$\delta_t^* = \frac{-\sigma_t + \sqrt{a_t\sigma_t^2 + 2(a_t-1)^2}}{\sigma_t(1-a_t)}$$

with $a_t = \exp(\lambda\sigma_t - \sigma_t^2/2)$. Therefore, by (2.2) we have

$$\Lambda_t^{ess} = \prod_{k=1}^{t}(1 - \frac{(\delta_k^*\sigma_k)^2}{2})\exp\{\delta_k^*(R_k - r - \lambda\sigma_k + \frac{\sigma_k^2}{2})\}.$$

Next, we derive the change of measure process of the extended Girsanov principle Λ_t^{egp}. Since

$$\frac{\mathrm{M}_t}{\mathrm{M}_{t-1}} = \frac{\exp(R_t)}{E_{t-1}[\exp(R_t)]} = \exp(R_t - \mu_t - \gamma_t),$$

where $\gamma_t = -\log(1-\sigma_t^2/2)$ with $\sigma_t^2 < 2$, thus the pdf of $Y = \mathrm{M}_t/\mathrm{M}_{t-1}$ is

$$\phi(y) = \frac{\exp\{-\sqrt{2}|\log y + \gamma_t|/\sigma_t\}}{(y\sigma_t\sqrt{2})} \quad \text{for} \quad y > 0.$$

By (2.3), we have

$$\Lambda_t^{egp} = \prod_{k=1}^{t}\exp\{-\sqrt{2}(|R_k + \gamma_k - r| - |R_k - \mu_k|)/\sigma_k\}.$$

□

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