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Notes on Entropy for Concomitants of Record Values in Farlie-Gumbel-Morgenstern (FGM) Family

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Abstract: Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of bivariate random variables from a continuous distribution. If $\{R_n, n \geq 1\}$ is the sequence of record values in the sequence of X's, then the Y which corresponds with the nth-record will be called the concomitant of the nth-record, denoted by $R_{[n]}$. In FGM family, we determine the amount of information contained in $R_{[n]}$ and compare it with amount of information given in R_n. Also, we show that the Kullback-Leibler distance among the concomitants of record values is distribution-free. Finally, we provide some numerical results of mutual information and Pearson correlation coefficient for measuring the amount of dependency between R_n and $R_{[n]}$ in the copula model of FGM family.

Key words: Concomitants, Farlie-Gumbel-Morgenstern family, Kullback-Leibler distance, mutual information, record values, Shannon entropy.

1. Introduction

Let $(X_1, Y_1), (X_2, Y_2), \cdots$ be a sequence of bivariate random variables from a continuous distribution. If $\{R_n, n \geq 1\}$ is the sequence of record values in the sequence of X's, then the Y which corresponds with the nth-record will be called the concomitant of the nth-record, denoted by $R_{[n]}$. The concomitants of record values arise in a wide variety of practical experiments such as industrial stress testing, life time experiments, meteorological analysis, sporting matches and some other experimental fields. For other important applications of record values and their concomitants see Arnold *et al.* (1998) and Ahsanullah (1995). Some properties from concomitants of record values were discussed in Houchens (1984), Nevzorov and Ahsanullah (2000). The cumulative distribution function (cdf) for the FGM family is given by Johnson and Kotz (1975) as

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))], \quad -1 \leq \alpha \leq 1, \qquad (1)$$

where $F_X(x)$, $F_Y(y)$ are marginal cdf of X and Y, respectively. The copula model for this family is defined by Nelson (1999) as follows:

$$f_{X,Y}(x, y) = [1 + \alpha(-1 + 2x)(-1 + 2y)], \quad 0 \leq x, y \leq 1. \qquad (2)$$

Houchens (1984) has obtained the probability density function (pdf) of concomitant of nth-record value for $n \geq 1$ arising in (1) as

$$h_{[n]}(y) = \int_{-\infty}^{+\infty} f(y|x) g_n(x) dx = f_Y(y)[1 + \alpha_n(2F_Y(y) - 1)], \tag{3}$$

where $\alpha_n = \alpha(1 - 2^{1-n})$ and pdf of R_n is

$$g_n(x) = \frac{1}{(n-1)!}[-\ln(1 - F_X(x))]^{n-1} f_X(x). \tag{4}$$

Also, the joint density function of $R_n, R_{[n]}$ for the FGM copula is given by

$$f_{n,[n]}(x, y) = [1 + \alpha(1 - 2x)(1 - 2y)] \frac{[-\ln(1 - x)]^{n-1}}{(n-1)!}. \tag{5}$$

The information measures for record values have been investigated by several authors, including, Zahedi and Shakil (2006), Baratpour *et al.* (2007), and Madadi and Tata (2009). Amini and Ahmadi (2007) investigate the properties of Fisher information in the sequence of the first n records and their concomitants. Tahmasebi and Behboodian (2012) obtained some results of information measures for concomitants of order statistics. Recently, Fashandi and Ahmadi (2012) studied characterizations of symmetric distributions based on Rényi entropy of concocomiants. Shannon's entropy of a continuous random variable X, with pdf $f_X(x)$, is given by

$$H(X) = -\int_{-\infty}^{+\infty} f_X(x) \ln f_X(x) dx. \tag{6}$$

This is a mathematical measure of information which measures the average reduction of uncertainty of X. The organization of this article is as follows. In Section 2, we determine the amount of information contained in $R_{[n]}$ and compare it with amount of information given in R_n. In Section 3, we show that the Kullback-Leibler distance between concomitants of nth- and mth- record values in FGM family is free from marginal distributions. Also, we present analytical expressions and some numerical results of mutual information and Pearson correlation coefficient between R_n and $R_{[n]}$ in the copula model of FGM family.

2. Entropy for Concomitants of Record Values in FGM Family

Theorem 2.1. Let (X_i, Y_i), $i = 1, 2, \cdots$ be a sequence of independent observations from (1). If $R_{[n]}$ is the concomitant of the nth-record value on the X sequence of observations, then the Shannon entropy of $R_{[n]}$ for $n > 1$, and $\alpha \neq 0$ is given by

$$H(R_{[n]}) = C_\alpha(n) + H(Y)(1 - \alpha_n) - 2\alpha_n \phi(f), \tag{7}$$

where

$$C_\alpha(n) = \frac{1}{8\alpha_n}\{(1-\alpha_n)^2[2\ln(1-\alpha_n)-1] - (1+\alpha_n)^2[2\ln(1+\alpha_n)-1]\}, \quad (8)$$

and

$$\phi(f) = \int_0^1 u \ln f_Y(F_Y^{-1}(u))du.$$

Proof. From (3) and (6), we get

$$\begin{aligned} H(R_{[n]}) &= -E_{h_{[n]}}[\ln f_Y(Y)] - E_{h_{[n]}}[\ln(1+\alpha_n(2F_Y(Y)-1))] \\ &= H(Y)(1-\alpha_n) - 2\alpha_n \int_0^1 u \ln f_Y(F_Y^{-1}(u))du \\ &\quad - E_{h_{[n]}}[\ln(1+\alpha_n(2F_Y(Y)-1))]\,. \end{aligned} \quad (9)$$

Now, we need to find $E_{h_{[n]}}[\ln(1+\alpha_n(2F_Y(Y)-1))]$. First, we write

$$\begin{aligned} T(r) &= E_{h_{[n]}}[(1+\alpha_n(2F_Y(Y)-1))^r] = \int_{-\infty}^{+\infty} f_Y(y)[1+\alpha_n(2F_Y(y)-1)]^{r+1}dy \\ &= \frac{1}{2\alpha_n}[\frac{(1+\alpha_n)^{r+2}-(1-\alpha_n)^{r+2}}{r+2}]. \end{aligned} \quad (10)$$

Since the function under the integral sign in (10) is bounded by an integrable function. So, we have

$$\begin{aligned} &\frac{-\partial T(r)}{\partial r}|_{r=0} = -E_{h_{[n]}}[\ln(1+\alpha_n(2F_Y(Y)-1))] = C_\alpha(n) \\ &= \frac{1}{8\alpha_n}[(1-\alpha_n)^2(2\ln(1-\alpha_n)-1) - (1+\alpha_n)^2(2\ln(1+\alpha_n)-1)]. \end{aligned} \quad (11)$$

Putting (11) in (9) the result follows. □

The difference between entropy of $(n+1)$th- and nth- concomitant of record values is obtained as

$$\Delta(n) = H(R_{[n+1]}) - H(R_{[n]}) = C_\alpha(n+1) - C_\alpha(n) - \frac{\alpha}{2^n}[H(Y) + 2\phi(f)].$$

Note that $\lim_{n\to\infty} \Delta(n) = 0$. A general expression for the entropy of the nth-record value R_n is presented by Zahedi and Shakil (2006) as

$$\begin{aligned} H(R_n) &= \ln(\Gamma(n)) - (n-1)\psi(n) \\ &\quad - \frac{1}{\Gamma(n)} \int_{-\infty}^{+\infty} [-\ln(1-F_X(x))]^{n-1} f_X(x) \ln(f_X(x))dx, \end{aligned} \quad (12)$$

where $\psi(n)$ is the digamma function. More recently, Baratpour *et al.* (2007)

have explored the properties of $H(R_n)$. In the following examples we compare $H(R_{[n]})$ with $H(R_n)$ in FGM family.

Example 2.1. Let $Z_{[n]}$ be concomitant of nth-record value from (2), then, by using (3), the density function of $Z_{[n]}$ is

$$p_{[n]}(u) = 1 + \alpha_n(2u - 1).$$

Now, by using (7) and (12), we can easily show that $H(Z_{[n]})$ with $H(Z_n)$ for the copula model of FGM family have the following properties

(i) $H(Z_{[n]}) = C_\alpha(n) = C_{-\alpha}(n)$, $\forall n > 1$, and $\alpha \neq 0$.

(ii) $H(Z_n) = \log(\Gamma(n)) - (n-1)\psi(n)$.

(iii) $H(Z_n) \leq H(Z_{[n]})$.

(iv) $.5 - \ln(2) < H(Z_{[n]}) \leq 0$, for $n \geq 1$, and $-1 \leq \alpha \leq 1$.

(v) for $n > 1$, $H(Z_{[n]})$ is increasing (decreasing) in α for $-1 \leq \alpha < 0$ $(0 < \alpha \leq 1)$.

(vi) $H(Z_{[n]})$ is decreasing in n for $\alpha \neq 0$.

The relative differential entropy index, $\eta_\alpha(m, n)$, between mth- and nth- $(1 < m < n)$ concomitant of record values, is given by

$$\eta_\alpha(m, n) = \frac{H(Z_{[n]}) - H(Z_{[m]})}{H(Z_{[n]}) - H(Z_{[1]})} = \frac{C_\alpha(n) - C_\alpha(m)}{C_\alpha(n)}.$$

Our numerical computations indicate when n and α are fixed, then, $\eta_\alpha(m, n)$ for the copula model of FGM family is decreasing in m.

Remark 1. Using FGM example of Ebrahimi *et al.* (2010), insightful expression for $H(R_{[n]})$ is given by

$$H(R_{[n]}) = H(Z_{[n]}) - E_{h_{[n]}}[\ln f_Y(Y)], \tag{13}$$

where $H(Z_{[n]}) = C_\alpha(n)$ is the entropy for nth- concomitant of record value in the copula model of FGM family.

Remark 2. The following representation gives the Rényi entropy of order β for $R_{[n]}$ in FGM family as

$$R^*_\beta(R_{[n]}) = \frac{1}{1-\beta} \ln E_U\{[p_{[n]}(U)]^\beta [f_Y(F_Y^{-1}(U))]^{\beta-1}\}, \tag{14}$$

where U is a uniform random variable with parameters 0 and 1.

Example 2.2. Let $(X_i,Y_i), i=1,2,\cdots$ be a sequence of independent observations from Gumbel's bivariate exponential distribution with cdf

$$F(x,y)=(1-\exp(\frac{-x}{\theta_1}))(1-\exp(\frac{-y}{\theta_2}))[1+\alpha\exp(\frac{-x}{\theta_1}-\frac{y}{\theta_2})],$$
$$-1\leq\alpha\leq1,\ \ x,y>0,\ \ \theta_1,\theta_2>0.$$

Then in this case by using (3), the pdf of $R_{[n]}$ is

$$h_{[n]}(y)=\frac{1}{\theta_2}e^{\frac{-y}{\theta_2}}[1+\alpha_n(1-2e^{\frac{-y}{\theta_2}})].$$

Now, with using (7), we have

$$H(R_{[n]})=C_\alpha(n)+(1+\ln(\theta_2))+\frac{\alpha_n}{2}. \tag{15}$$

If we put $\theta_1=\theta_2$, then by using (12), (15) and numerical computations, we have

$$B_\alpha(n)=H(R_{[n]})-H(R_n)=C_\alpha(n)+1+\frac{\alpha_n}{2}-\ln(\Gamma(n))+(n-1)\psi(n)-n>0.$$

It is easy to check the following properties regarding $H(R_{[n]})$ from FGM type Gumbel's bivariate exponential distribution.

(i) $H(R_{[n]})$ is monotone increasing in α, for $n>1$; $\forall\theta_2>0$.

(ii) $H(R_{[n]})$ is an increasing concave function of θ_2, for $n>1$, $\alpha\neq0$.

(iii) $H(R_{[n]})$ is increasing (decreasing) in n, for $0<\alpha\leq1$ ($-1\leq\alpha<0$).

Example 2.3. Suppose (X_i,Y_i), $i=1,2,\cdots$ be a sequence of independent observations from (1) with cdf

$$F(x,y)=(1-x^{-\lambda_1})(1-y^{-\lambda_2})[1+\alpha(x^{-\lambda_1})(y^{-\lambda_2})],\ \ x,y>1,\ \ \lambda_1,\lambda_2>0.$$

In this case, by using (3), the pdf of $R_{[n]}$ is

$$h_{[n]}(y)=\lambda_2 y^{-(\lambda_2+1)}[1+\alpha_n(1-2y^{-\lambda_2})],\ \ y>1.$$

By using (7), we get

$$H(R_{[n]})=C_\alpha(n)-\ln\lambda_2+\frac{\lambda_2+1}{\lambda_2}(1+\frac{\alpha_n}{2}). \tag{16}$$

Now, if we put $\lambda_1=\lambda_2=\lambda$, then by using (12), (16) and numerical computations, we have

$$J_\alpha(n)=H(R_{[n]})-H(R_n)=C_\alpha(n)-\ln\Gamma(n)+(n-1)\psi(n)+\frac{\lambda+1}{\lambda}(1+\frac{\alpha_n}{2}-n)<0.$$

It is easy to check the following properties of $H(R_{[n]})$ as:

(i) $H(R_{[n]})$ is monotone decreasing in λ_2, for $n > 1$, $\alpha \neq 0$.

(ii) $H(R_{[n]})$ is monotone increasing in α, for $n > 1$; $\forall \lambda_2 > 0$.

(iii) $H(R_{[n]})$ is increasing (decreasing) in n, for $0 < \alpha \leq 1$ $(-1 \leq \alpha < 0)$.

In the following theorem, we provide entropy bounds for concomitants of record values in FGM family.

Theorem 2.2. Let $R_{[n]}$ be the concomitant of the nth-record value in FGM family. Then for $n > 1$, and $\alpha \neq 0$, we have

$$C_\alpha(n) + H(Y)[1 - |\alpha|(1 - 2^{1-n})] \leq H(R_{[n]}) \leq C_\alpha(n) + H(Y)[1 + |\alpha|(1 - 2^{1-n})]. \tag{17}$$

Proof. From Theorem 2.1, we have

$$H(R_{[n]}) = C_\alpha(n) + H(Y) + D_\alpha(n), \tag{18}$$

where

$$D_\alpha(n) = \alpha(1 - 2^{1-n}) \int_{-\infty}^{+\infty} (1 - 2F_Y(y)) f_Y(y) \ln f_Y(y) dy,$$

and $C_\alpha(n)$ is defined in (8). Since $-1 \leq 1 - 2F_Y(y) \leq 1$, we have

$$-|\alpha|(1 - 2^{1-n})H(Y) \leq D_\alpha(n) \leq |\alpha|(1 - 2^{1-n})H(Y). \tag{19}$$

Thus, by (18) and (19), the proof is clear. □

3. Kullback-Leibler Distance

The Kullback-Leibler distance for two continuous random variables Z_1 and Z_2 with pdf's f_1 and f_2, respectively, is given by

$$K(Z_1, Z_2) = \int_{-\infty}^{+\infty} f_1(z) \ln(\frac{f_1(z)}{f_2(z)}) dz = E_1(\ln \frac{f_1(Z)}{f_2(Z)}), \tag{20}$$

where E_1 denotes the expectation with respect to f_1. $K(Z_1, Z_2) \geq 0$, and equality holds if and only if $f_1(z) = f_2(z)$ almost everywhere. It generalizes two measures of information, entropy and mutual information for communication theory.

Theorem 3.1. Let $R_{[n]}$ and $R_{[m]}$ be the concomitants of nth- and mth- record values in FGM family. Then the Kullback-Leibler distance between $R_{[n]}$ and $R_{[m]}$ is

$$\begin{aligned} K(R_{[n]}, R_{[m]}) &= K(Z_{[n]}, Z_{[m]}) \\ &= -C_\alpha(n) + (2^{-n} - 2^{-m})U_\alpha(m) + \frac{1 - 2^{1-n}}{1 - 2^{1-m}} C_\alpha(m), \end{aligned} \tag{21}$$

where

$$U_\alpha(m) = \frac{1}{\alpha(1-2^{1-m})^2}\{(1-\alpha_m)[\ln(1-\alpha_m)-1] \\ -(1+\alpha_m)[\ln(1+\alpha_m)-1]\}, \quad (22)$$

and $C_\alpha(n)$ is defined in (8).

Proof. By using the invariance property of Kullback -Leibler information, the proof is easy. □

Corollary 3.1. If $n = 1$ and $m > 1$, then, we obtain

$$K(R_{[1]}, R_{[m]}) = -\int f_Y(y)\ln[1+\alpha_m(2F_Y(y)-1)]dy = V_\alpha(m), \quad (23)$$

where

$$V_\alpha(m) = \frac{1}{2\alpha_m}\{(1-\alpha_m))[\ln(1-\alpha_m)-1] - (1+\alpha_m)[\ln(1+\alpha_m)-1]\}.$$

It is easy to see that $K(R_{[1]}, R_{[m]})$ for $m \geq 1$, and $-1 \leq \alpha \leq 1$ has the following properties:

(i) $0 \leq K(R_{[1]}, R_{[m]}) < 1 - \ln 2$ for $m \geq 1$, and $-1 \leq \alpha \leq 1$.

(ii) for fixed $m > 1$, $K(R_{[1]}, R_{[m]})$ is decreasing (increasing) in α for $-1 \leq \alpha < 0$ $(0 < \alpha \leq 1)$.

(iii) $V_\alpha(m)$ is increasing in m for $\alpha \neq 0$.

(iv) $V_\alpha(m) = V_{-\alpha}(m)$, for $m > 1$, and $\alpha \neq 0$.

Corollary 3.2. If $m = 1$ and $n > 1$, then, we have

$$K(R_{[n]}, R_{[1]}) = \int h_{[n]}(y)\ln(\frac{h_{[n]}(y)}{f_Y(y)})dy = -C_\alpha(n),$$

where $.5 - \ln 2 < C_\alpha(n) \leq 0$. Now, by using the results of Example 2.1 and Corollary 3.1, we can conclude that

$$\min_{1\leq n} K(R_{[n]}, R_{[1]}) = \min_{1\leq m} K(R_{[1]}, R_{[m]}) = 0.$$

For a bivariate random variable (X, Y) with density function $f_{X,Y}(x, y)$ the mutual information is defined as

$$I(X, Y) = H(Y) - H(Y|X), \quad (24)$$

where $H(Y|X)$ is the conditional entropy of Y given X. The mutual information is a generalization of the coefficient of determination, ρ^2_{XY}, which unifies a variety of problems. We know that $I(X,Y) \geq 0$, and equality holds if and only if X and Y are statistically independent. Also, it has the invariance property under one-to-one transformation of (X,Y).

Corollary 3.3. Let $R_n = F^{-1}(Z_n)$ be the nth-record value and $R_{[n]} = F^{-1}(Z_{[n]})$ be its concomitant obtained by a sequence from (1). Then, the mutual information between R_n and $R_{[n]}$ for $n > 1$ is distribution-free and is given by

$$\begin{aligned} I_\alpha(R_n, R_{[n]}) &= I_\alpha(Z_n, Z_{[n]}) \\ &= C_\alpha(n) + \int_{-1}^{1}\int_{-1}^{1} \ln(1+\alpha wv)(1+\alpha wv)\frac{[-\ln(\frac{1+w}{2})]^{n-1}}{4(n-1)!}dvdw \\ &= C_\alpha(n) + E_{f_{n,[n]}}[\ln(1+\alpha WV)], \end{aligned} \tag{25}$$

where $W = 1-2X$ and $V = 1-2Y$.

Proof. By using the invariance property of mutual information, the proof is clear. □

We can also obtain an explicit expression for the Pearson correlation coefficient between R_n and $R_{[n]}$ in the copula model of FGM family. By using (5) and after some simple algebra, we get

$$\rho_\alpha(R_n, R_{[n]}) = \frac{\alpha}{3}\sqrt{\frac{4^n - 3^n}{12^{n-1}[1 - \frac{\alpha^2}{3}(2^{1-n}-1)^2]}}. \tag{26}$$

Table 1 provides the values of $I_\alpha(R_n, R_{[n]})$ and $\rho_\alpha(R_n, R_{[n]})$ as a function of n and α, for $n = 1(1)9$, and $\alpha = .2, .4, .8, 1$. These values are derived by using (25), (26) and Maple software. Table 1 and easy computations show that $I_\alpha(R_n, R_{[n]})$ and $\rho_\alpha(R_n, R_{[n]})$ for the copula model of FGM family have the following properties:

(i) $I_\alpha(R_n, R_{[n]}) = I_{-\alpha}(R_n, R_{[n]})$,

(ii) $\rho_\alpha(R_n, R_{[n]}) = -\rho_{-\alpha}(R_n, R_{[n]})$,

(iii) $I_\alpha(R_n, R_{[n]}) < |\rho_\alpha(R_n, R_{[n]})|$,

(iv) $I_\alpha(R_n, R_{[n]})$ increases as $|\alpha|$ increases,

(v) $I_\alpha(R_n, R_{[n]}) \leq I_\alpha(X,Y) \leq 0.0599$,

(vi) $|\rho_\alpha(R_n, R_{[n]})| \leq \alpha/3$,

(vii) For fixed $\alpha \neq 0$, $I_\alpha(R_n, R_{[n]})$ is decreasing in n.

Table 1: $I_\alpha(R_n, R_{[n]})$ and $\rho_\alpha(R_n, R_{[n]})$ for the copula model of FGM family

	$I_\alpha(R_n, R_{[n]})$				$\rho_\alpha(R_n, R_{[n]})$			
	α				α			
n	0.2	0.4	0.8	1	0.2	0.4	0.8	1
1	0.0022	0.0089	0.0371	0.0599	0.0666	0.1333	0.2666	0.3333
2	0.0013	0.0052	0.0227	0.0383	0.0510	0.1025	0.2093	0.2659
3	0.0005	0.0023	0.0109	0.0202	0.0339	0.0686	0.1440	0.1874
4	0.0002	0.0009	0.0047	0.0097	0.0213	0.0433	0.0927	0.1229
5	0.00008	0.0003	0.0018	0.0043	0.0130	0.0265	0.0574	0.0769
6	0.00003	0.0001	0.0007	0.0018	0.0078	0.0159	0.0346	0.0467
7	0.000010	0.00004	0.0002	0.0007	0.0046	0.0094	0.0206	0.0279
8	0.000003	0.000015	0.000092	0.0003	0.0027	0.0055	0.0121	0.0164
9	0.000001	0.000005	0.00003	0.0001	0.0015	0.0032	0.0071	0.0096

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References

Ahsanullah, M. (1995). *Record Statistics.* Nova Science Publishers, New York.

Amini, M. and Ahmadi, J. (2007). Fisher information in record values and their concomitants about dependence and correlation parameters. *Statistics and Probability Letters* **77**, 964-972.

Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998). *Records.* Wiley, New York.

Baratpour, S., Ahmadi, J. and Arghami, N. R. (2007). Entropy properties of record statistics. *Statistical Papers* **48**, 197-213.

Ebrahimi, N., Hamedani, G. G., Soofi, E. S. and Volkmer, H. (2010). A class of models for uncorrelated random variables. *Journal of Multivariate Analysis* **8**, 1859-1871.

Fashandi, M. and Ahmadi, J. (2012). Characterizations of symmetric distributions based Rényi entropy. *Statistics and Probability Letters* **82**, 798-804.

Houchens, R. L. (1984). *Record Value Theory and Inference.* Ph.D. Dissertation, University of California, Riverside, California.

Johnson, N. L. and Kotz, S. (1975). On some generalized Farlie-Gumbel-Morgenstern distributions. *Communication in Statistics - Theory and Methods* **4**, 415-427.

Madadi, M. and Tata, M. (2011). Shannon information in record data. *Metrika* **74**, 11-31.

Nelson, R. B. (1999). *An Introduction to Copulas.* Lecture Notes in Statistics 139, Springer, New York.

Nevzorov, V. B. and Ahsanullah, M. (2000). Some distributions of induced records. *Biometrical Journal* **42**, 1069-1081.

Tahmasebi, S. and Behboodian, J. (2012). Information properties for concomitants of order statistics in Farlie-Gumbel-Morgenstern (FGM) family. *Communication in Statistics - Theory and Methods* **41**, 1954-1968.

Zahedi, H. and Shakil, M. (2006). Properties of entropies of record values in reliability and life testing context. *Communication in Statistics - Theory and Methods* **35**, 997-1010.

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